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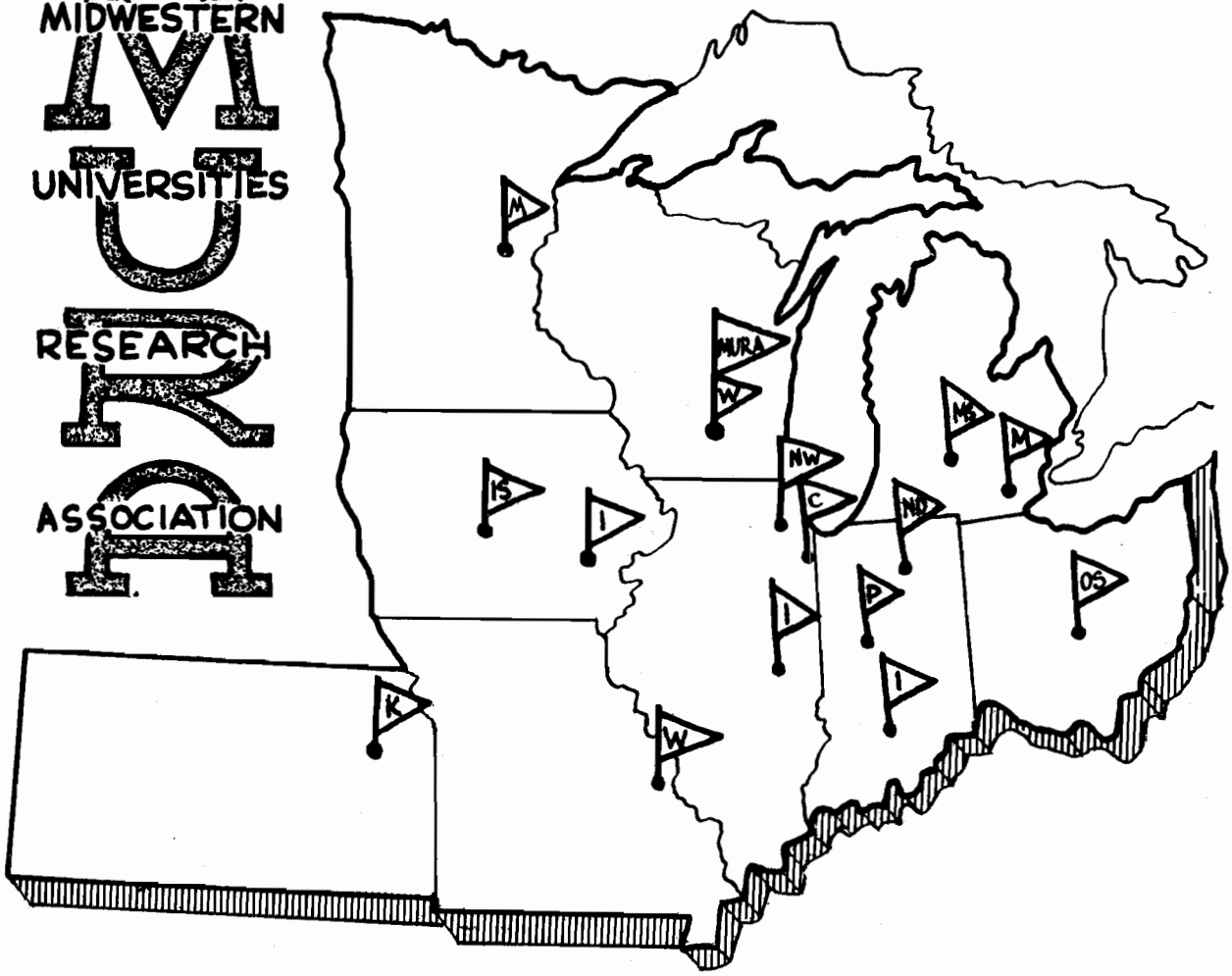

THEORY OF THE AG SYNCHROTRON

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ABSTRACT

This report applies the general results obtained in the report MURA-397 for an accelerator having a general magnetic field to the case of the AG synchrotron. Results are found for the properties of the equilibrium orbits, the tune, and the betatron oscillations as a function of the particle momentum. The theoretical results are expected to be good within an error of 10% for presently considered machines. The theoretical results are compared with those found on the IBM 704 computer. A summary of the results is given in Section II. It is also pointed out that it is possible to choose the parameters of the linear AG machine so that either ν_r or ν_z is independent of energy to first order in the energy change.

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**On leave from the University of Notre Dame.

I. INTRODUCTION

In this report we will apply the general results obtained in the report MURA-397 for an accelerator having a general magnetic field to the case of the AG synchrotron. The AG synchrotron has been treated in great detail by many people. Some of the more thorough treatments of the AG synchrotron are those of Courant and Snyder,[1] M. Hamermesh,[2] and A. Schoch.[3] Most of the results presented in this report can be found in the above references.

We would like to use the AG synchrotron as an illustration of the general results found in MURA-397. With the approximations used in MURA-397, the results found in this report should be good within an error of about 10%, for presently considered AG machines.

This report should also serve the purpose of collecting in one place a list of orbit theory results for the AG synchrotron in a simple approximate form.

In this report we have not considered the case where part of the z-focusing is due to spiralling of the magnets, or to wedge focusing as in the Argonne accelerator. However, the general results of MURA-397 could be applied to this case also.

II. SUMMARY OF RESULTS

In this section we would like to summarize some of the more important results for the AG synchrotron. The derivations are given in Section III, where more detailed results are also given. For the sake of simplicity, the results listed here are sometimes less accurate than those found in Section III.

We assume that the magnetic field in the median plane of the AG synchrotron has the form,

$$H_z = -B_0 \left\{ 1 + \frac{r - R_0}{R_0} K(\theta) \right\}, \tag{2.1}$$

where K (θ), the field gradient, is given by the Fourier series

$$K(\theta) = \sum_n K_n e^{inN\theta} . \tag{2.2}$$

We will assume that K_0 and K_1 are real. If the field gradient K (θ) is, as is often assumed, a square wave with the value $+n_1$ in the radially converging magnets and the value $-n_2$ in the radially diverging magnets, then

$$K_0 = \frac{1}{2}(n_1 - n_2) , \tag{2.3a}$$

$$K_1 = K_{-1} = \frac{2}{\pi}(n_1 + n_2) . \tag{2.3b}$$

The Equilibrium Orbit

Let p_0 be the momentum of the particle corresponding to the circular orbit which has the radius R_0 .

$$p_0 = \frac{1}{c} e R_0 B_0 . \tag{2.4}$$

The equilibrium orbit corresponding to the momentum $p = p_0 + \Delta p$ is not circular. It can be written as

$$r = R(1 + x(\theta)), \tag{2.5}$$

where R is the average radius of the equilibrium orbit and $x(\theta)$ represents the oscillation of the orbit about the average radius, R.

In the following we will expand our results in powers of $\Delta p / p_0$, and give just the leading term. Usually, the $\Delta p / p_0$ of interest is so small that the leading term is sufficient.

