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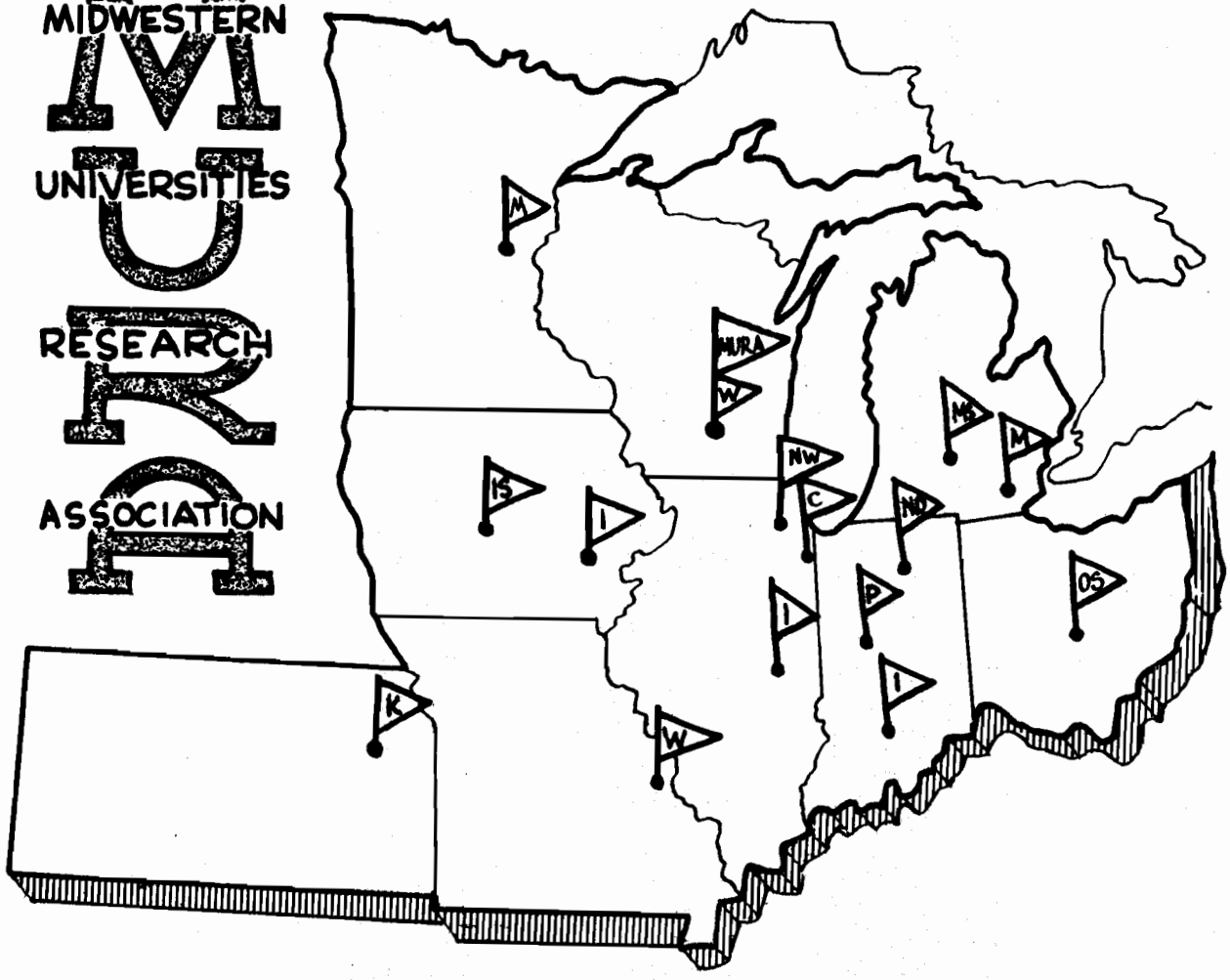

LIOUVILLE'S THEOREM FOR A CONTINUOUS MEDIUM WITH CONSERVATIVE INTERACTIONS

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REPORT **NUMBER** 433

MURA 433

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LIOUVILLE'S THEOREM FOR A CONTINUOUS MEDIUM WITH CONSERVATIVE INTERACTIONS**

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October 8, 1958

ABSTRACT

It is shown that for a continuous medium with conservative interactions the density in six-dimensional phase space is preserved as one follows the motion of the medium.

*AEC Research and Development Report. Research supported by the Atomic Energy Commission, Contract No. AEC AT (11-1) 384.

