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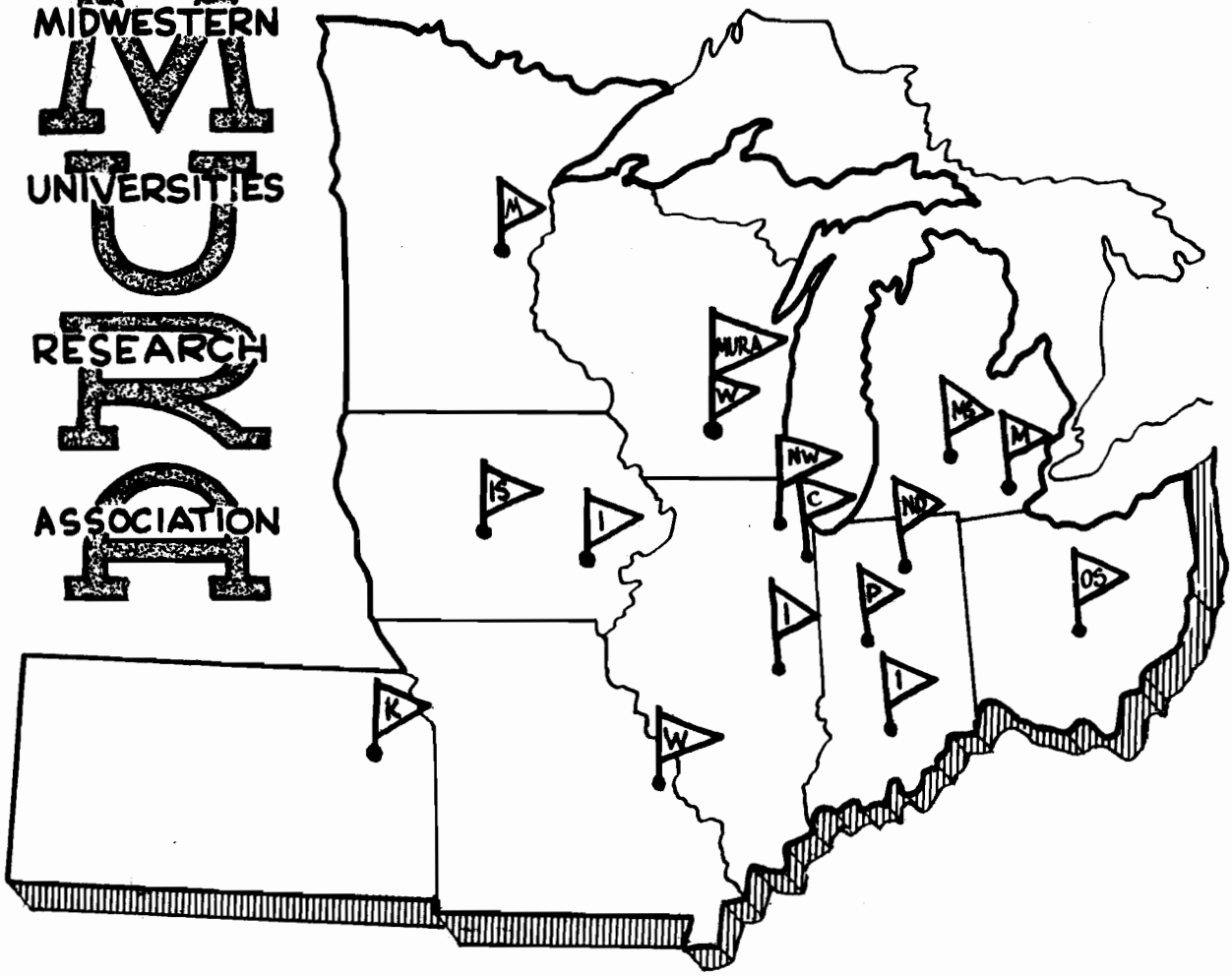

COMPUTATIONAL SOLUTIONS OF THE TWO-DIMENSIONAL WAVE-EQUATION FOR A CROSS-SHAPED REGION

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COMPUTATIONAL SOLUTIONS OF THE TWO-DIMENSIONAL WAVE-EQUATION FOR A CROSS-SHAPED REGION

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ABSTRACT

Approximate computational solutions are obtained for the two-dimensional wave-equation , with $\psi = 0$ on the boundary, for a cross-shaped region. The differential equation was replaced by a set of difference equations and the even eigenfunctions, with their associated eigenvalues, found by use of the GENDARME program. Solutions were obtained in this way for several ratios of arm-length to arm-width. When this ratio exceeded two or three, solutions for the lowest mode were found to be closely similar in the central region of the cross. Guided by the computational results, a simple and reasonable form for a trial function is proposed which, by a variational method, permits a good analytic estimate to be made of the lowest eigenvalue and affords an approximate description of the associated wave function.

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I. INTRODUCTION

There has been some interest[1] in the solutions of the two-dimensional wave-equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 \psi = 0,$$

for the "drum-head" problem illustrated in Fig. 1. In particular, interest has been directed to the character of the lowest mode, for which one would expect the eigensolution to possess even symmetry about the ——— – ——— lines of Fig. 1, and specifically in the ratio of the values of the dependent variable ψ at the two points B and A.