The average radius R of the orbit is given, to first order in $\Delta p / p_0$ by

$$\frac{R - R_0}{R_0} = \frac{1}{\nu^2 + K_0 + 1} \frac{\Delta p}{p_0}, \tag{2.6}$$

where

$$\nu^2 = 2 \frac{K_1^2}{N^2}. \tag{2.7}$$

The oscillation $x(\theta)$ is given by

$$x(\theta) = \frac{2}{\nu^2 + K_0 + 1} \frac{\Delta p}{p_0} \frac{K_1}{N^2} \cos N\theta. \tag{2.8}$$

<u>Tune</u>

The r-tune ν_r for the circular orbit is

$$\nu_r^2 = \nu^2 + K_0 + 1, \tag{2.9}$$

and the z-tune of the circular orbit is

$$\nu_z^2 = \nu^2 - K_0. \tag{2.10}$$

The r-tune varies with the momentum according to

$$\nu_r^2 = (\nu^2 + K_0 + 1)\left\{1 - \frac{\nu^2(2\nu^2 - 7) + K_0(3\nu^2 + K_0 - 1)}{(\nu^2 + K_0 + 1)^2}\,\frac{\Delta p}{p_0}\right\}. \tag{2.11}$$

The z-tune varies with momentum according to

$$\nu_z^2 = (\nu^2 - K_0)\left\{1 - \frac{\nu^2(2\nu^2 - 1) + K_0(\nu^2 + 1 - K_0)}{(\nu^2 - K_0)(\nu^2 + K_0 + 1)}\,\frac{\Delta p}{p_0}\right\}. \tag{2.12}$$

Betatron Oscillations

The linear betatron oscillations about the equilibrium orbit are given by

$$u = a\, u_\nu(\theta) + a^*\, u_\nu^*(\theta), \tag{2.13a}$$

$$y = b\, y_\nu(\theta) + b^*\, y_\nu^*(\theta), \tag{2.13b}$$

where $u_\nu(\theta)$ and $y_\nu(\theta)$ are the Floquet solutions of the linear equations for the r and z motion, and a and b depend on the initial amplitude of the motion. u_ν and y_ν for the circular orbit are given by

$$u_\nu(\theta) = e^{i\nu_r\theta}\left\{1 + \frac{K_1}{N^2}\,\frac{1}{1 + 2\nu_r/N}\,e^{iN\theta} + \frac{K_1}{N^2}\,\frac{1}{1 - 2\nu_r/N}\,e^{-iN\theta}\right\}, \tag{2.14a}$$

$$y_\nu(\theta) = e^{i\nu_z\theta}\left\{1 - \frac{K_1}{N^2}\,\frac{1}{1 + 2\nu_z/N}\,e^{iN\theta} - \frac{K_1}{N^2}\,\frac{1}{1 - 2\nu_z/N}\,e^{-iN\theta}\right\} \tag{2.14b}$$

The maximum oscillation at any angle θ is proportional to the form factors $F_r(\theta) = |\mu_\nu(\theta)|^2$, $F_z = |y_\nu(\theta)|^2$ and F_r and F_z for the circular orbit are given by

$$F_r(\theta) = 1 + 4\frac{K_1}{N^2}\cos N\theta, \tag{2.15a}$$

$$F_z(\theta) = 1 - 4\frac{K_1}{N^2}\cos N\theta. \tag{2.15b}$$

Transition Energy

The transition energy E_t for the circular orbit is given by

$$\frac{E_t}{mc^2} = \sqrt{\nu^2 + K_0 + 1}, \tag{2.16}$$

III. DERIVATION OF RESULTS

We will apply the general results of MURA-397 for a general accelerator to find the orbit properties of the AG synchrotron.

We assume that the magnetic field in the median plane of the AG synchrotron has the form

$$H_z = -B_0\left\{1 + \frac{r - R_0}{R_0}K(\theta)\right\}, \tag{3.1a}$$

where $K(\theta)$, the field gradient, is given by the Fourier series

$$K(\theta) = \sum_n K_n e^{i\omega_n\theta}, \tag{3.1b}$$

$$\omega_n = nN.$$

If the field gradient K (θ) is a square wave as shown in Fig. 1, then

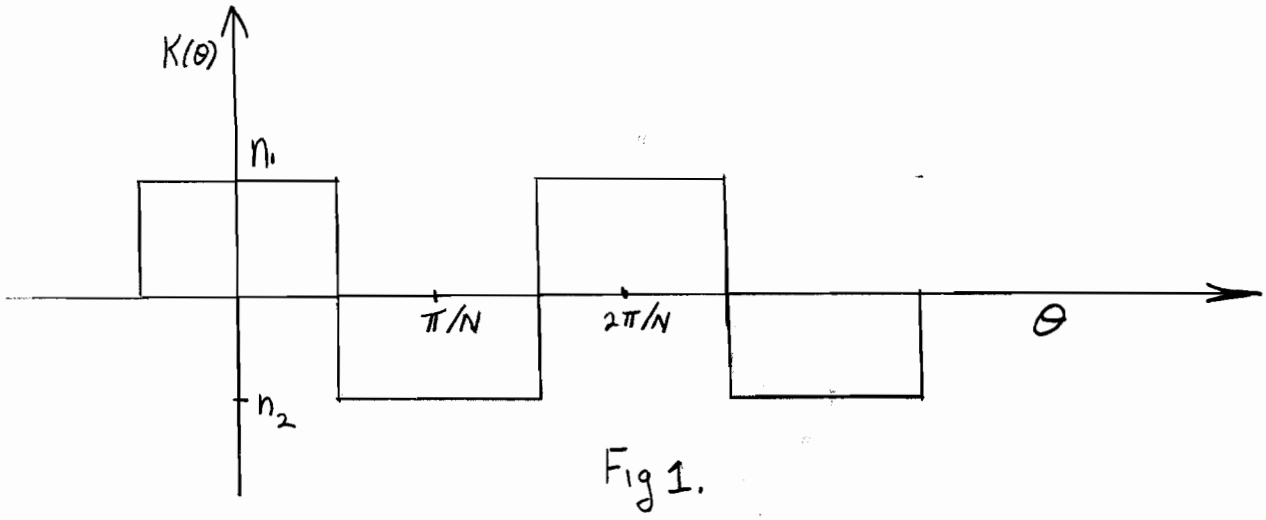

Fig 1.

$$K(\theta) = \frac{1}{2}(n_1 - n_2) + \frac{n_1 + n_2}{2} \sum_{n=0}^{\infty} \frac{4}{\pi} \frac{(-1)^n}{2n+1} \cos (2n+1) N \theta \tag{3. 2a}$$

so that

$$K_0 = \frac{1}{2}(n_1 - n_2), \tag{3. 2b}$$

$$K_1 = K_{-1} = \frac{2}{\pi}(n_1 + n_2). \tag{3. 2c}$$

In MURA-397 the magnetic field in the median plane is assumed to have the form

$$H_z = -\sum_n G_n(r)\, e^{i \omega_n \theta} \tag{3. 3}$$

Thus for the AG synchrotron,

$$H_0(r) = B_0 \left(1 + K_0 \frac{r - R_0}{R_0}\right), \tag{3. 3a}$$

$$H_n(r) = B_0 \, |K_n| \, \frac{r - R_0}{R_0}, \quad n \neq 0 \tag{3. 3b}$$

where

$$G_n(r) = H_n(r)\, e^{i\beta_n(r)} \tag{3.3c}$$

H_n and β_n being real. For the case of no spiralling or wedge focusing, the β_n are independent of r. We will assume that there is no spiralling.

The Equilibrium Orbit

We will first find the equilibrium orbits of the AG synchrotron corresponding to different particle momenta. According to MURA-397, the average radius of the equilibrium orbit, R, depends on the momentum according to the relation

$$p = \frac{eR}{2c}\left\{ H_0 + \sqrt{H_0^2 + 8\sum_{n\geq 1} \frac{1}{\omega_n^2}\left(R H_n' H_n + 2H_n^2\right)}\right., \tag{3.4}$$

where the $H_n(r)$ are evaluated at $r = R$.