**Research supported in part by the National Science Foundation.

I. INTRODUCTION

The study of the motion of particles in an accelerator becomes a many-body problem when the interactions between particles are taken into account. It is thus important to investigate the possibility of establishing the validity--or approximate validity--of general dynamical theorems applicable to the n-body problem. Such a powerful theorem is the theorem proved here to be rigorously valid for continuous media, and asserted to be an extremely good approximate theorem for particles in an accelerator.

Liouville's theorem is a theorem which asserts that in a $2fN$ dimensional space (f is the number of degrees of freedom of one particle) spanned by the coordinates and momenta of all particles (called γ space), the density in phase is a constant as one moves along with any phase point. It is thus a statement about the density of points; each point representing a dynamical system. The systems constitute an ensemble and of course do not interact.

The theorem proven here refers to a system of many interacting particles, and asserts that in the $2f$-dimensional space spanned by a single system of coordinates and momenta (called a μ space), the density in phase is a constant as one moves along with any phase point. It is thus a statement about the behavior of interacting particles, and thus really quite different from Liouville's theorem.

The validity of the theorem, as well as the limits of its validity, may readily be seen by the following intuitive argument:

Consider first a system of many particles, N. Suppose these particles are subject to external forces (which may even be time dependent), but there

are no interactions between the particles. Clearly density in phase in μ space is a constant of the motion as one follows the motion of a phase point. This follows then immediately from Liouville's Theorem in γ space, since with no interactions between particles μ space for N particles is simply γ space for a single particle.

Consider now a system of a great many particles N, with interactions between the particles. Imagine that the solution has been obtained so that we know the motion of all the particles as a function of time. Concentrate now on a "small" number of particles n, which initially are localized in μ space. We will define what "small" means shortly. Let all the other particles move along the trajectories appropriate to the solution of the N-body problem. If the interactions between one of the particles and the n particles can be neglected compared to the interactions between the N-n particles and one particle, then these particles are subject to "external forces" and by the first case the density in μ space is a constant as one moves along with the sample group of n particles. This is clearly true for any sample, and hence the theorem is established.

That is, as long as one has sufficient particles N, that a sample can be obtained of sufficiently small number of particles n, that the interactions between these particles and one of their number is negligible compared to the interactions between one of these particles and the N-n particles, while at the same time n is sufficiently large that fluctuation phenomena can be neglected, then the theorem is valid. In the rigorous proof given in the next section, the limit of a continuous medium is taken so that fluctuation phenomena

do not exist. For applications to particle accelerators where we consider a number of particles $N \simeq 10^{13}$ this approximation is very valid, corresponding to neglect of particle-particle collisions which throw a particle out of the accelerator, but not neglecting long range electromagnetic interactions which are responsible for space-charge limits, plasma oscillations, beam-beam interactions, and possible two-stream amplification mechanisms.

The practical importance of the theorem can be readily seen by limiting one's attention to systems which initially have a constant density in a restricted region of μ space, and no particles outside this region. (This is determined by the injection mechanism, and is a reasonable approximation to most situations). In this case, the N-body problem is completely characterized by the behavior of the boundary surface as a function of time. This surface satisfies a partial differential integral equation of the first order in at most 2f independent variables, so that the N-body problem((f N) differential equations of the second order) is greatly simplified. In particular, for problems involving one degree of freedom, the equation for the boundary curve as a function of time and one coordinate is quite amenable to analysis.

II. FORMAL PROOF

Let λ_i (i = 1, ---, 2f,) be parameters labelling the particles of the medium (2f dimensional phase space; this is the μ space).

$dn = \sigma\, d\lambda_1 \ldots d\lambda_{2f}$ = number of particles in 'volume' element $d\lambda$.

σ = constant 'density' with respect to λ .

Let $\pi_\alpha(\lambda)$ = momentum density

$= \sigma p_\alpha$

$q_\alpha(\lambda)$ = position of particle λ .

The Hamiltonian is

$$H = \sigma \int h[q(\lambda), \pi(\lambda)]\, d\lambda + \sigma^2 \iint v[q(\lambda), \pi(\lambda), q(\lambda')\pi(\lambda')]\, d\lambda\, d\lambda',$$

where h and v do not depend explicitly on λ , and the equations of motion are

$$\dot{q}_\alpha(\lambda) = \frac{\delta H}{\delta \pi_\alpha(\lambda)} = \sigma \frac{\partial h[\lambda]}{\partial \pi_\alpha(\lambda)} + \sigma^2 \int \frac{\partial}{\partial \pi_\alpha(\lambda)} v[\lambda, \lambda']\, d\lambda'$$

$$\dot{\pi}_\alpha(\lambda) = -\frac{\delta H}{\delta q_\alpha(\lambda)} = -\sigma \frac{\partial h[\lambda]}{\partial q_\alpha[\lambda]} - \sigma^2 \int \frac{\partial}{\partial q_\alpha(\lambda)} v[\lambda, \lambda']\, d\lambda'.$$