Rather than undertaking to investigate this problem computationally by a direct relaxation procedure, similar to that employed with the FOROCYL program[2] by Dr. Symon and his associates in the study of "eigenpoles,"[3, 4] it appeared preferable to proceed directly to a solution of the eigenvalue problem for the corresponding difference equations by use of GENDARME.[5] In order to exploit the symmetry of the solutions of interest here (and so reduce the computational work to an acceptable amount) while limited by the GENDARME program to solution of eigenvalue problems with a symmetric matrix, it was necessary to redefine the dependent variable (eigenvector elements) in a way which will be indicated below.

II. METHOD

The region of interest was divided into square cells, of dimensions t on a side, and the basic "4-point" algorism,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \doteq (1/t^2) \left[\sum_4 \psi_{i,j} - 4\, \psi_o \right]$$

employed, where the summation $\sum_4 \psi_{i,j}$ is taken over the values of ψ for the four points which are nearest neighbors to the central point with which ψ_0 and $\nabla^2\psi$ are associated. The basic problem thus becomes an eigenvalue problem for difference equations of the form

$$\frac{1}{4}\sum_4 \psi = \left[1 - \frac{k^2 t^2}{4}\right]\psi ,$$

with eigenvalues

$$\lambda_m = 1 - \frac{(k^2)_m t^2}{4} .$$

Two different dimensions, $t = w/3$ and $t = w/2$, were employed in the computations reported here, with the identification of points shown in Figs. 2 and 3 for these respective cases. As noted previously, it was expedient to introduce new dependent variables in order to present the computer with a symmetric matrix for diagonalization. The new variables were defined as follows:

<u>CASE I;</u> (Fig. 2, $t = w/3$):

$\psi_1 = 2\,u_1$

$\psi_2 = u_2$

$\psi_3 = u_3$

$\psi_4 = u_4$

$\psi_5 = 2^{-1/2}\,u_5$

$\psi_6 = u_6$

$\psi_7 = u_7$

$\psi_8 = 2^{-1/2}\,u_8$

$\psi_9 = 2^{-1/2}\,u_9$

$\psi_{10} = u_{10}$

<u>CASE II;</u> (Fig. 3, $t = w/2$):

$\psi_1 = 2\,u_1$

$\psi_2 = u_2$

$\psi_3 = u_3$

$\psi_4 = u_4$

$\psi_5 = 2^{-1/2}\,u_5$

$\psi_6 = u_6$

$\psi_7 = 2^{-1/2}\,u_7$

$\psi_8 = u_8$

$\psi_9 = 2^{-1/2}\,u_9$

$$\psi_{11} = 2^{-1/2}\, u_{11}$$
$$\psi_{12} = 2^{-1/2}\, u_{12}$$
$$\psi_{13} = u_{13}$$
$$\vdots$$
$$\psi_{46} = u_{46}$$
$$\psi_{47} = 2^{-1/2}\, u_{47}$$
$$\psi_{48} = 2^{-1/2}\, u_{48}$$
$$\psi_{49} = u_{49}$$
$$\psi_{50} = 2^{-1/2}\, u_{50}$$
$$\psi_{51} = 2^{-1/2}\, u_{51}$$

$$\psi_{10} = u_{10}$$
$$\vdots$$
$$\psi_{40} = u_{40}$$
$$\psi_{41} = 2^{-1/2}\, u_{41}$$
$$\psi_{42} = u_{42}$$
$$\psi_{43} = 2^{-1/2}\, u_{43}\ ,$$

in which the factor $2^{-1/2}$ appears for those elements which do <u>not</u> lie on one of the boundary lines of symmetry.

By making use of the symmetry of the solutions sought, and noting that the values of ψ at the true boundaries are $\psi = 0$, the difference equations for the cases with which we are concerned here are found to be represented by the following matrix equations.

CASE I

$$
\begin{pmatrix}
0h0000000000 & & & & & & & & & & & \\
h0hq00000000 & & & & & & & & & & & \\
0h00r0000000 & & & & & & & & & & & \\
0q00r0q00000 & & & & & & & & & & & \\
00rr0r0q0000 & \cdots & & & & & & & & & & \\
0000r000r000 & & & & & & & & & & & \\
000q000r0q00 & & & & & & & & & & & \\
0000q0r0q0q0 & & & & & & & & & & & \\
00000r0q000q & & & & & & & & & & & \\
000000q000r0 & q & & & & & & & & & & \\
0000000q0r0q & & q & & & & & & & & & \\
00000000q0q0 & & & q & & & & & & & & \\
 & & & & q & 0r & q & & & & & \\
 & . & & & & q & r0q & q & & & & \\
 & . & & & & & q & q0 & q & & & \\
 & . & & & & & & q & 0r & & & \\
 & & & & & & & & q & r0q & & \\
 & & & & & & & & & q & q0 &
\end{pmatrix}
\cdot \vec{u} = \lambda \vec{u}
$$

and CASE II

$$
\begin{pmatrix}
0h00000 & & & & & & \\
h0hq000 & & & & & & \\
0h00r00 & \cdots & & & & & \\
0q00rq0 & & & & & & \\
00rr00q & & & & & & \\
000q00rq & & & & & & \\
0000qr0 & q & & & & & \\
 & . & q & 0rq & & & \\
 & . & & qr0 & q & & \\
 & . & & & q & 0r & \\
 & & & & & qr0 &
\end{pmatrix}
\cdot \vec{u} = \lambda \vec{u} \ ,
$$

where $h = 0.5$, $q = 0.25$, $r = 2^{-3/2} = 0.3535\ 5339\ 059$, and, as noted above, we interpret $\lambda = 1 - \frac{k^2 t^2}{4}$. The problems in this form may be presented to the MURA GENDARME program[5] for the I. B. M. 704 electronic digital computer and may be expected to be solved quite rapidly by virtue of the fact that all matrix elements remote from the diagonal vanish.