We write

$$H_0 = B_0\,(1 + K_0 s), \tag{3.5a}$$

$$H_n = B_0\,|K_n|\, s \quad ; \; n \neq 0, \tag{3.5b}$$

$$R H_n' = B_0\,|K_n|\,(1+s), \; n \neq 0, \tag{3.5c}$$

where

$$R = R_0\,(1 + s) \tag{3.5d}$$

and thus find from Eq. (3.4) that

$$p = \frac{e B_0 R_0}{2c}(1+s)\left\{(1+K_0 s) + \left((1+K_0 s)^2 + \frac{8}{N^2}\sum_{n\geq 1}\frac{|K_n|^2}{n^2}\left[s(1+s)+2s^2\right]\right)^{\frac{1}{2}}\right\}$$

(3. 6a)

or

$$p = p_0 \frac{1+s}{2}\left\{1+K_0 s + \left(1+s\left(2K_0+\frac{8\bar{K}^2}{N^2}\right)+s^2\left(K_0^2+\frac{24\bar{K}^2}{N^2}\right)\right)^{\frac{1}{2}}\right\}$$

(3. 6b)

where $p_0 = e R_0 B_0 / c$ is the momentum of the circular orbit, and

$$\bar{K}^2 = \sum_{n\geq 1}\frac{1}{n^2}|K_n|^2. \tag{3. 6c}$$

It may be noticed that usually

$$\bar{K} \cong |K_1|. \tag{3. 6d}$$

Eq. (3. 6) can be solved for s as a function of p which will give R as a function of p. For most purposes, it is sufficient to consider the case when $s \ll 1$. We then find from Eq. (3. 6b) that

$$p = p_0 \frac{1+s}{2}\left\{2 + s\left(2K_0 + 4\frac{\bar{K}^2}{N^2}\right)\right\}, \tag{3. 7a}$$

or

$$p = p_0 \left\{1 + s\left(\nu^2 + K_0 + 1\right)\right\}, \tag{3. 7b}$$

where

$$\nu^2 = 2\,\bar{K}^2/N^2 \tag{3.7c}$$

It will turn out that both r-tune, ν_r, and z-tune ν_z are usually nearly equal to ν.

We can solve Eq. (3.7b) for s,

$$s = \frac{1}{\nu^2+K_0+1}\left(\frac{p}{p_0}-1\right). \tag{3.8}$$

The equilibrium orbit $r(\theta)$ is given by

$$r(\theta) = R\,(1+x(\theta)), \tag{3.9}$$

where, according to MURA-397, $x(\theta)$ is given by

$$x(\theta) = \frac{eR}{pc}\sum_{n\neq 0}\frac{1}{\omega_n^2}\,H_n(R)\,e^{i\omega_n\theta}. \tag{3.10}$$

Thus we find for $x(\theta)$,

$$x(\theta) = \frac{eRB_0}{pc}\,s\sum_{n\neq 0}\frac{1}{\omega_n^2}\,K_n\,e^{i\omega_n\theta}. \tag{3.11}$$

Now by Eq. (3.7b)

$$\frac{eRB_0}{pc} = \frac{eR_0B_0}{p_0c}\,(1+s)\,(1-s(\nu^2+K_0+1)), \tag{3.12a}$$

$$\frac{eRB_0}{pc} = 1 - s\,(\nu^2+K_0). \tag{3.12b}$$

Thus we find that $\chi(\theta)$ depends on the average radius of the orbit, $R = R_0(1+s)$ according to

$$\chi(\theta) = s \sum_{n \neq 0} \frac{1}{\omega_n^2} K_n e^{i\omega_n \theta}, \tag{3.13a}$$

and depends on the momentum according to

$$\chi(\theta) = \frac{1}{\nu^2 + K_0 + 1}\left(\frac{p}{p_0} - 1\right) \sum_{n \neq 0} \frac{1}{\omega_n^2} K_n e^{i\omega_n \theta}. \tag{3.13b}$$

Tune

We will now find the r-tune ν_r and the z-tune ν_z of the betatron oscillations corresponding to different particle momenta. According to MURA-397, the r-tune is given by[4]

$$\nu_r^2 = \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 + 2\left(\frac{eR}{pc}\right)^2 \sum_{n \geq 1} \frac{1}{\omega_n^2}\left\{R^2 H_n'' H_n + R^2 H_n'^2 + 7RH_n' H_n + 4H_n^2\right\}. \tag{3.14}$$

It is simpler to first find ν_r as a function of the average radius of the equilibrium orbit $R = R_0(1+s)$, then we can use Eq. (3.8) to find ν_r as a function of the particle momentum, p.

Using Eqs. (3.5) for H_n and H_n', and Eq. (3.12) for eRB_0/pc, we find that

$$\nu_r^2 = \left[1 - s(\nu^2 + K_0)\right]\left[K_0(1+s) + 2(1 + K_0 s)\right] - 1$$
$$+ 2\left[1 - 2s(K_0 + \nu^2)\right] \sum_n \frac{1}{\omega_n^2} \Big\{ K_n^2 (1+s)^2$$
$$+ 7 K_n^2 s(1+s)$$
$$+ 4 K_n^2 s^2 \Big\}$$

$$\nu_r^2 = \left[1 - s(K_0 + \nu^2)\right]\left[2 + K_0 + s(3K_0)\right] - 1 \tag{3.15a}$$
$$+ 2\left[1 - 2s(K_0 + \nu^2)\right] \frac{\bar{K}^2}{N^2}\left[1 + 9s\right],$$

$$\nu_r^2 = (2 + K_0) - 1 + \nu^2 \tag{3.15b}$$
$$+ s\Big\{ 3K_0 - (2 + K_0)(K_0 + \nu^2)$$
$$+ \nu^2 \left(9 - 2(K_0 + \nu^2)\right)\Big\},$$

$$\nu_r^2 = \nu^2 + K_0 + 1 \tag{3.15c}$$
$$+ s\Big[3K_0 - 2(K_0 + \nu^2) - K_0(K_0 + \nu^2)$$
$$+ \nu^2(9 - 2\nu^2) - 2\nu^2 K_0\Big],$$

(3.15d)

$$\nu_r^2 = \nu^2 + K_0 + 1 + s\left[\nu^2(7-2\nu^2) + K_0 - K_0^2 - 3K_0\nu^2\right]. \tag{3.15e}$$

Thus the radial tune ν_r depends on s according to

$$\nu_r^2 = (\nu^2 + K_0 + 1)\left\{1 - \frac{\nu^2(2\nu^2-7) - K_0(1-3\nu^2-K_0)}{\nu^2+K_0+1}\, s\right\}. \tag{3.16}$$

Using Eq. (3.8), we find that ν_r depends on the particle momentum according to

$$\nu_r^2 = (\nu^2+K_0+1)\left\{1 - \frac{\nu^2(2\nu^2-7) - K_0(1-3\nu^2-K_0)}{(\nu^2+K_0+1)^2}\left(\frac{p}{p_0}-1\right)\right\}. \tag{3.17}$$

We now treat the z-tune ν_z. According to MURA-397, the z-tune is given by[4] (for the case of no spiralling)

$$\nu_z^2 = 2\left(\frac{eR}{p_0}\right)^2 \sum_{n\geq 1} H_n^2 - \frac{eR}{p_0} R H_0' + 2\left(\frac{eR}{p_0}\right)^2 \sum_{n\geq 1} \frac{1}{\omega_n^2}\left\{-2nH_n' H_n + R^2 H_n'^2 - R^2 H_n'' H_n\right\}. \tag{3.18}$$

Thus for the AG machine, we find that

$$\nu_z^2 = 2\left[1-2s(K_0+\nu^2)\right]\sum_{n\geq 1} K_n^2 s^2$$
$$-\left[1-s(\nu^2+K_0)\right]K_0(1+s)$$
$$+2\left[1-2s(K_0+\nu^2)\right]\frac{\bar{K}^2}{N^2}\left[-s(1+s)+(1+s)^2\right], \tag{3.19a}$$

$$\nu_z^2 = 2s^2\sum_{n\geq 1} K_n^2$$
$$-K_0\left[1-s(K_0+\nu^2)\right]\left[1+s\right]$$
$$+\nu^2\left[1-2s(K_0+\nu^2)\right]\left[1+s\right], \tag{3.19b}$$

$$\nu_z^2 = \nu^2-K_0 + s\left[-K_0(1-K_0-\nu^2)-2\nu^2(K_0+\nu^2)+\nu^2\right]$$
$$+s^2\,2\sum_{n\geq 1} K_n^2, \tag{3.19c}$$

$$\nu_z^2 = \nu^2-K_0 - s\left[2\nu^4+K_0-K_0^2+K_0\nu^2-\nu^2\right]$$
$$+s^2\,2\sum_{n\geq 1} K_n^2 \tag{3.19d}$$

