423

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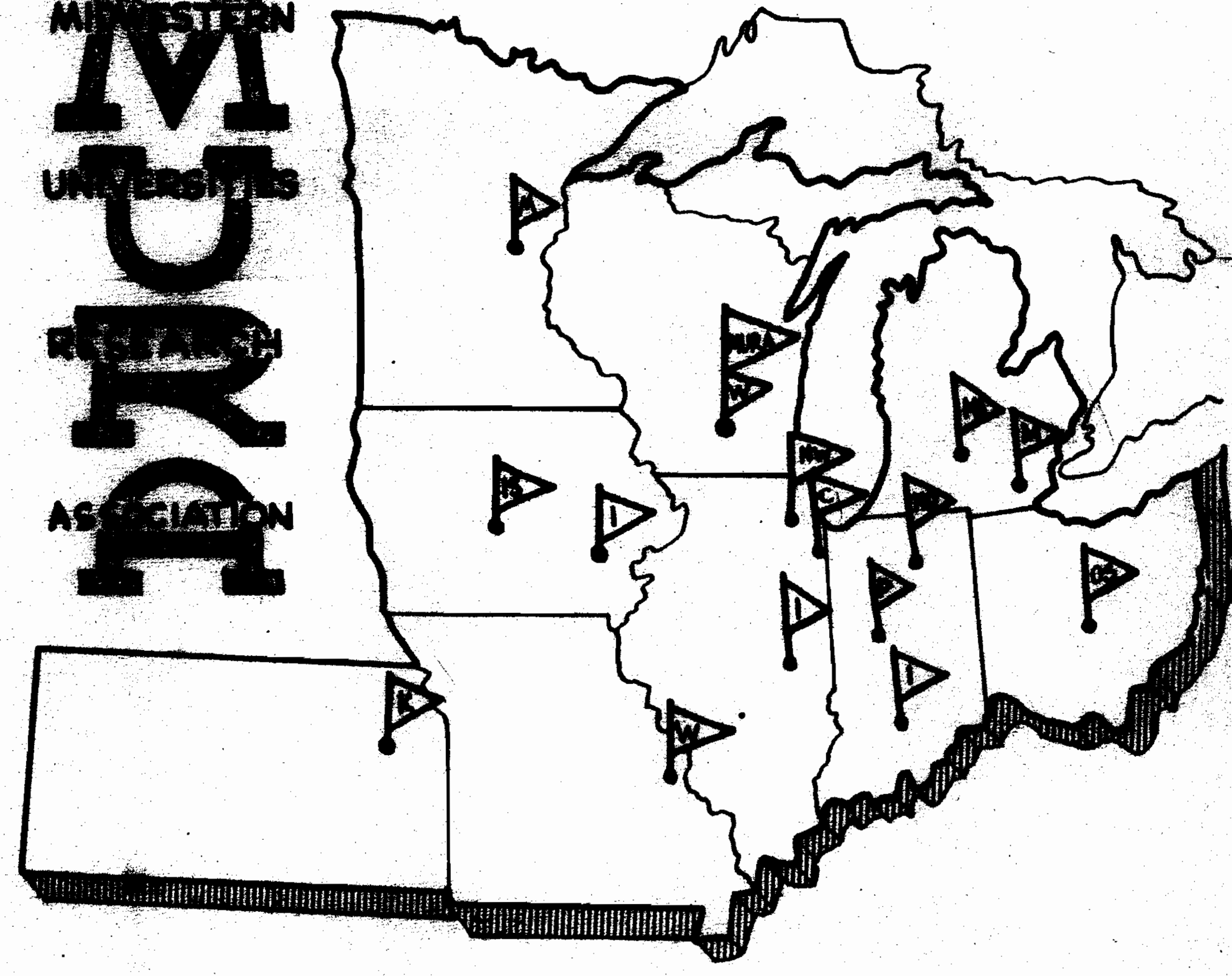

A STUDY OF THE RF PHASE PLANE NEAR TRANSITION WITH FREQUENCY MODULATION

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REPORT NUMBER 423

MURA-423

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August 6, 1958

ABSTRACT

The theory given in MURA-106 of rf acceleration near the transition energy is generalized to apply to moving coordinates synchronous with a frequency modulated rf voltage. Formulas are developed for fixed points and separatrices in the rf phase plane, and for frequencies of phase oscillations. Digital computer studies of the acceleration process have been made and resulting rf phase plots are presented. Computed bucket areas are given as functions of the parameters η , Γ introduced in MURA-106. A criterion is formulated for the applicability of the adiabatic approximation in the neighborhood of the transition energy.

*AEC Research and Development Report. Research supported by the Atomic Energy Commission, Contract No. AEC AT(11-1) 384.

Symon and Sessler[1] have shown that the behavior of a particle in an idealized accelerator can be described by a Hamiltonian function of the form

$$H^* = 2\pi E(W) - \frac{2\pi\nu}{h} W + \frac{V}{h} \cos(h\theta^*) \tag{1}$$

where $$W = \int \frac{dE}{f} \tag{2}$$

h is the harmonic number

ν is the frequency of the oscillator

V is the voltage across the gap

θ^* is the angular coordinate in a cylindrical phase space rotating at synchronous frequency

$f(E)$ is the frequency of revolution of a particle at energy E.