III. RESULTS

A. Check of Computations:

In the simple case pertaining to the oscillations of a square drum head ($s = 0$), the eigenvalue problem for the difference equations is sufficiently simple that a hand calculation can easily be made to check the computational results; the corresponding solutions of the exact differential equation can, moreover, be written down analytically in this case and compared with the solution obtained from our algorithmic procedure. In Tables I and II we give the results of these comparisons with respect to the lowest mode.

TABLE I

EIGENVALUE AND EIGENVECTOR FOR LOWEST MODE
Eigenvector normalized to unity at center (ψ_1)
CASE I: $t = w/3$, with $s = 0$ ($N = 6$).

	From Differential Equation (Analytic Solution)	From Difference Equations	
		Hand Calculation	Computer
$(kw)^2$	$\pi^2/2 = 4.93480\ 22005$	4.82308 54638	4.82308 54661
$(kt)^2$	$\pi^2/18 = 0.54831\ 13558$	0.53589 83849	0.53589 83851 2
λ	$1 - \pi^2/72 = 0.86292\ 21611$	$(3/4)^{1/2} = 0.86602\ 54038$	0.86602 54037 2
ψ_1	(1)	(1)	(1)
ψ_2	0.86602 54038	0.86602 54038	0.86602 5 4961
ψ_3	0.75	0.75	0.75000 0 3309
ψ_4	0.5	0.5	0.49999 9 0268
ψ_5	0.43301 27019	0.43301 27019	0.43301 2 2454
ψ_6	0.25	0.25	0.24999 9 6396

TABLE II

EIGENVALUE AND EIGENVECTOR OF LOWEST MODE
Eigenvector normalized to unity at center (ψ_1)
CASE II: $t = w/2$, with $s = 0$ $(N = 3)$.

	From Differential Equation (Analytic Solution	From Difference Equations	
		Hand Calculation	Computer
$(kw)^2$	$\pi^2/_2 = 4.93480\ 22005$	4.68629 15011	4.68629 14995
$(kt)^2$	$\pi^2/_8 = 1.23370\ 05501$	1.17157 28753	1.17157 28749
λ	$1 - \pi^2/_{32} = 0.69157\ 48625$	$(1/2)^{1/2} = 0.70710\ 67812$	0.70710 67812 8
ψ_1	(1)	(1)	(1)
ψ_2	0.70710 67812	0.70710 67812	0.70710 67609
ψ_3	0.5	0.5	0.49999 99 817

B. Final Results:

We list below in Table III the values of ψ_B / ψ_A for the individual runs made. The entire set of values of ψ along the symmetry line AA' (Fig. 1) for all the runs may be found in Appendices I and II. The results listed in Table III are also illustrated graphically in Fig. 4. The data of Appendices I and II, after normalization of ψ to a central value of unity, are likewise illustrated in Figs. 5 and 6.

TABLE III

AMPLITUDE RATIO ψ_B / ψ_A
FOR LOWEST MODE

$\frac{s}{w}$	$\frac{s}{s+w}$	ψ_B / ψ_A	
		CASE I	CASE II
0	1	0	0
1/3 = 0.333···	3/4 = 0.75	0.368 873	
1/2 = 0.5	2/3 = 0.666···		0.452 037
2/3 = 0.666···	3/5 = 0.6	0.529 093	
1	1/2 = 0.5	0.598 579	0.579 500
4/3 = 1.333···	3/7 = 0.428571···	0.630 758	
3/2 = 1.5	2/5 = 0.4		0.620 856
2	1/3 = 0.333···	0.654 771	0.636 223
3	1/4 = 0.25	0.662 757	0.645 072
4	1/5 = 0.2	0.664 150	
9/2 = 4.5	2/11 = 0.1818···		0.646 958
5	1/6 = 0.1666···	0.664 384	
7	1/8 = 0.125		0.647 158
10	1/11 = 0.0909···		0.647 142

IV. CONCLUSION AND DISCUSSION

One would not expect, of course, the results in the two cases studied to be in exact agreement, either with each other or with the true solution of the differential equation. The results of Case II indeed might be expected to

be somewhat crude, since the mesh dimension t was taken as large as w/2. In fact, however, the results are seen to differ rather little from one another.

The computational results are in agreement in suggesting that, once s is some two or three times w, the values of ψ at points B and A are related essentially in the ratio 2 to 3, with kw close to 1.3.

For s/w large (for example, equal to 5), the values of ψ along the ——— – ——— line AA' appear to be closely represented by a single hyperbolic function ($\psi \propto \sinh 0.85_2 \frac{\xi}{w}$, where $\xi \equiv s + w - x$ is measured from the end of the arm where $\psi = 0$).

The nature of the lowest-mode solution, as inferred from the foregoing work, suggests a means whereby one may estimate relatively simply the salient features of the solution in question. In particular it is possible to estimate the eigenvalue by use of the relationship

$$k^2 = \frac{\int (\nabla \psi)^2}{\int \psi^2} ,$$

the stationary character[6] of which insures that good estimates of k^2 can be obtained through the use of rather crude trial functions for ψ if the boundary conditions and continuity are not violated.