We have kept a term in s^2 in the above because it has an exceptionally large coefficient. We thus find that the z-tune depends on s according to

$$\nu_z^2 = (\nu^2 - K_0)\left\{1 - s\,\frac{\nu^2(2\nu^2-1) + K_0(1-K_0+\nu^2)}{\nu^2 - K_0} + s^2\,\frac{1}{\nu^2-K_0}\,2\sum_{n\geq 1} K_n^2\right\}. \tag{3.20}$$

ν_z depends on p according to

$$\nu_z^2 = (\nu^2 - K_0)\left\{1 - \left(\frac{p}{p_0}-1\right)\frac{\nu^2(2\nu^2-1) + K_0(1-K_0+\nu^2)}{(\nu^2-K_0)(\nu^2+K_0+1} + \left(\frac{p}{p_0}-1\right)^2\frac{2\sum_{n\geq 1} K_n^2}{(\nu^2-K_0)(\nu^2+K_0+1}\right\}. \tag{3.21}$$

Betatron Oscillations

The linear betatron oscillations about the equilibrium orbit are given by

$$u = a\,u_\nu(\theta) + a^*\,u_\nu^*(\theta), \tag{3.22a}$$

$$y = b\,y_\nu(\theta) + b^*\,y_\nu^*(\theta), \tag{3.22b}$$

where $u_\nu(\theta)$ and $y_\nu(\theta)$ are the Floquet solutions of the linear equations of the r and z motion, and a and b depend on the inital amplitude of the motion. If $u_\nu(\theta)$ and $y_\nu(\theta)$ are known, then all properties of the betatron oscillations may be found. According to MURA-397, $u_\nu(\theta)$ and $y_\nu(\theta)$ are given by

$$u_\nu(\theta) = e^{i\nu_r\theta}\left\{1 - \sum_{n\neq 0} \frac{g_n}{\omega_n^2}\,\frac{1}{1+2\nu_r/\omega_n}\,e^{i\omega_n\theta}\right\}, \tag{3.23a}$$

$$y_\nu(\theta) = e^{i\nu_z\theta}\left\{1 - \sum_{n\neq 0} \frac{f_n}{\omega_n^2}\,\frac{1}{1+2\nu_z/\omega_n}\,e^{i\omega_n\theta}\right\}, \tag{3.23b}$$

where

$$g_n = -\frac{eR}{pc}\left(RH_n' + 2H_n\right), \tag{3.24a}$$

$$f_n = \frac{eR}{pc}\,RH_n'. \tag{3.24b}$$

For the AG machine

$$g_n = -\left[1 - s(K_0 + \nu^2)\right]\left[K_n(1+s) + 2K_n s\right],$$

$$g_n = -K_n\left[1 - s(\nu^2 + K_0 - 3)\right], \tag{3.25a}$$

$$f_n = K_n\left[1 - s(\nu^2 + K_0 - 1)\right]. \tag{3.25b}$$

Thus the Floquet functions are given by

$$u_\nu(\theta) = e^{i\nu_r\theta}\left\{1 + \sum_{n\neq 0}\left[1 - s(\nu^2 + K_0 - 3)\right]\frac{K_n}{\omega_n^2(1+2\nu_r/\omega_n)}\,e^{i\omega_n\theta}\right\}, \tag{3.26a}$$

$$y_\nu(\theta) = e^{i\nu_z\theta}\left\{1 - \sum_{n\neq 0}\left[1 - s(\nu^2 + K_0 - 1)\right]\frac{K_n}{\omega_n^2(1+2\nu_z/\omega_n)}\, e^{i\omega_n\theta}\right\} \tag{3.26b}$$

The form of the betatron oscillations is described by the functions $F_r(\theta)$, $F_z(\theta)$ where

$$F_r = |u_\nu(\theta)|^2, \tag{3.27a}$$

$$F_z = |y_\nu(\theta)|^2. \tag{3.27b}$$

The r-motion will have its maximum and minimum displacements where $F_r(\theta)$ has its maximum and minimum. For the circular orbit ($s = 0$), $F_r(\theta)$ and $F_\nu(\theta)$ are given by

$$F_r(\theta) = 1 + 4\sum_{n\geq 1}\frac{K_n}{\omega_n^2}\cos\omega_n\theta, \tag{3.28a}$$

$$F_z(\theta) = 1 - 4\sum_{n\geq 1}\frac{K_n}{\omega_n^2}\cos\omega_n\theta, \tag{3.29b}$$

where we assumed that the K_n are real.

The $F(\theta)$ function is essentially the $\beta(\theta)$ function of Courant and Snyder.[1]

Transition Energy

The going around frequency, $\omega/2\pi$, of the particle depends on the energy of the particle E. The energy E for which

$$\frac{d\omega}{dE} = 0 \tag{3.30}$$

is called the transition energy.

With the approximations used in MURA-390,

$$\omega = \frac{v}{R} \tag{3.31}$$

where v is the particle velocity.

Thus we find that

$$\frac{d\omega}{dE} = \frac{1}{pR}\left[\left(\frac{m}{E}\right)^2 - \frac{p}{R}\frac{dR}{dp}\right], \tag{3.32}$$

and the transition energy, E_t, is given by

$$\left(\frac{E_t}{mc^2}\right)^2 = \frac{R}{p}\frac{dp}{dR} \tag{3.33}$$

From Eq. (3.8), we see that for the circular orbit of the AG synchrotron,

$$\frac{p}{R}\frac{dR}{dp} = \frac{1}{K_0 + \nu^2 + 1} = \frac{1}{\nu_r^2} \tag{3.34}$$

where ν_r is the tune of the circular orbit.

Thus the transition energy occurs at

$$\frac{E_t}{mc^2} = \nu_r, \tag{3.35a}$$

or

$$\frac{E_t}{mc^2} = \sqrt{\nu^2 + K_0 + 1}. \tag{3.35b}$$

IV. COMPARISON WITH NUMERICAL RESULTS

In order to illustrate the accuracy of the results obtained in Section III, we have applied them to two AG machines whose orbit properties have been investigated on the MURA 704 computer.

Machine I

The parameters of this AG machine were chosen to approximate the Brookhaven and CERN AG machines. The magnetic field in the median plane is assumed to have the form

$$H_z = -B_0\left(1 + 2K_1 \frac{r-R_0}{R_0} \cos N\theta\right), \tag{4.1}$$

where $N = 50$, $K_1 = 356$.

The first two graphs, Figs. 1 and 2, deal with the equilibrium orbits. The equilibrium orbit may be thought of as described by two parameters, by R, the average radius of the equilibrium orbit, and by A, the amplitude of the oscillation of the orbit about the average radius.

In Fig. 1 we have plotted the average radius of the equilibrium orbit as given by $s = (R - R_0)/R_0$ as a function of the momentum change $(p - p_0)/p_0$. The straight line is the theoretical curve; the dots are the computer points.

In Fig. 2, we have plotted A, the amplitude of the oscillation of equilibrium orbit about its average radius in units of R, as a function of the momentum change. The straight line is the theoretical curve, the dots are computer points.

The results for the tune ν_r and ν_z for the circular orbit for which $p = p_0$ are as follows:

	Theory	Computer
ν_r	10.2	10.92
ν_z	10.1	10.86

Figs. 3 and 4 represent the tune ν_r and ν_z as a function of momentum. In Fig. 3, we plot the precentage r-tune change $\Delta\nu_r/\nu_r$ where ν_r is the tune of the circular orbit, against percentage change in momentum, $\Delta p/p_0$. In Fig. 4, we plot $\Delta\nu_z/\nu_z$ in the same way. On both graphs the straight line is the theoretical curve and the dots are computer points.