One can perform an additional canonical transformation specified by the generating function

$$S = \theta^*(W^* + W_s) \tag{3}$$

where $$W^* = W - W_s(t) \tag{4}$$

and W_s isthe synchronous value of W.

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1. MURA-106

The Hamiltonian is then rewritten as

$$H^{**} = 2\pi E(W) - \frac{2\pi\nu}{h} W^{*} + \frac{V}{h}\cos(h\theta^{*}) + \frac{\dot{W}_s}{h}(h\theta^{*}) \tag{5}$$

Now expand E(W) in a Taylor series about W_s

$$E(W) = E_s + f_s W^{*} + \frac{1}{2} f_s' W^{*2} + \frac{1}{6} f_s'' W^{*3} + \cdots$$

From Eq. (53) of MURA-106 we have near transition

$$f = f_t \left(1 - \frac{(E - E_t)^2}{k E_t^2}\right) \doteq f_t \left(1 - \frac{(W - W_t)^2 f_t^2}{k E_t^2}\right) \tag{6}$$

Let

$$W_t^{*} = W_t - W_s(t) \tag{7}$$

$$f_s \doteq f_t \left(1 - \frac{W_t^{*2} f_t^2}{k E_t^2}\right) \tag{8}$$

$$f_s' \doteq f_t \left(-2 \frac{(W - W_t) f_t^2}{k E_t^2}\right)\Bigg|_{W = W_s} = \frac{2 W_t^{*} f_t^3}{k E_t^2} \tag{9}$$

$$f_s'' \doteq \frac{f_t^3}{k E_t^2}(-2) \tag{10}$$

The Hamiltonian now takes the form:

$$H^{**} = 2\pi f_t W^* - \frac{2\pi f_t^3}{k E_t^2} W_t^{*2} W^* + \frac{2\pi f_t^3}{k E_t^2} W_t^* W^{*2} - \frac{2\pi}{3} \frac{f_t^3 W^{*3}}{k E_t^2} - \frac{2\pi \nu W^*}{h} + \frac{\dot{W}}{h}(h\theta^*) + \frac{V}{h}\cos(h\theta^*) \tag{11}$$

If we introduce the variables

$$y = a(W^* - W_t^*) \tag{12}$$

$$\phi = \pi - h\theta^* \tag{13}$$

$$\tau = aVt \tag{14}$$

where

$$a = \left(\frac{4\pi h}{k E_t^2 V}\right)^{\frac{1}{3}} f_t \tag{15}$$

and make the abbreviations

$$\eta = B\left(1 - \frac{\nu}{h f_t}\right) \tag{16}$$

$$\Gamma = \frac{\dot{W}_s}{V} \tag{17}$$

where

$$B = (2\pi^2 k)^{\frac{1}{3}} \left(\frac{h E_t}{V} \right)^{2/3} \tag{18}$$

then the Hamiltonian $C(y, \varphi, \tau)$ has the form

$$C = -a h \left(\frac{1}{aV}\right) H^{**} = -\frac{h}{V} H^{**} = \frac{1}{6} y^3 - \eta y + \cos\varphi + r\varphi \tag{19}$$

where we have omitted terms independent of y and φ. We see that y and φ are canonical variables after this change of scale, for

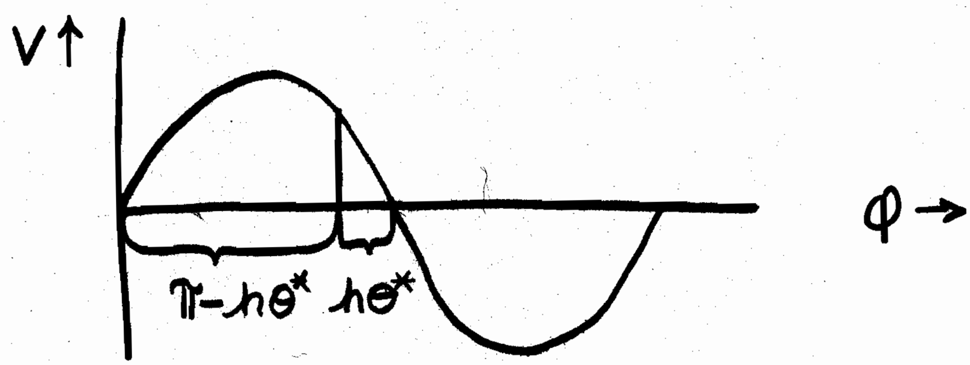
$$\frac{\partial y}{\partial \tau} = \frac{\partial y}{\partial W^*} \frac{\partial W^*}{\partial t} \frac{\partial t}{\partial \tau} = -a \frac{\partial H^{**}}{\partial \theta^*} \frac{1}{aV} = -\frac{\partial C}{\partial \varphi} \tag{20}$$

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial \varphi}{\partial \theta^*} \frac{\partial \theta^*}{\partial t} \frac{\partial t}{\partial \tau} = -h \frac{\partial H^{**}}{\partial W^*} \frac{1}{aV} = \frac{\partial C}{\partial y} \tag{21}$$

Note the change in the definition of φ here from MURA-106, where $\varphi = h\theta^*$. The coordinate of the reference point revolving in the accelerator at frequency ν/h is $\theta^* = 0$. From the equation of motion it can be seen that the reference point crosses the gap when $V \sin h\theta^* = 0$ and the voltage is decreasing. For a particle at the reference point,

$$\frac{\partial W}{\partial A} = -\frac{\partial H^*}{\partial \theta^*} = 0 = V \sin h\theta^* \tag{22}$$

For small positive values of $h\theta^*$ we want $\frac{\partial W}{\partial t}$ to be positive. A particle with a coordinate $\theta^* > 0$ arrives at the gap before the reference particle. In order for the voltage to be positive at this time, the reference particle must cross the gap when the voltage is decreasing. If we set $\varphi = \pi - h\,\theta^*$, φ has the physical meaning that it is the phase of the voltage when the particle crosses the gap. This can be clearly seen from the following diagram.