For the present problem it seems natural to choose as a trial function within the arms (say specifically the right-hand arm) an expression of the form

$$A \frac{\sinh \left[\left(\frac{\pi}{2w}\right)^2 - k^2\right]^{1/2} \xi}{\sinh \left[\left(\frac{\pi}{2w}\right)^2 - k^2\right]^{1/2} s} \cos \frac{\pi y}{2w} ,$$

since the presence of a hyperbolic function will clearly permit a lower

eigenvalue than would be associated with a solution involving the circular function alone. The trial function to be used within the square portion of the cross must, then, match up with the function $A \cos \frac{\pi y}{2w}$ at $x = \pm w$ (and with $A \cos \frac{\pi x}{2w}$ at $y = \pm w$). For this we select the form

$$A \left[\cos \frac{\pi x}{2w} + \cos \frac{\pi y}{2w} + a \cos \frac{\pi x}{2w} \cos \frac{\pi y}{2w} \right] ,$$

in which the constant "a" may serve as an adjustable parameter. It may be noted that the selection of $a = -1$ would imply a trial function which retains the constant value A along each of the coordinate axes and a less negative value would be appropriate to represent the increased amplitude near the origin.

If, for the sake of simplicity and relevance to the present problem, we apply these trial functions to the case $s/w \rightarrow \infty$, we obtain

$$(kw)^2 \cong \frac{\frac{\pi^2}{2} - (kw)^2 + \frac{\pi^2}{2}\left[1 + \frac{4a}{\pi} + \frac{a^2}{2}\right]\left[\left(\frac{\pi}{2}\right)^2 - (kw)^2\right]^{1/2}}{1 + 2\left[\left(1 + \frac{8}{\pi^2}\right) + \frac{4a}{\pi} + \frac{a^2}{4}\right]\left[\left(\frac{\pi}{2}\right)^2 - (kw)^2\right]^{1/2}} .$$

The right-hand side of this expression is quite insensitive to the value of k, which entered through its inclusion in the trial function, and is also, for that matter, not critically dependent on the parameter "a".

A hand calculation suggests that a value of "a" in the neighborhood of $a = -0.6$ is close to optimum and that the result $(kw)^2 \doteq 1.68$ is then obtained. This value, which corresponds to kw just under 1.30, is seen to be in close agreement with the algorithmic computations. The permissible crudity of the trial functions employed in this procedure precludes our making readily

a correspondingly accurate estimate of the ratio ψ_B / ψ_A —— with $a = -0.6$ we find for this ratio $1/(2 + a) = 1/1.4 = 0.71$, in contrast to the computer result 0.66.

V. ACKNOWLEDGMENT

We would like to express our appreciation of the cooperation which the MURA Computer Section willingly extended to us in arranging for the computations, and we in particular wish to extend our thanks to Mr. Mel Storm, Mr. Paul Marty, and Mrs. A. Frazier for their assistance.

VI. REFERENCES AND NOTES

1. K. Ruedenberg (private communication).

2. FOROCYL Program (MURA IBM Program #13), MURA-221, Int. (Feb.-March, 1957).

3. K. R. Symon, MURA General Conference, Minutes #13 (January 4-5, 1957); MURA General Conference, Minutes #30 (May 10-11, 1957).

4. D. W. Kerst, MURA-249 (March 29, 1957).

5. James Kister, GENDARME Program (MURA IBM Program #124), MURA-392, Int. (March, 1958). This program diagonalizes a N x N symmetric matrix by the Jacobi method. In the present work the convergence criterion employed $\delta = 2^{-58}$ [value at address 166 entered as 4096 (i.e., $2^{70}\delta$)].

6. See, f. ex., P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953), Part II, Sect. 9.4, esp. pp. 1112-3.

APPENDIX I

LOWEST EIGENVALUE AND VALUES OF ASSOCIATED EIGENVECTOR ALONG SYMMETRY LINE

CASE I: $t = w/3$ $\left[kw = 6\,(1 - \lambda)^{1/2}\right]$

s/w	0	1/3	2/3	1	4/3	2	3	4	5
λ	.866 025	.919 795	.938 779	.946 368	.949 773	.952 273	.953 099	.953 238	.953 263
kw	2.196	1.699	1.485	1.390	1.345	1.311	1.299	1.297	1.297
$\psi_1 = 2\,u_1$	.666 667	.515 882	.449 622	.415 806	.396 423	.377 078	.367 193	.364 577	.363 940
ψ_2	.577 350	.474 507	.422 097	.393 512	.376 507	.359 081	.349 968	.347 535	.346 927
ψ_4	.333 333	.358 472	.345 764	.332 475	.322 465	.310 597	.303 702	.301 754	.301 260
ψ_7	(0)	.190 295	.237 892	.248 893	.250 047	.246 900	.243 360	.242 134	.241 796
ψ_{10}		(0)	.119 973	.162 761	.178 690	.186 892	.187 517	.187 127	.186 963
ψ_{13}			(0)	.080 009	.114 034	.136 137	.141 698	.142 239	.142 265
ψ_{16}				(0)	.055 392	.094 268	.105 683	.107 298	.107 533
ψ_{19}					(0)	.059 181	.077 691	.080 564	.081 038
ψ_{22}						.028 513	.055 802	.060 214	.060 967
ψ_{25}						(0)	.038 355	.044 694	.045 798
ψ_{28}							.023 979	.032 775	.034 333
ψ_{31}							.011 526	.023 510	.025 647
ψ_{34}							(0)	.016 148	.019 028
ψ_{37}								.010 095	.013 954
ψ_{40}								.004 853	.010 009
ψ_{43}								(0)	.006 871
ψ_{46}									.004 286
ψ_{49}									.002 061
(ψ_{52})									(0)