Figs. 5 and 6 deal with the betatron oscillations around the equilibrium orbit. In Fig. 5, we have plotted the horizontal betatron oscillation, u, as a function of θ for the case where $u = 1\times10^{-4}$ and $u' = 0$ at $\theta = 0$. u is measured in units of R and θ is in units of $2\pi/N$. Fig. 6 is the corresponding plot for the vertical betatron oscillations. The solid curve is the theoretical result and the dots are the computer points. In computing the theoretical results, the computer results for the tune were used. If the theoretical results for the tune are used, the two curves, computer and theoretical, slip out of phase and comparing the two is more difficult.

Machine II

The parameters of this machine were chosen to illustrate the effect that for a machine which has $\nu_r \simeq \sqrt{4.5}$, the r-tune, ν_r, is independent of the momentum to first order in $\Delta p/p_0$.

For this machine the field in the median plane is given by

$$H_z = -B_0 \left(1 + 2K_1 \frac{r - R_0}{R_0} \cos N\theta\right)$$

where $N = 50$, $K_1 = 70.5$.

Since for this machine $\nu^2 = 2K_1^2/N^2 = 4$, Eq. (3.15) for the r-tune ν_r predicts that $\nu_r = \sqrt{5} = 2.24$ for the circular orbit.

The results for the tune ν_r and ν_r for the circular orbit for which $p = p_0$ are as follows:

	Theory	Computer
ν_r	2.24	2.236
ν_z	2.00	1.999

In Fig. 3, the crosses indicate the variation of ν_r with $\Delta p/p_0$ for Machine II. The tune ν_r of Machine II changes with energy at a rate which is about ten times slower than that of Machine I.

In Fig. 7, the z-tune ν_z is plotted against $\Delta p/p_0$. The solid curve is the theoretical curve and the dots are computer points. The curve is parabolic and not linear because the quadratic term in Eq. (3.21) is not negligible here.

One may notice that according to Eq. (3.17), that ν_r is independent of momentum when $\nu = \sqrt{3.5} = 1.87$ and $K_0 = 0$, which means that $\nu_r = 2.12$ and $\nu_z = 1.87$. Also, according to Eq. (3.21), ν_z can be made independent of momentum when $\nu = \sqrt{.5} = .707$ and $K_0 = 0$, which means that $\nu_r = 1.22$ and $\nu_z = .707$. It does not seem possible to choose the parameters so that both ν_r and ν_z are independent of momentum. By choosing an intermediate case like $\nu = 1.4$ and $K_0 = 0$, both ν_r and ν_z can be made to vary somewhat more slowly with momentum. It should also be noted that in the case of z-tune, the term which is second order in the momentum change may become important particularly when $\nu \simeq .5$.

REFERENCES

1. E. Courant and H. Snyder, Annals of Physics 3, 1, (1958).

2. M. Hamermesh, Argonne National Laboratory, 1955.

3. A. Schoch, CERN 57 - 21.

4. The formuli used in calculating the tune ν_r , ν_z are slightly different from those given in MURA-397. They include a small correction which will be presented in a future report.

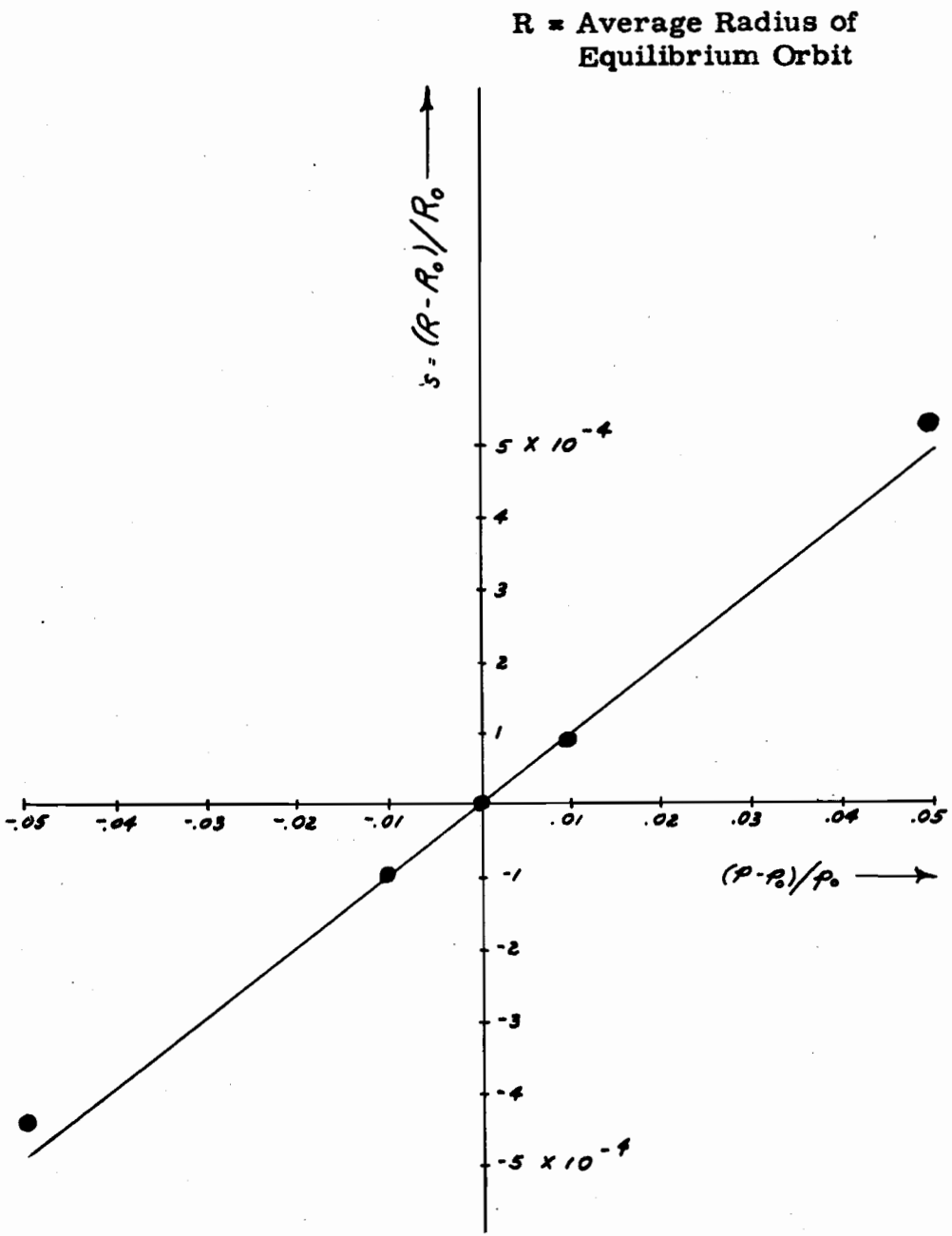
R = Average Radius of Equilibrium Orbit
$s = (R - R_0)/R_0$
5×10^{-4}
4
3
2
1
-1
-2
-3
-4
-5×10^{-4}
-.05
-.04
-.03
-.02
-.01
.01
.02
.03
.04
.05
$(p - p_0)/p_0$