We can now investigate the properties of frequency modulated buckets near transition.

The condition for a fixed point is

$$\frac{\partial C}{\partial \varphi} = \frac{\partial C}{\partial y} = 0 \tag{23}$$

By differentiating Eq. (19) we find that the values of φ and y at the fixed points are

$$\varphi = \sin^{-1} \Gamma \tag{24}$$

$$y = \pm\sqrt{2\eta} \tag{25}$$

From Eqs. (6), (12), (14), (16), and (17) together with

$$\frac{\nu_s}{h} = f_s \tag{26}$$

one can see that the value of y at the fixed point is just y_s.

By expanding Eq. (19) in a Taylor series about the fixed points, we can determine the type of fixed point. If the Taylor series has elliptical form, the point is a stable fixed point. If the Taylor series has hyperbolic form, it is an unstable fixed point. It can thus be seen that the unstable fixed point is in the second quadrant below transition and in the first quadrant above transition. The fixed points are plotted as a function of Γ in Fig. 1.

Substitution of the unstable fixed points in Eq. (19) gives the following values of C on the separatrices.

$$C = \cos \phi_u + \Gamma \phi_u - \frac{2}{3}\sqrt{2}\, \eta^{3/2} \qquad \text{above transition} \tag{27}$$

$$C = \cos \phi_u + \Gamma \phi_u + \frac{2}{3}\sqrt{2}\, \eta^{3/2} \qquad \text{below transition} \tag{28}$$

where ϕ_u is the value of ϕ at the unstable fixed point.

The properties of these frequency-modulated buckets near transition have been examined by means of digital computer studies, using the TTT program. This program computes the energy increment gained by the particle each time it crosses a gap.

$$\Delta E = V_j(t_j N) \sin \phi_{jN} \tag{29}$$

where t_{jN} is the time of arrival of the particle at the j^{th} oscillator on the N^{th} revolution and ϕ_{jN} is the phase of the j^{th} oscillator at the moment when the particle passes the gap on the N^{th} revolution.

The graphs of the buckets were obtained by using two gaps, one oscillating sinusoidally with constant frequency and a maximum voltage V_o and the other not oscillating but with a constant voltage of $-\Gamma V_o$. In this way we can obtain graphs of a moving bucket in a fixed coordinate system. This can be seen by differentiating Eq. (5) with respect to θ^*.

$$\frac{dW^*}{dt} = -\frac{\partial H^*}{\partial \theta^*} = \sin h\theta^* - \dot{W}_s = V\sin\varphi - \dot{W}_s \tag{30}$$

In an accelerator with two gaps as described above, the increase in energy per unit time is

$$\frac{dE}{dt} = fV\sin\Phi - f\Gamma V \tag{31}$$

$$\frac{dW}{dt} = V\sin\Phi - \Gamma V = V\sin\Phi - \dot{W}_s \tag{32}$$

Thus if $\Phi = \varphi$, the motion of the particles in a fixed coordinate system in this accelerator with two gaps, one with constant frequency and the other non-oscillatory, is equivalent to the motion of particles in a coordinate system moving with a rate $\dot{W}_s$ in an accelerator with one frequency modulated gap of the same voltage with $\dot{W}_s = \Gamma V$.

We have obtained graphs for buckets with a Γ of .5, but the following discussion is applicable to buckets with $0 \leq \Gamma < 1$.

For large values of η there exist separate buckets above and below the transition energy as shown in Fig. 2 for Γ = .5.

These buckets correspond to $\dot{W}_s > 0$. One should note that these two buckets do not exist in a real accelerator at the same time however. Below transition $\dot{\gamma} > 0$ when $\dot{W}_s > 0$, but above transition, $\dot{\gamma} < 0$ when $\dot{W}_s > 0$. If $\dot{W}_s < 0$, then these buckets would be reflected about the point ($W = W_s$, $\varphi = 0$) as in Fig. 3. In an accelerator then, one of the buckets in Fig. 2 would be reflected about the point ($W = W_s$, $\varphi = 0$). If $\dot{\gamma} > 0$, the bucket with $E_s > E_t$ is reflected and both buckets move toward transition. If $\dot{\gamma} < 0$ the bucket with $E_s < E_t$ is reflected and both buckets move away from transition.

There is a value of $\eta = \eta_c(\Gamma)$ for which the two values of C become equal and the separate buckets merge, as shown in Fig. 4 for $\Gamma = .5$. η_c may be found by setting the values of C in Eqs. (27) and (28) equal to each other and solving graphically for η. A better method is to consider only one of the buckets, either above or below transition, and solve Eq. (19) for the values of y where $\varphi = \varphi_s$, where φ_s is the value of φ at the stable fixed point. Note that in a cubic equation of the form

$$x^3 + ax + b = 0$$

if $\frac{b^2}{4} + \frac{a^3}{27} < 0$ there will be three real and unequal roots

if $\frac{b^2}{4} + \frac{a^3}{27} = 0$ there will be three real roots of which two at least are equal. The latter case corresponds to $\eta = \eta_c$.