APPENDIX II
LOWEST EIGENVALUE AND VALUES OF ASSOCIATED EIGENVECTOR ALONG SYMMETRY LINE
CASE II: $t = w/2$ $\left[kw = 4\,(1 - \lambda)^{1/2}\right]$

s/w	0	1/2	1	3/2	2	3	9/2	7	10
λ	.707 107	.845 001	.877 179	.887 136	.890 784	.892 873	.893 318	.893 362	.893 363
kw	2.165	1.575	1.402	1.344	1.322	1.309	1.306	1.306	1.306
$\psi_1 = 2\,u_1$	1.000 000	.726 551	.637 312	.597 152	.576 535	.559 398	.553 457	.552 505	.552 485
ψ_2	.707 107	.613 937	.559 038	.529 752	.513 571	.499 473	.494 411	.493 593	.493 574
ψ_4	(0)	.328 427	.369 322	.370 745	.366 805	.360 852	.358 063	.357 558	.357 537
ψ_6		(0)	.179 705	.227 287	.241 044	.246 061	.246 037	.245 867	.245 860
ψ_8			(0)	.107 171	.144 132	.163 077	.166 314	.166 506	.166 509
ψ_{10}				(0)	.067 211	.104 298	.111 608	.112 239	.112 243
ψ_{12}					(0)	.061 617	.074 353	.075 535	.075 553
ψ_{14}						.028 569	.048 863	.050 798	.050 839
ψ_{16}						(0)	.031 138	.034 142	.034 197
ψ_{18}							.018 358	.022 925	.023 004
ψ_{20}							.008 499	.015 347	.015 472
ψ_{22}							(0)	.010 215	.010 414
ψ_{24}								.006 713	.007 004
ψ_{26}								.004 277	.004 712
ψ_{28}								.002 522	.003 165
ψ_{30}								.001 169	.002 124
ψ_{32}								(0)	.001 420
ψ_{34}									.000 949
ψ_{36}									.000 624
ψ_{38}									.000 401
ψ_{40}									.000 234
ψ_{42}									.000 106
ψ_{44}									(0)