Fig. 1

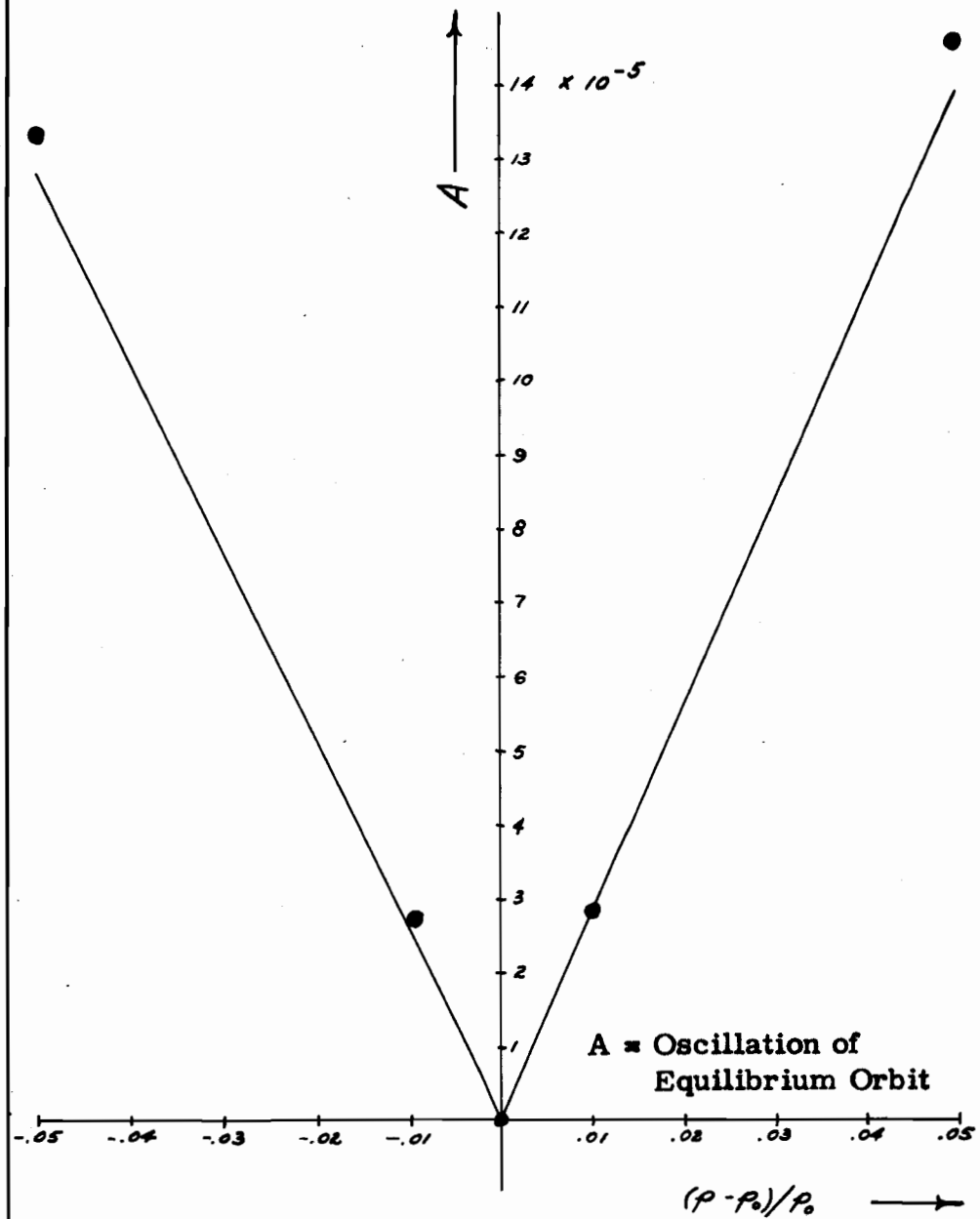
A
14×10^{-5}
13
12
11
10
9
8
7
6
5
4
3
2
1
A = Oscillation of Equilibrium Orbit
-.05
-.04
-.03
-.02
-.01
.01
.02
.03
.04
.05
$(p - p_0)/p_0$

Fig. 2

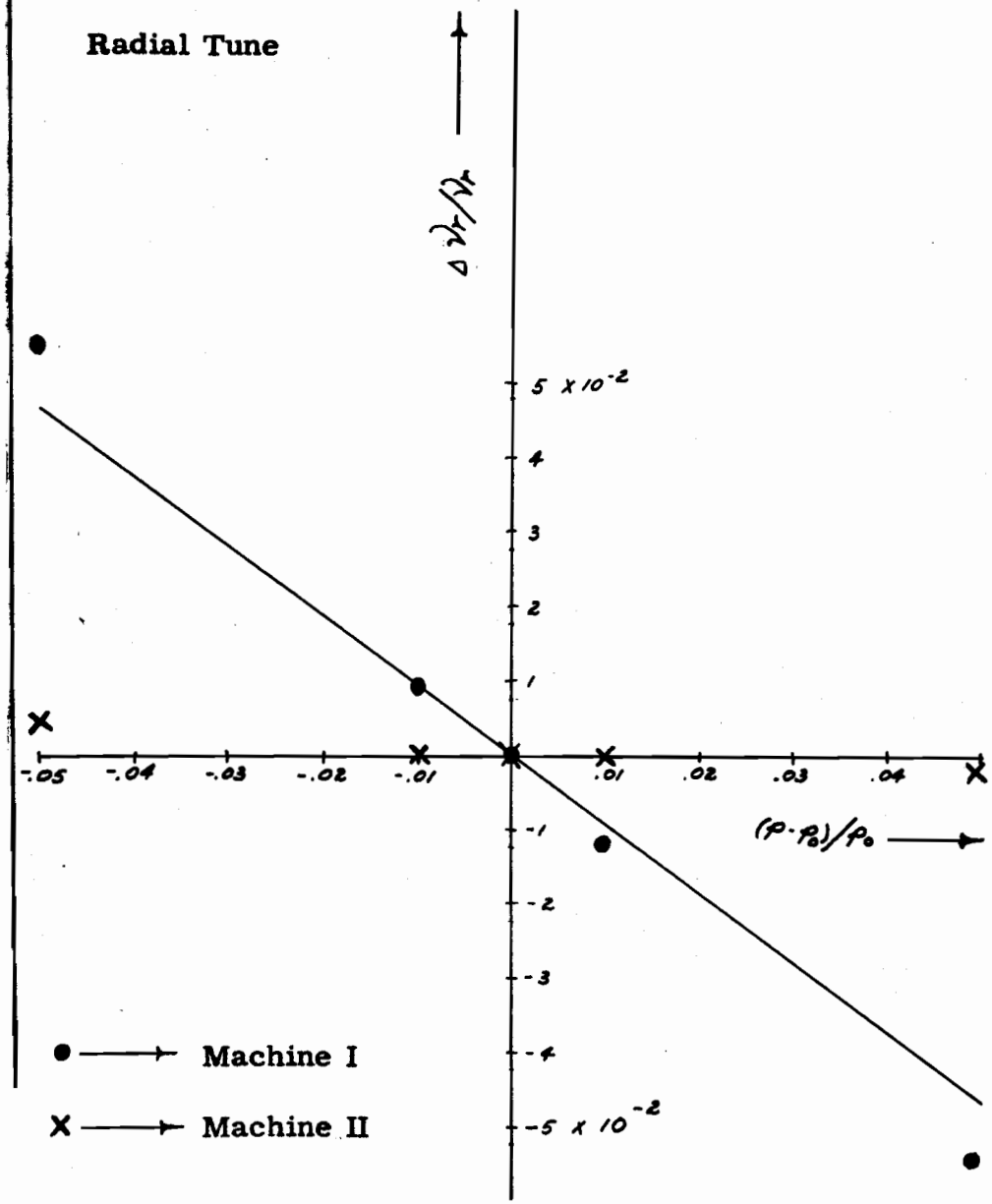
Radial Tune
$\Delta \nu_r/\nu_r$
5 x 10^-2
4
3
2
1
-1
-2
-3
-4
-5 x 10^-2
-.05
-.04
-.03
-.02
-.01
.01
.02
.03
.04
$(p-p_0)/p_0$
Machine I
Machine II