Accordingly we consider the bucket above transition. Using Eqs. (19) and (27) we get

$$+\frac{1}{6}y^3 - \eta y + \cos\varphi_s + \Gamma\varphi_s = \cos\varphi_u + \Gamma\varphi_u - \frac{2}{3}\sqrt{2}\,\eta^{3/2}$$

$$+\frac{1}{6}y^3 - \eta y + \cos\phi_s + \Gamma\phi_s = \cos\phi_u + \Gamma\phi_u - \frac{2}{3}\sqrt{2}\,\eta^{3/2} \tag{33}$$

By rearranging terms and making the substitutions

$$\phi_u = \pi - \phi_s \tag{34}$$

$$\cos\phi_u = -\cos\phi_s \tag{35}$$

we get

$$y^3 - 6\eta y + 12\cos\phi_s + 6\Gamma(2\phi_s - \pi + 4\sqrt{2}\,\eta^{3/2} = 0 \tag{36}$$

At $\eta = \eta_c$

$$\left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2 = 0 = \eta_c^{\frac{3}{2}}\left[12\sqrt{2}(2\phi_s - \pi)\Gamma + 24\sqrt{2}\cos\phi_s\right]$$

$$+36\cos^2\phi_s + 9\Gamma^2(2\phi_s - \pi)^2 + 36\Gamma(2\phi_s - \pi)\cos\phi_s$$

and

$$\eta_c = \left[\frac{12\cos^2\phi_s + 3\Gamma^2(2\phi_s - \pi)^2 + 12\Gamma(2\phi_s - \pi)\cos\phi_s}{4\sqrt{2}(2\phi_s - \pi)\Gamma + 8\sqrt{2}\cos\phi_s}\right]^{2/3} \tag{37}$$

where ϕ_s is the value of ϕ at the stable fixed point above transition.

Equivalent results would have been obtained if we had considered the bucket below transition. The value of η_c is plotted as a function of Γ in Fig. 5.

For $0 < \eta < \eta_c$, the phase plot is similar to the one shown in Fig. 6. Below transition the particles in buckets execute oscillations in phase space in a counterclockwise direction. Above transition they oscillate in a clockwise direction. For $\eta = 0$, the phase plot obtained from the digital computer is as shown in Fig. 7. There is a small discrepancy here between digital computer results and the foregoing analytical treatment in that, whereas no buckets were predicted in the phase plot at $\eta \leq 0$, small buckets were found to exist in digital computations at $\eta = 0$, even for $\Gamma = 0$, i.e., static buckets. The discrepancy can perhaps be attributed to the fact that in the analytical treatment, the energy gained by the particle as it passes the gap is assumed to occur continuously as it travels around the orbit. Investigations were made on the digital computer for $\eta < 0$ at $\Gamma = 0$ and $\Gamma = .5$. It was found that small buckets continued to exist for both $\Gamma = 0$ and $\Gamma = .5$ from $0 < \eta < -.04$. For $\eta < -.06$ there were no buckets, as predicted by the analytical treatment. In Fig. 8 a graph is shown for the case $\eta = -1$, $\Gamma = .5$.

The area α of the buckets in $y - \varphi$ units as a function of η and Γ has been measured by the TTT program. The area in $W - \theta$ units is then given by the following formula

$$A = \frac{1}{a} \alpha(\eta, \Gamma) \tag{38}$$

where a is defined in Eq. (15), and the function $\alpha(\eta, \Gamma)$, the area in $y - \varphi$ units is plotted against η for various values of Γ in Fig. 9.

The frequency of phase oscillation may be found by expanding the Hamiltonian about the stable fixed point and comparing the resultant form with the Hamiltonian for a harmonic oscillator. The Hamiltonian expanded about the stable fixed point is

$$C = \frac{\sqrt{2\eta}}{2} y^2 + \frac{\sqrt{1-\Gamma^2}}{2} \phi^2 \tag{39}$$

The Hamiltonian for a harmonic oscillator is

$$\frac{1}{2}\frac{p^2}{m} + \frac{1}{2} k \phi^2 \tag{40}$$

and the frequency of a harmonic oscillator is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{41}$$

In comparison the frequency of phase oscillation in τ units for small buckets is

$$\nu_p = \frac{1}{2\pi}\left[\sqrt{1-\Gamma^2}\sqrt{2\eta}\right]^{\frac{1}{2}} \tag{42}$$

Since $\frac{1}{t} = \frac{aV}{\tau}$, the frequency of phase oscillation in t units is

$$\nu_p = \frac{1}{2\pi}\left[\sqrt{1-\Gamma^2}\sqrt{2\eta}\right]^{\frac{1}{2}} aV \tag{43}$$

where a is defined by Eq. (15)

The phase plots of frequency-modulated buckets do not represent the motion of the particles unless the parameters η, Γ change slowly in comparison with the frequency of phase oscillation. One can perhaps consider a plot of a bucket at a particular value of η meaningful if the time required for a phase oscillation at that η is equal to or greater than the time required to accelerate the bucket from that point to transition. By equating these two times we can determine a value of $\eta = \eta_d$ which we consider to be the division between the adiabatic and the sudden regions. That is, if $\eta >> \eta_d$, the particle behaves adiabatically with respect to the phase plots. If $\eta << \eta_d$, the transition to $\eta = 0$ can be regarded as sudden.