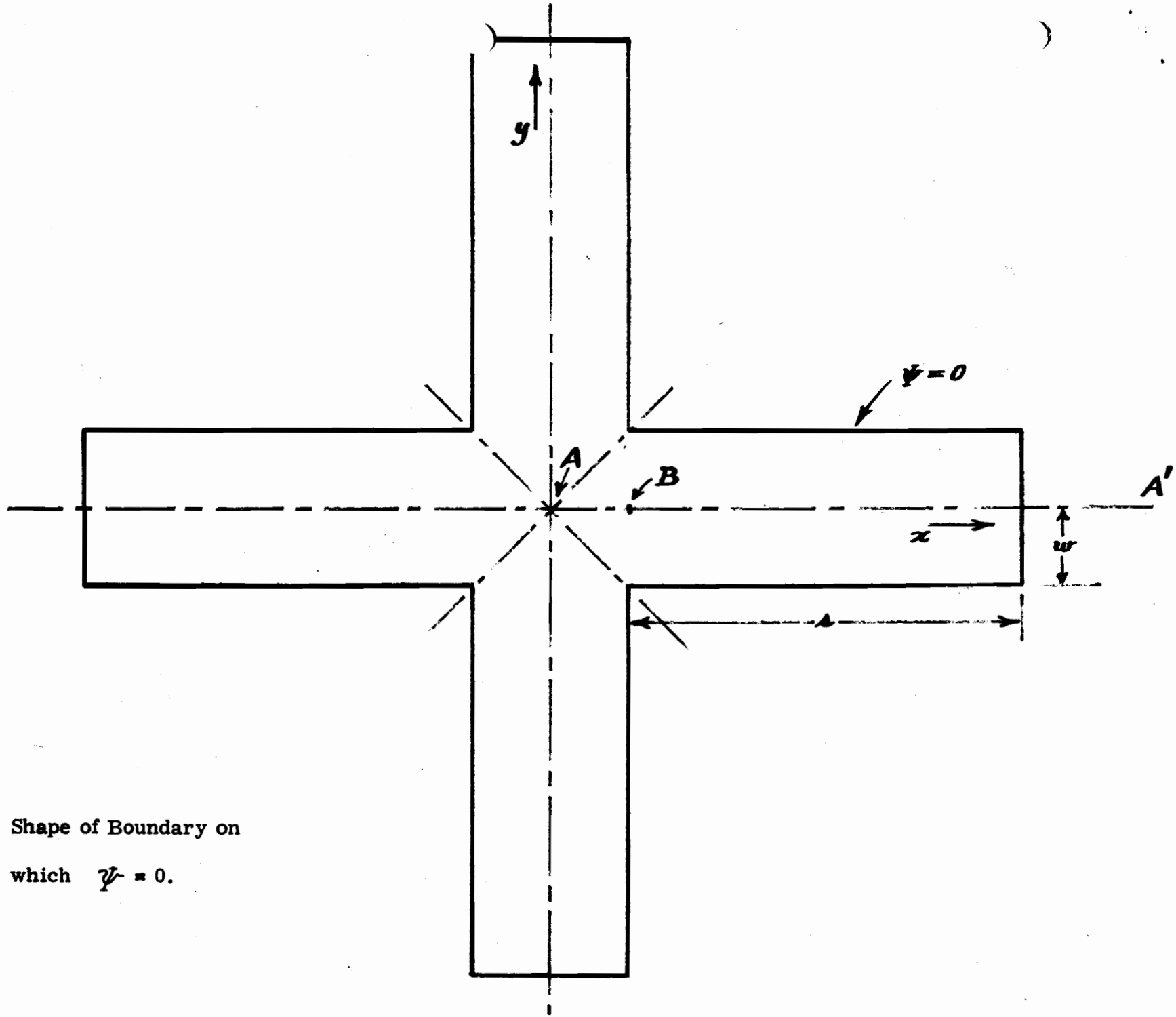


Fig. 1. Shape of Boundary on which $\psi = 0$.

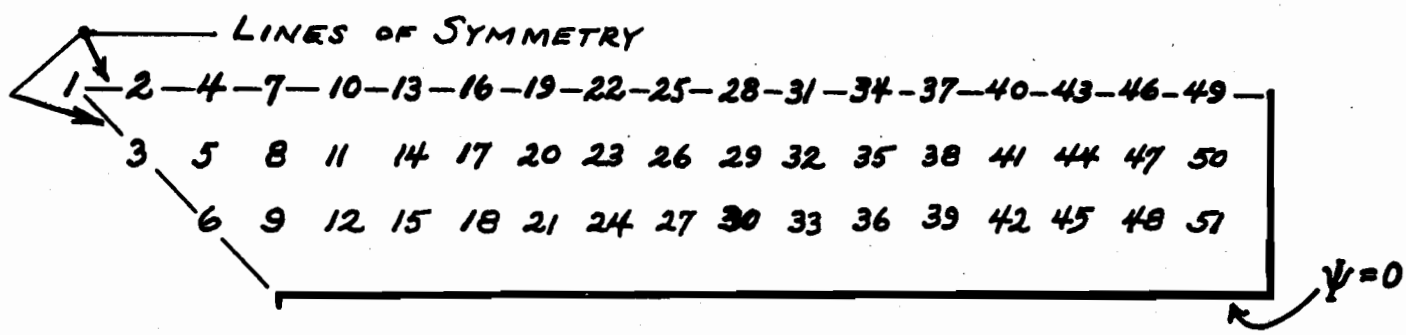


Fig. 2. Identification of Points within Region of Interest when t = w/3.

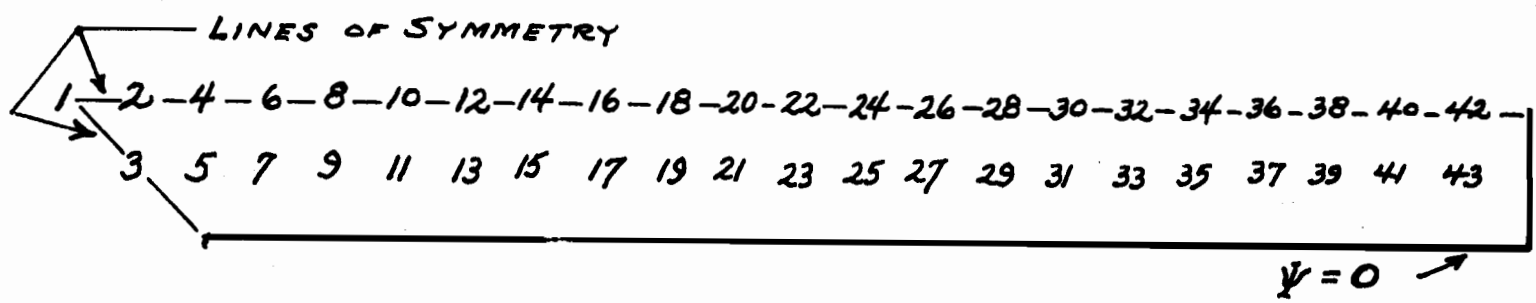


Fig. 3. Identification of Points within Region of Interest when t = w/2.