Fig. 3

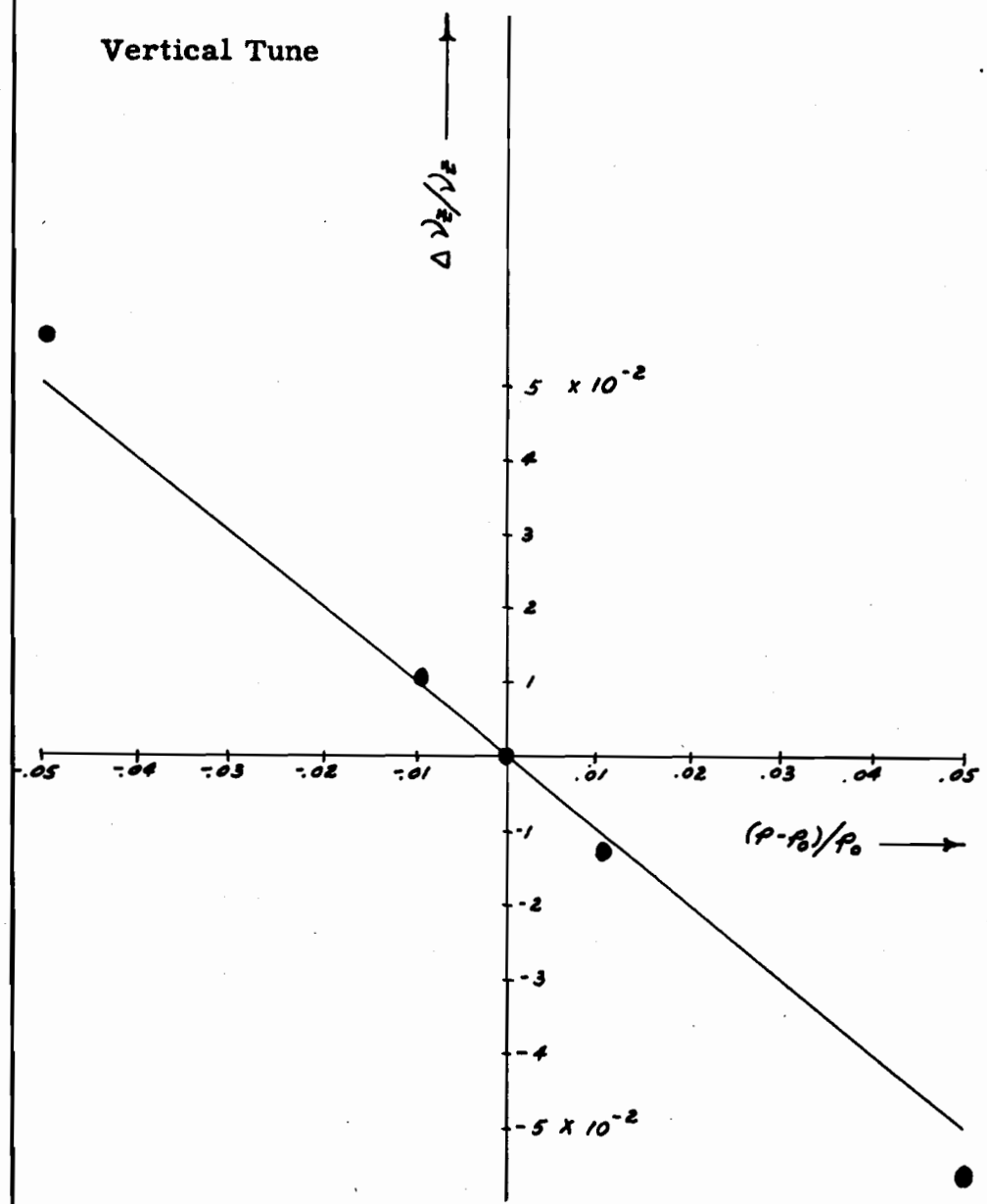
Vertical Tune
$\Delta \nu_z/\nu_z$
5 x 10^-2
4
3
2
1
-1
-2
-3
-4
-5 x 10^-2
-.05
-.04
-.03
-.02
-.01
.01
.02
.03
.04
.05
$(p-p_0)/p_0$

Fig. 4

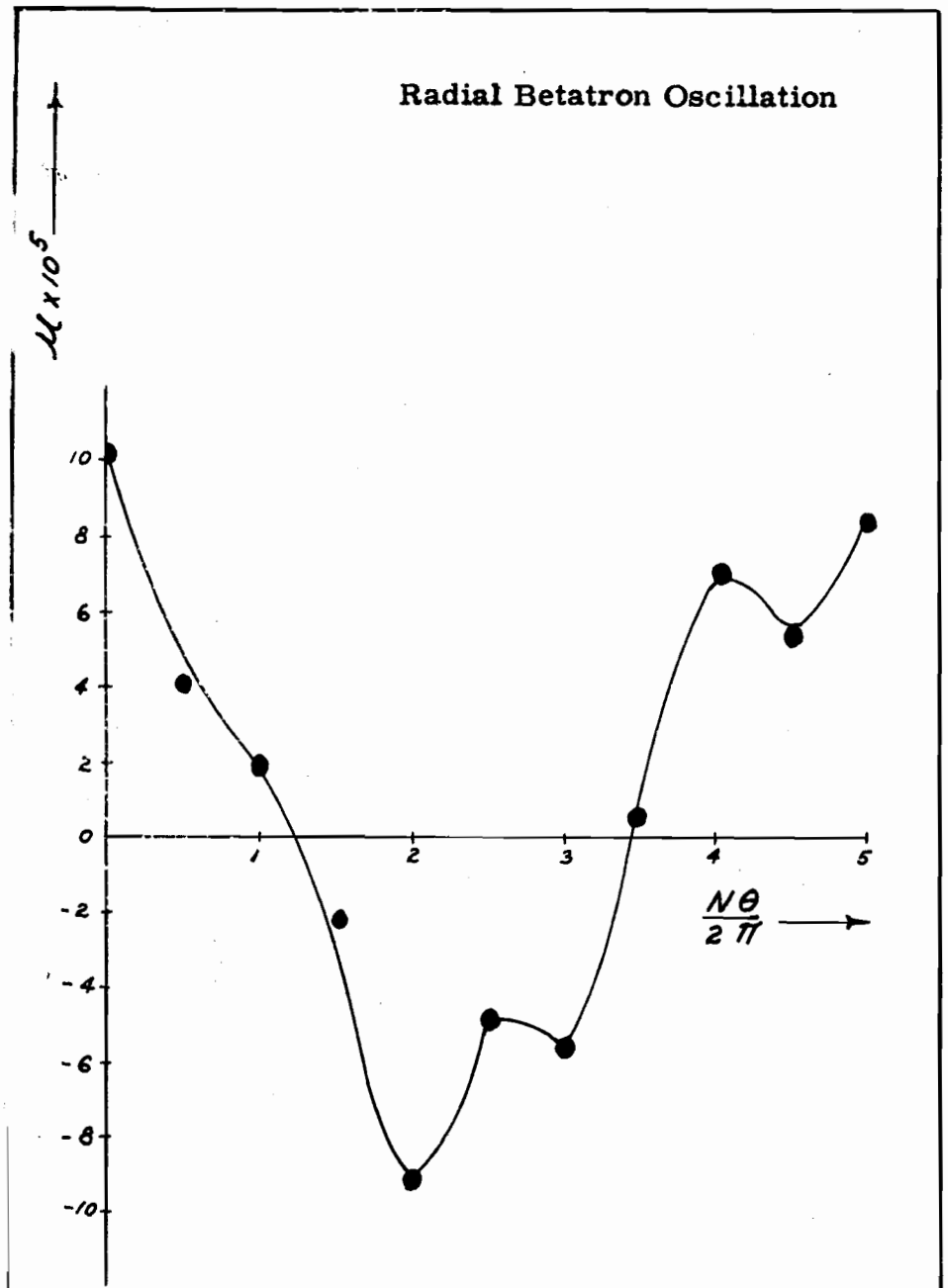
Radial Betatron Oscillation
$x \times 10^5$
10
8
6
4
2
0
-2
-4
-6
-8
-10
1
2
3
4
5
$\frac{N\theta}{2\pi}$