For a particle near transition, the time required for acceleration to transition is

$$t = \frac{\Delta W}{\Gamma V} \qquad \text{or} \qquad \tau = \frac{y_s}{\Gamma} \tag{44}$$

where y_s is given by Eq. (25):

$$\tau^2 = \frac{2\eta}{\Gamma^2} \tag{45}$$

The square of the time required for one phase oscillation is, from Eq. (42)

$$\tau_p = \frac{4\pi^2}{\sqrt{1-\Gamma^2}\sqrt{2\eta}} \tag{46}$$

Equating the times in Eqs. (45) and (46), we find

$$\eta_d = \left(\frac{2\pi^4 \Gamma^4}{1-\Gamma^2} \right)^{1/3} \tag{47}$$

This value of η is plotted against Γ in Fig. 10. By comparing Fig. 10 with Fig. 5, we see for example that the concept of a critical value η_c at which the buckets change their character has a physical meaning only if $\Gamma < 0.2$.

Fixed Points γ

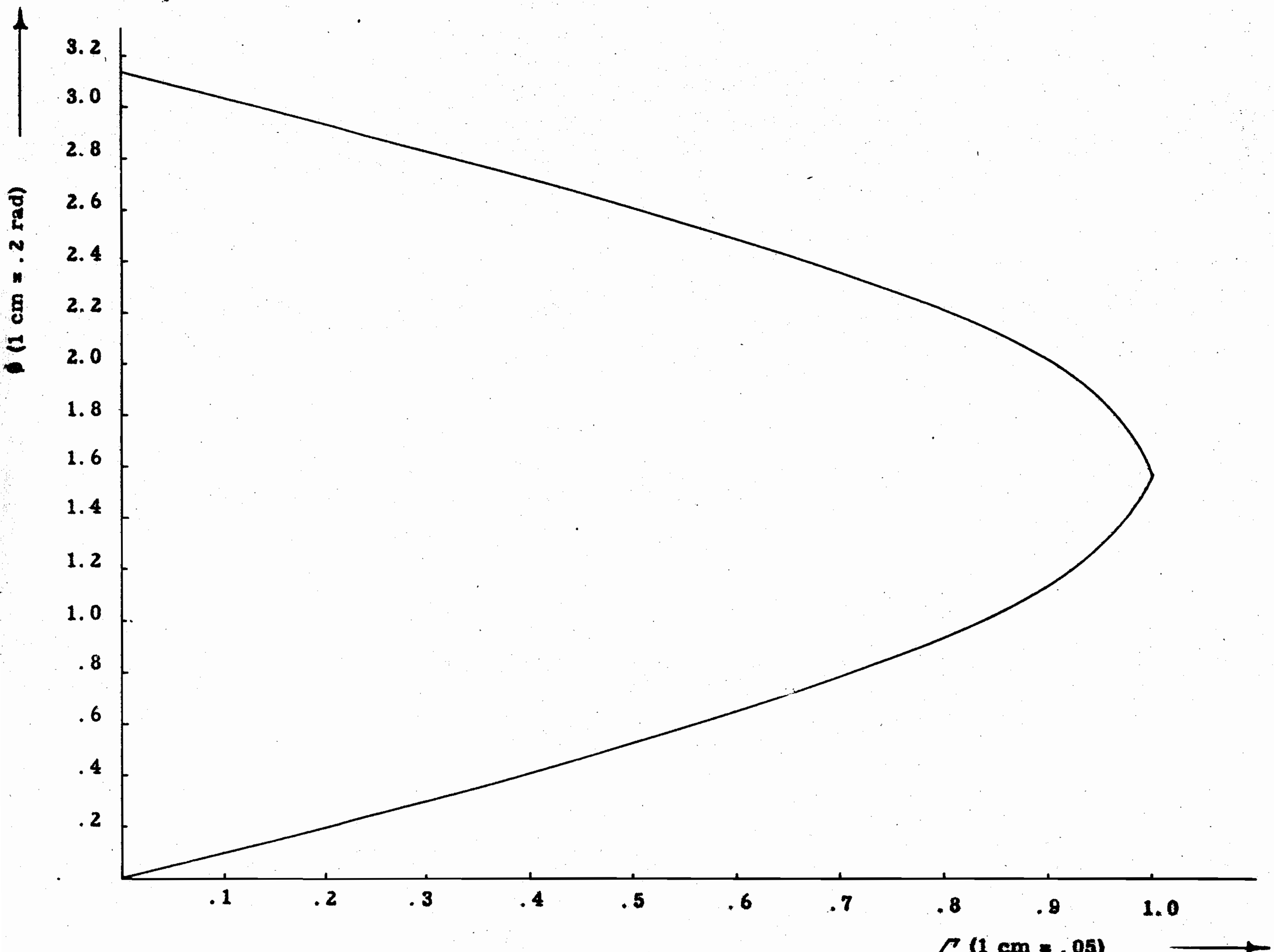


Fig. 1

Phase Plane at Transition - Moving Buckets

$\eta = 1.04$

$\Gamma = .5$