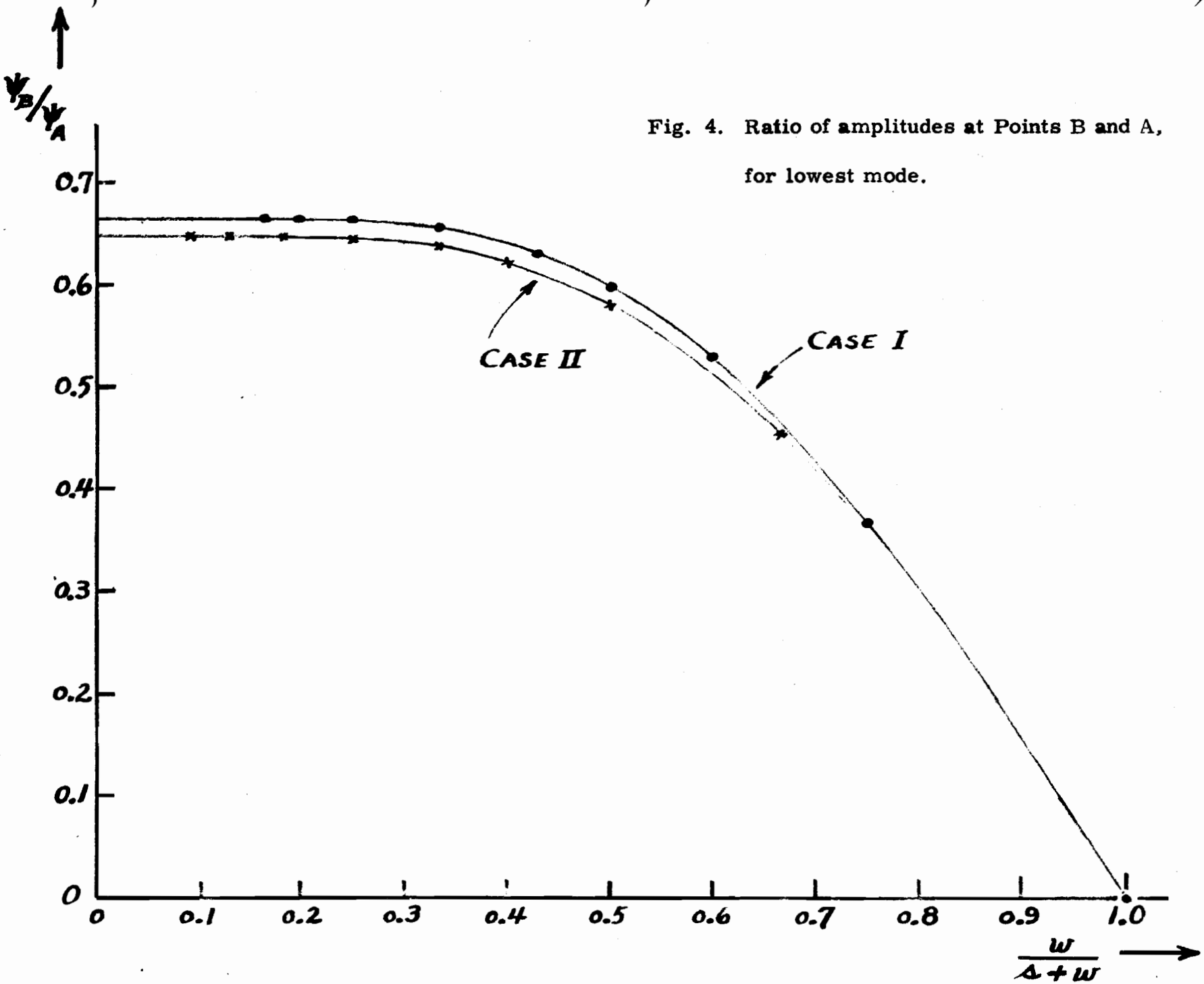
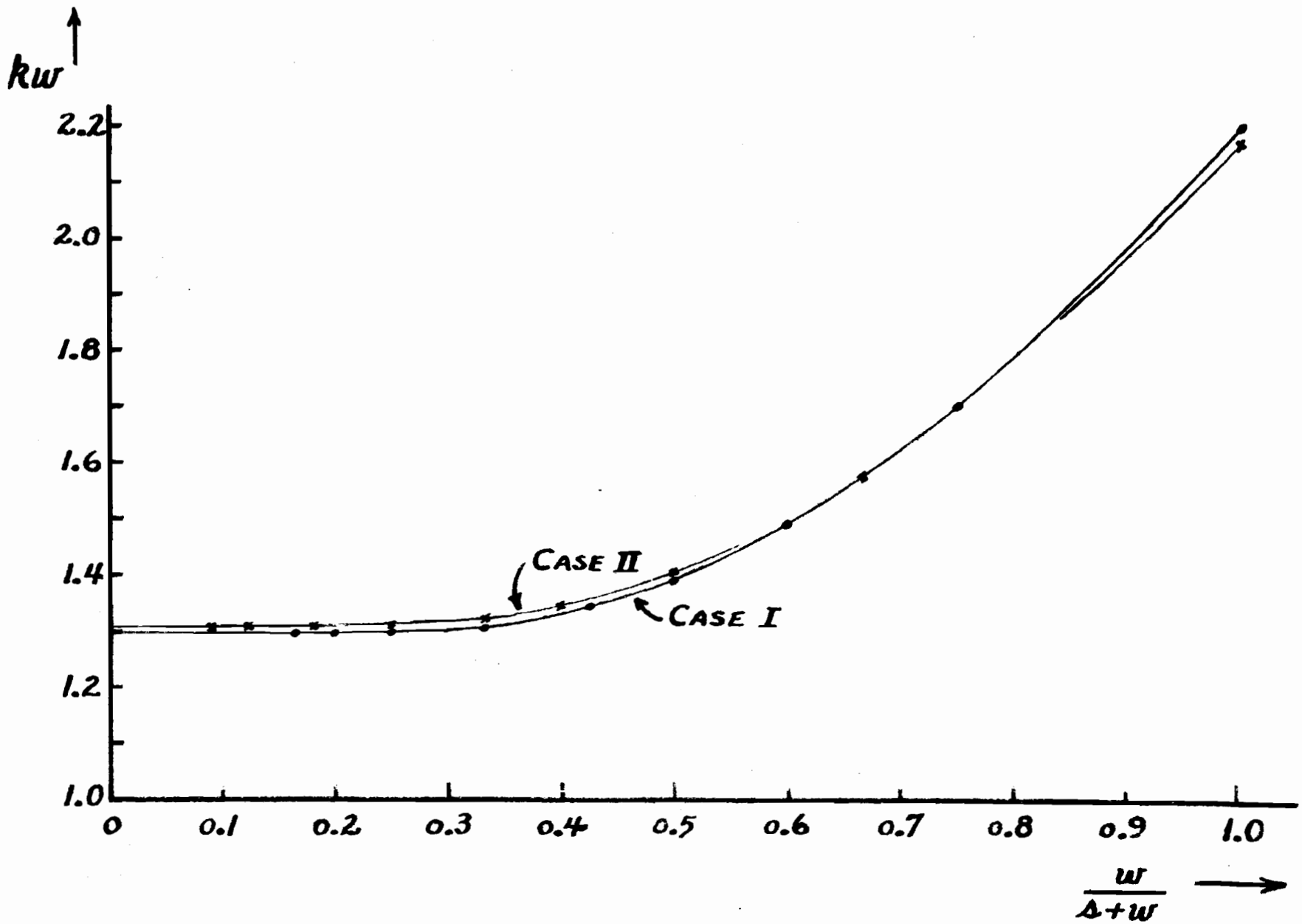