Fig. 5

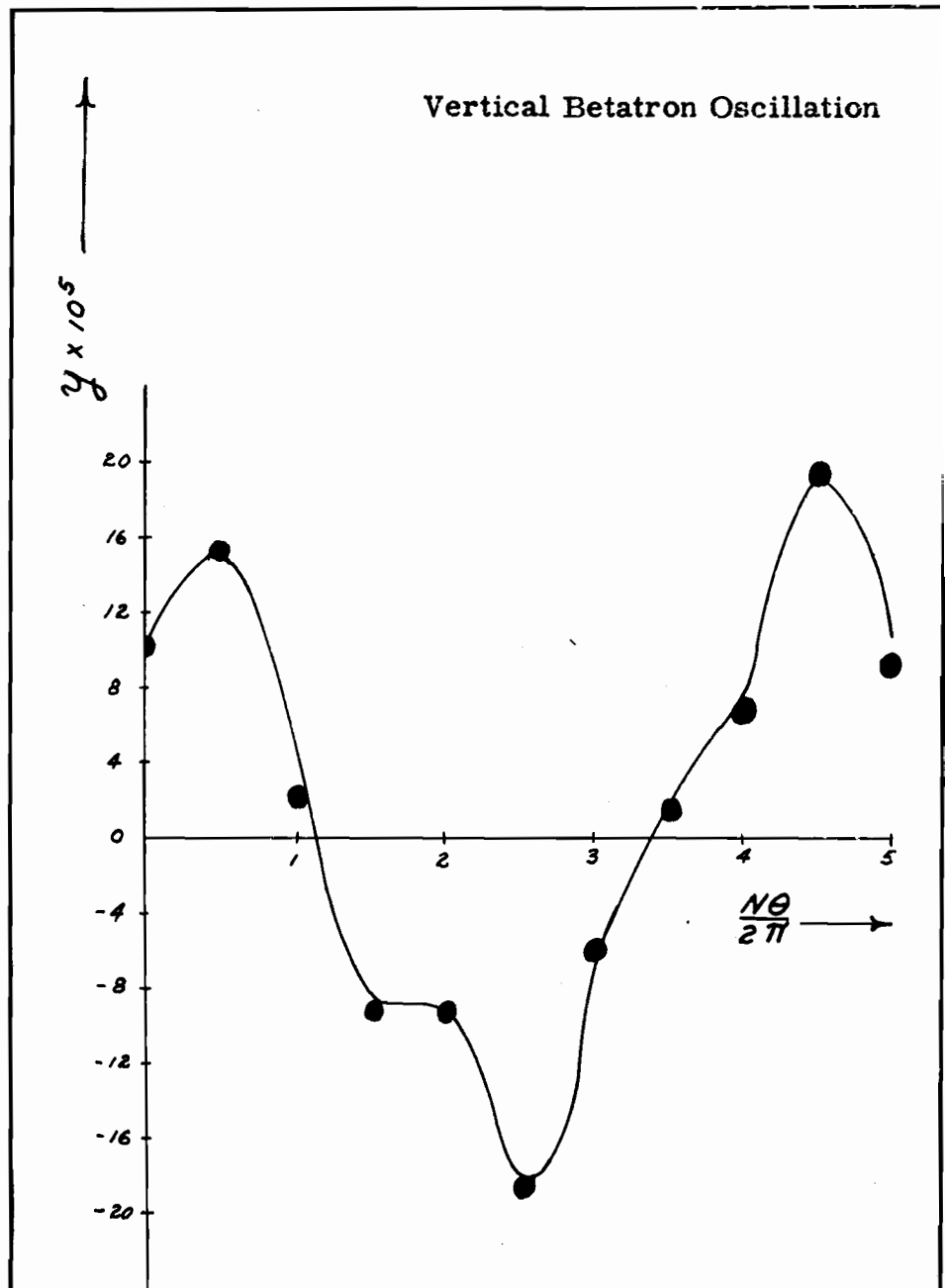
Vertical Betatron Oscillation
$y \times 10^5$
20
16
12
8
4
0
-4
-8
-12
-16
-20
1
2
3
4
5
$\frac{N\theta}{2\pi}$

Fig. 6

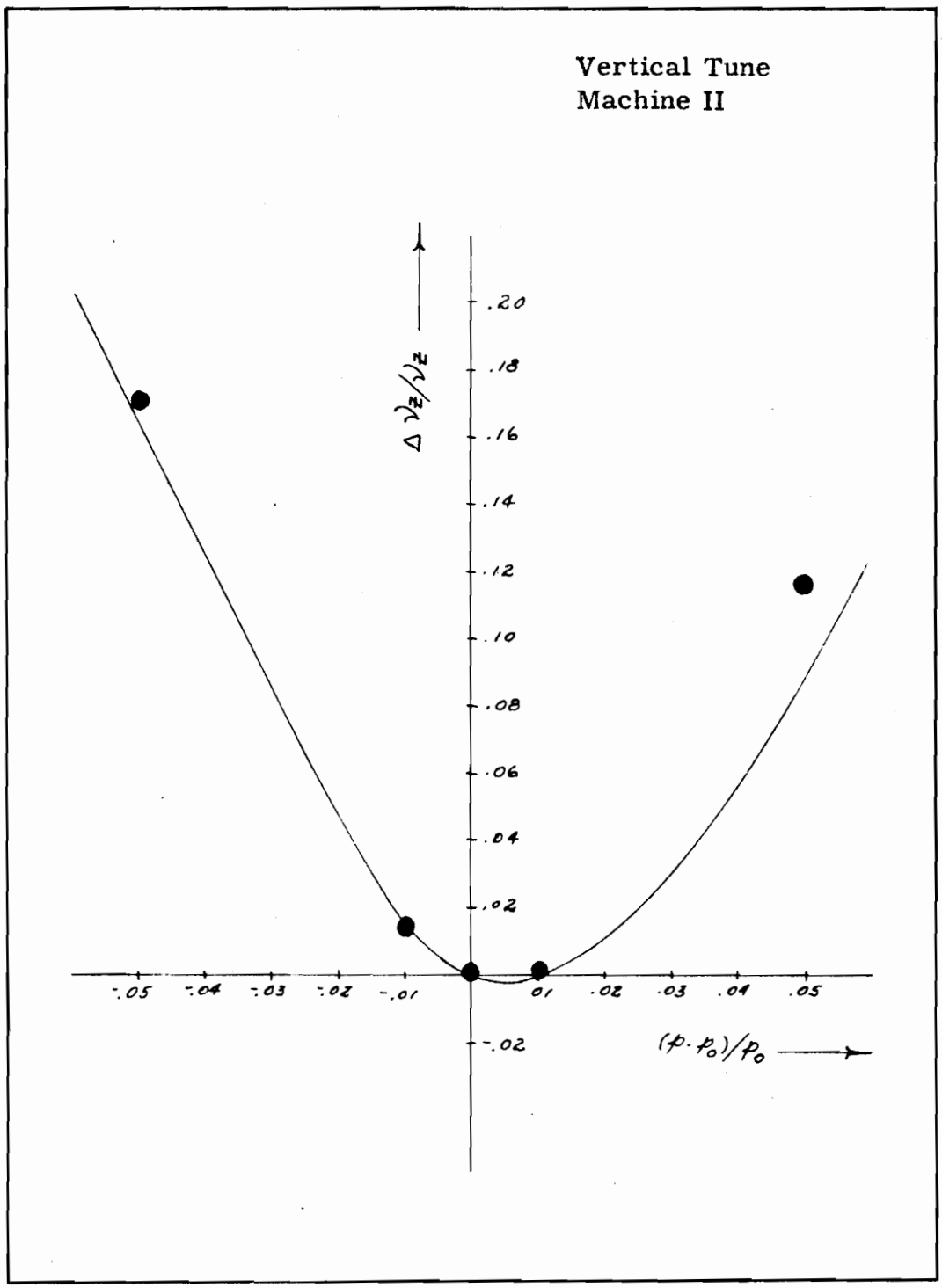
Vertical Tune
Machine II
$\Delta \nu_z/\nu_z$
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
-.02
-.05
-.04
-.03
-.02
-.01
.01
.02
.03
.04
.05
$(p-p_0)/p_0$

Fig. 7