Let us introduce a condensed notation:

$$\begin{aligned} \xi_s &= q_s \quad , \quad s = 1, \ldots, f \\ &= -\pi_s \quad , \quad s = f+1, \ldots, 2f. \end{aligned}$$

The equations of motion may be written

$$\dot{\xi}_s(\lambda) = \sum_t \epsilon_{st} \frac{\delta H}{\delta \xi_t(\lambda)} = \sum_t \epsilon_{st} \left[\sigma \frac{\partial h}{\partial \xi_t} + \sigma^2 \int \frac{\partial v}{\partial \xi_t}\, d\lambda' \right],$$

where

$$\begin{aligned} \epsilon_{s,f+s} &= 1 \quad , \quad s = 1, \ldots, f \\ \epsilon_{f+s,s} &= -1 \quad , \quad s = 1, \ldots, f \\ \epsilon_{st} &= 0 \quad , \quad \text{otherwise.} \end{aligned}$$

Clearly ϵ_{st} is antisymmetric. Now the density $\rho(q, \pi)$ in phase space is given by

$$\rho\, dq\, d\pi = \sigma\, d\lambda$$

or

$$\rho\, d\xi = \sigma\, d\lambda .$$

$$\frac{1}{\rho} = \frac{1}{\sigma} \times \text{Jacobian}$$

$$= \frac{1}{\sigma} \times \det\left|\frac{\partial \xi_s}{\partial \lambda_i}\right|$$

$$= \frac{1}{\sigma}\,\Delta, \text{ say.}$$

The inverse matrix to $\partial \xi_s / \partial \lambda_i$ is of course $\partial \lambda_i / \partial \xi_s$, and the adjoint to $\partial \xi_s / \partial \lambda_i$ is $\Delta \dfrac{\partial \lambda_i}{\partial \xi_s}$. Thus we have for the rate of change of $\dfrac{1}{\rho}$ along a trajectory (λ = constant):

$$\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{1}{\sigma} \left.\frac{\partial \Delta}{\partial t}\right|_{\lambda = \text{Constant}}$$

$$= \frac{1}{\sigma} \sum_{i,s} \left(\Delta \frac{\partial \lambda_i}{\partial \xi_s}\right) \frac{\partial^2 \xi_s}{\partial t\, \partial \lambda_i}$$

But

$$\frac{\partial^2 \xi_s}{\partial t\, \partial \lambda_i} = \sum_t \frac{\partial}{\partial \lambda_i} \epsilon_{st} \frac{\delta H}{\delta \xi_t(\lambda)}$$

$$= \sum_t \epsilon_{st} \frac{\partial}{\partial \lambda_i}\left[\sigma \frac{\partial h[\lambda]}{\partial \xi_t(\lambda)} + \sigma^2 \int \frac{\partial v[\lambda, \lambda']}{\partial \xi_t(\lambda)} d\lambda'\right]$$

$$= \sum_{r,t} \epsilon_{st} \frac{\partial \xi_r}{\partial \lambda_i} \left[\sigma \frac{\partial^2 h[\lambda]}{\partial \xi_t \, \partial \xi_r} + \sigma^2 \int \frac{\partial^2 v[\lambda, \lambda']}{\partial \xi_t(\lambda) \, \partial \xi_r(\lambda)} d\lambda' \right]$$

$$= \sum_{r,t} \epsilon_{st} \frac{\partial \xi_r}{\partial \lambda_i} \frac{\partial^2}{\partial \xi_t \, \partial \xi_r} \left[\sigma h[\lambda] + \sigma^2 \int v[\lambda, \lambda'] \, d\lambda' \right]$$

$$= \sum_{r,t} \epsilon_{st} \frac{\partial \xi_r}{\partial \lambda_i} M_{tr} \qquad \text{and} \qquad M_{tr} \text{ is symmetric}$$

So

$$\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{1}{\sigma} \Delta \cdot \sum_{i,s,t,r} \frac{\partial \lambda_i}{\partial \xi_s} \epsilon_{st} \frac{\partial \xi_r}{\partial \lambda_i} M_{tr}$$

$$= \frac{\Delta}{\sigma} \sum_{s,t,r} \delta_{r,s} \, \epsilon_{st} M_{tr}$$

$$= \frac{\Delta}{\sigma} \sum_{s,t} \epsilon_{st} M_{ts}$$

$$= 0$$

since ϵ is antisymmetric and M is symmetric. So the density in μ space does indeed remain constant along any trajectory.