Fig. 4. Ratio of amplitudes at Points B and A, for lowest mode.

Fig. 5. Eigenvalue for lowest mode.



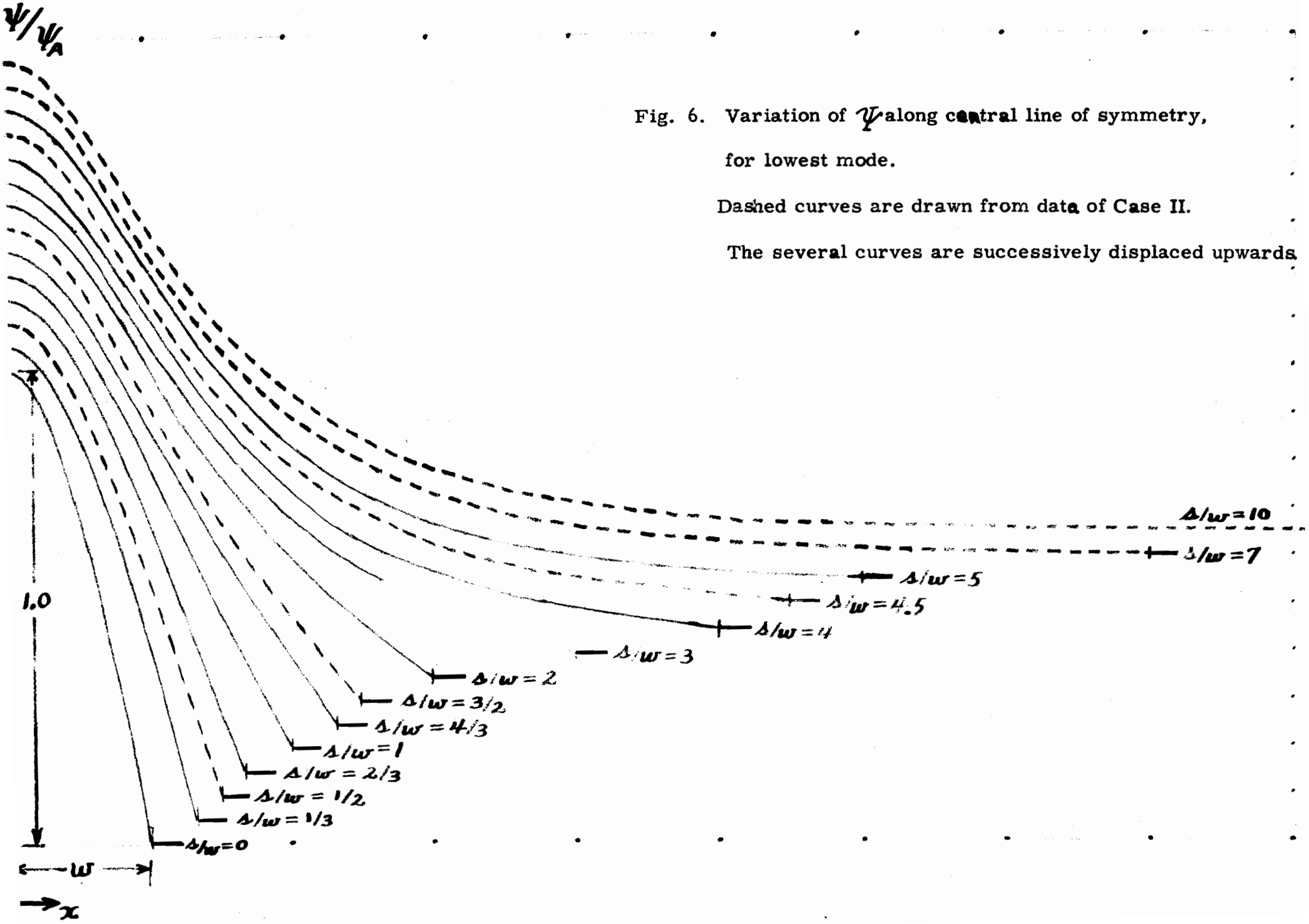


Fig. 6. Variation of ψ along central line of symmetry, for lowest mode.

Dashed curves are drawn from data of Case II.

The several curves are successively displaced upwards.