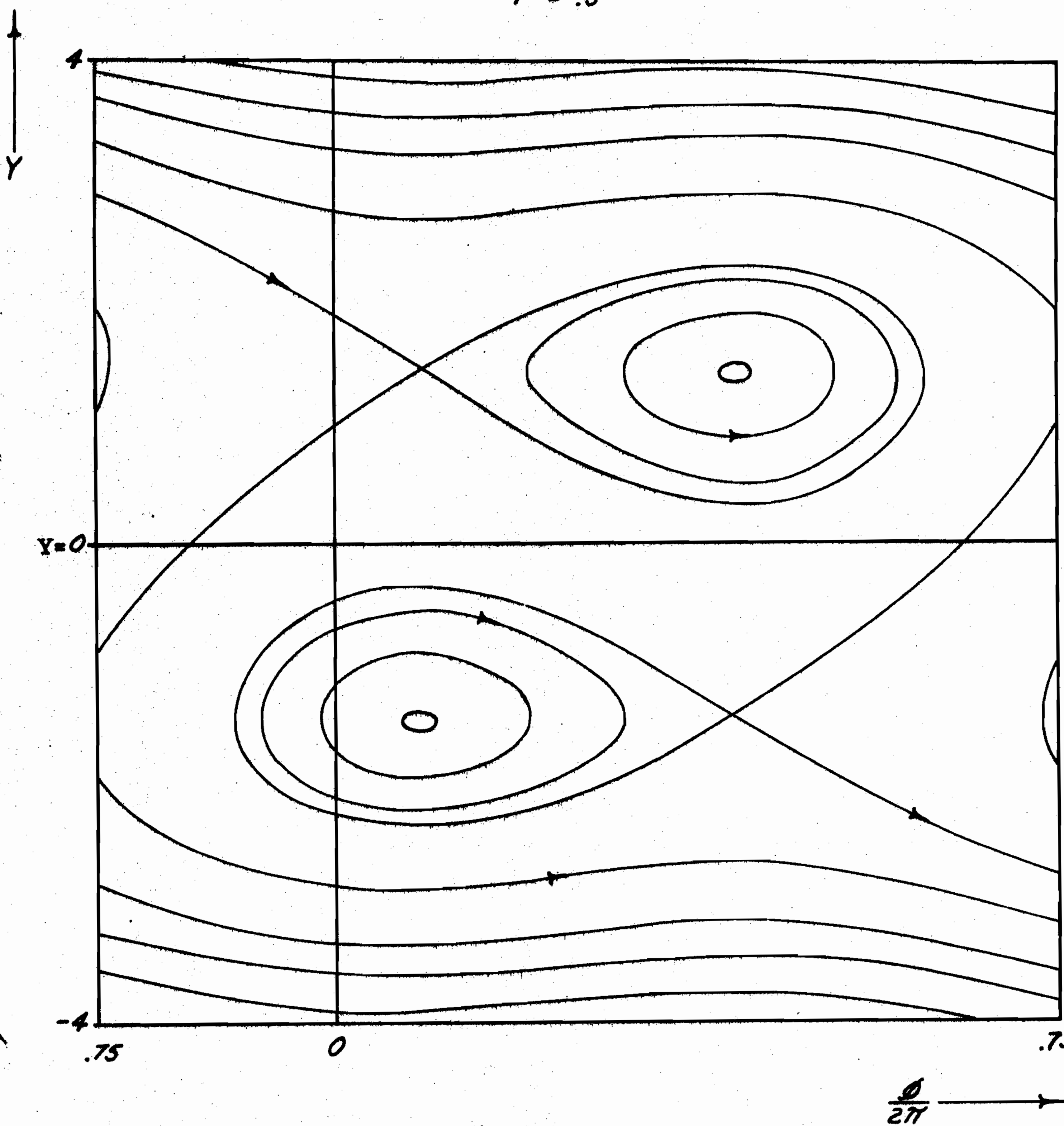


Fig. 2

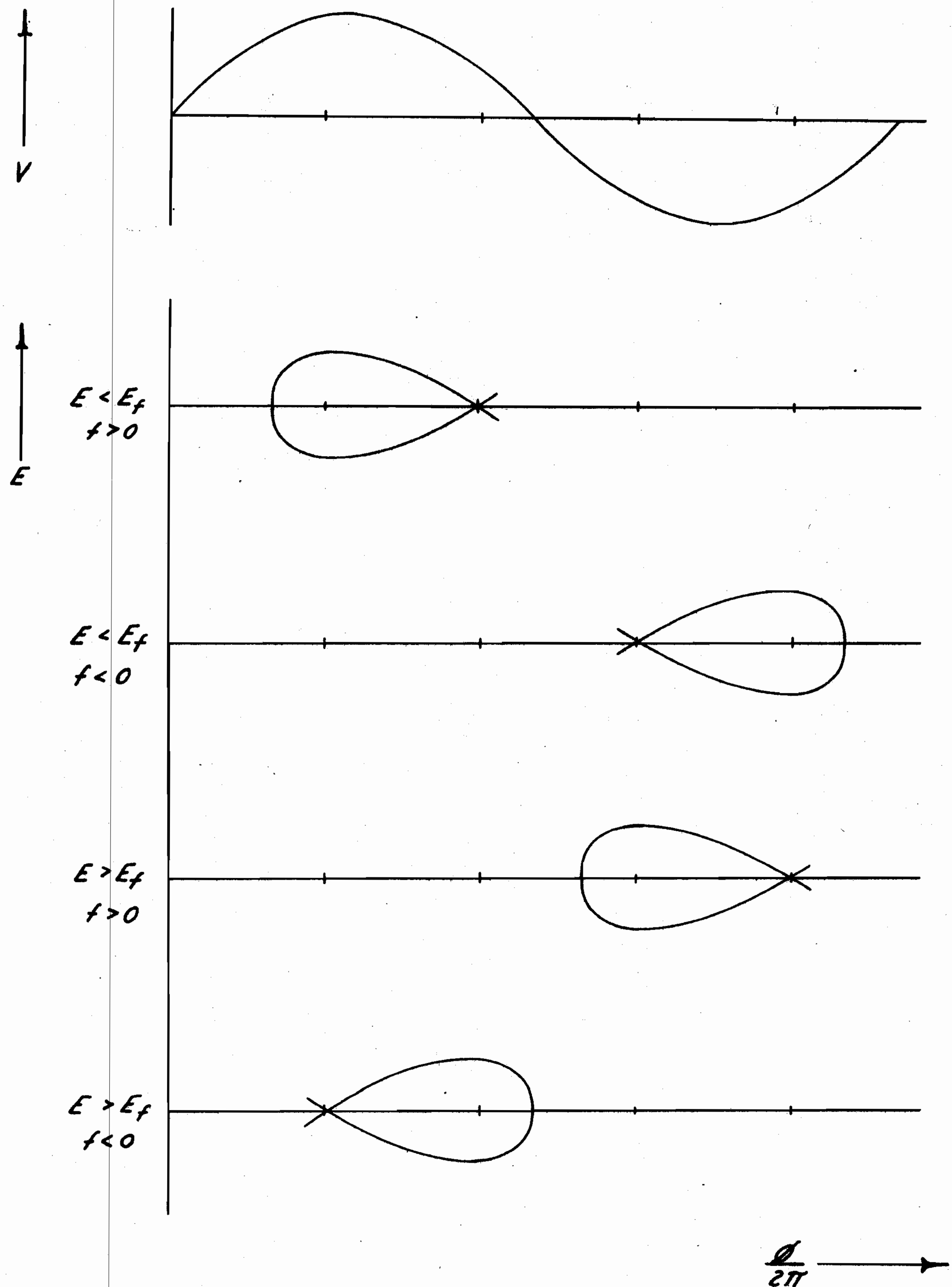


Fig. 3

Phase Plane at Transition - Moving Buckets

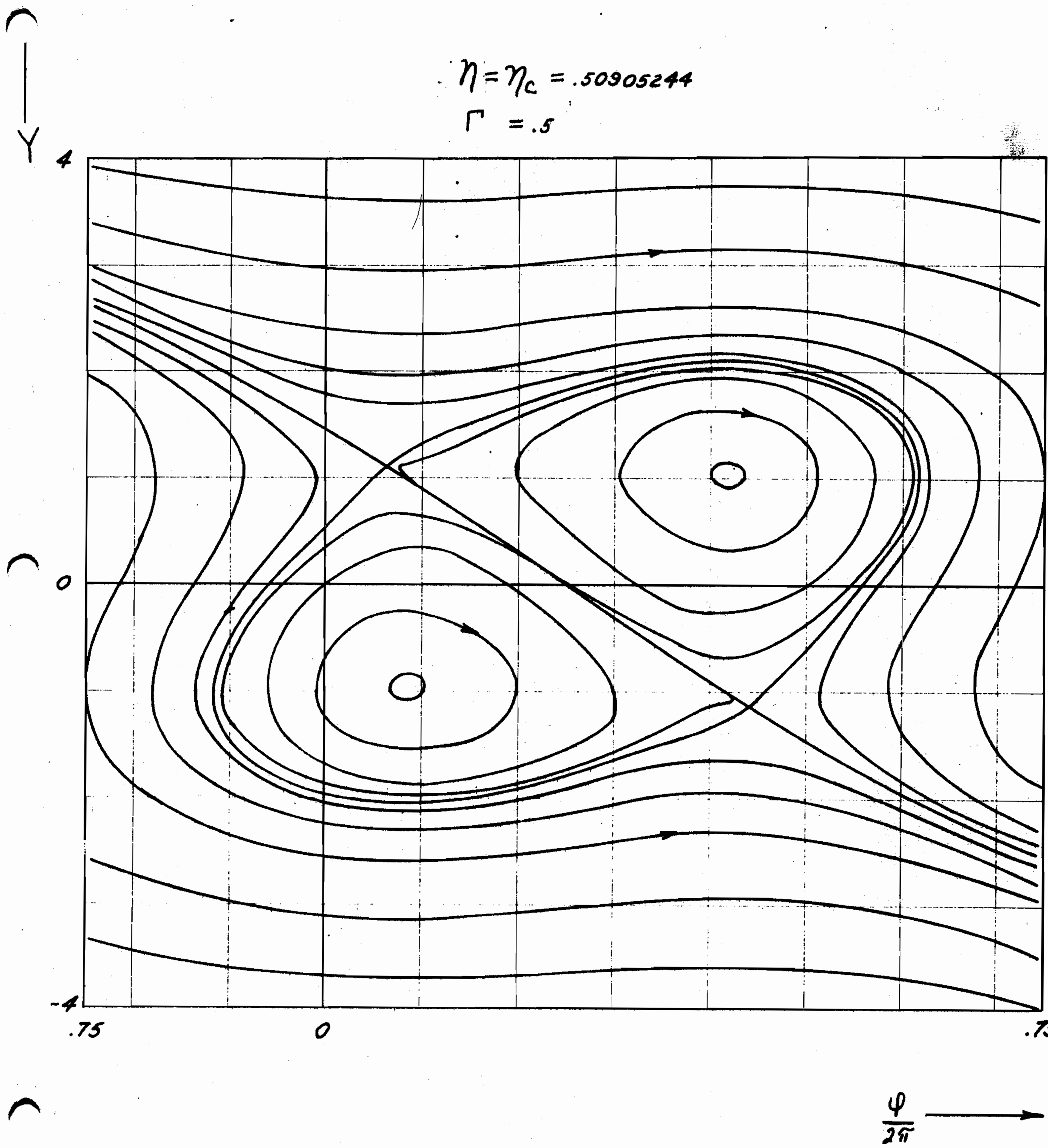


Fig. 4

η_c vs ρ

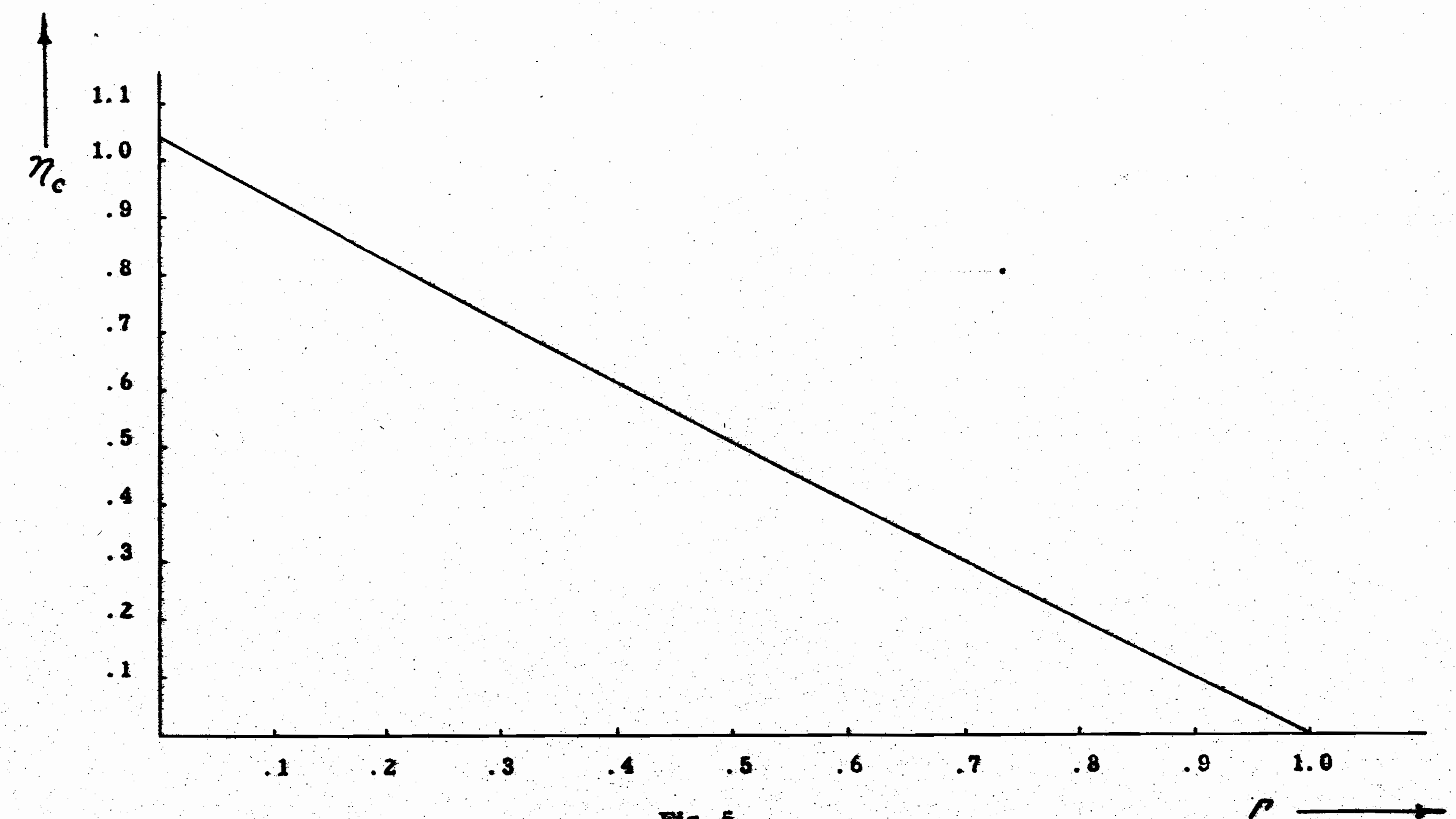


Fig. 5

Phase Plane at Transition - Moving Buckets

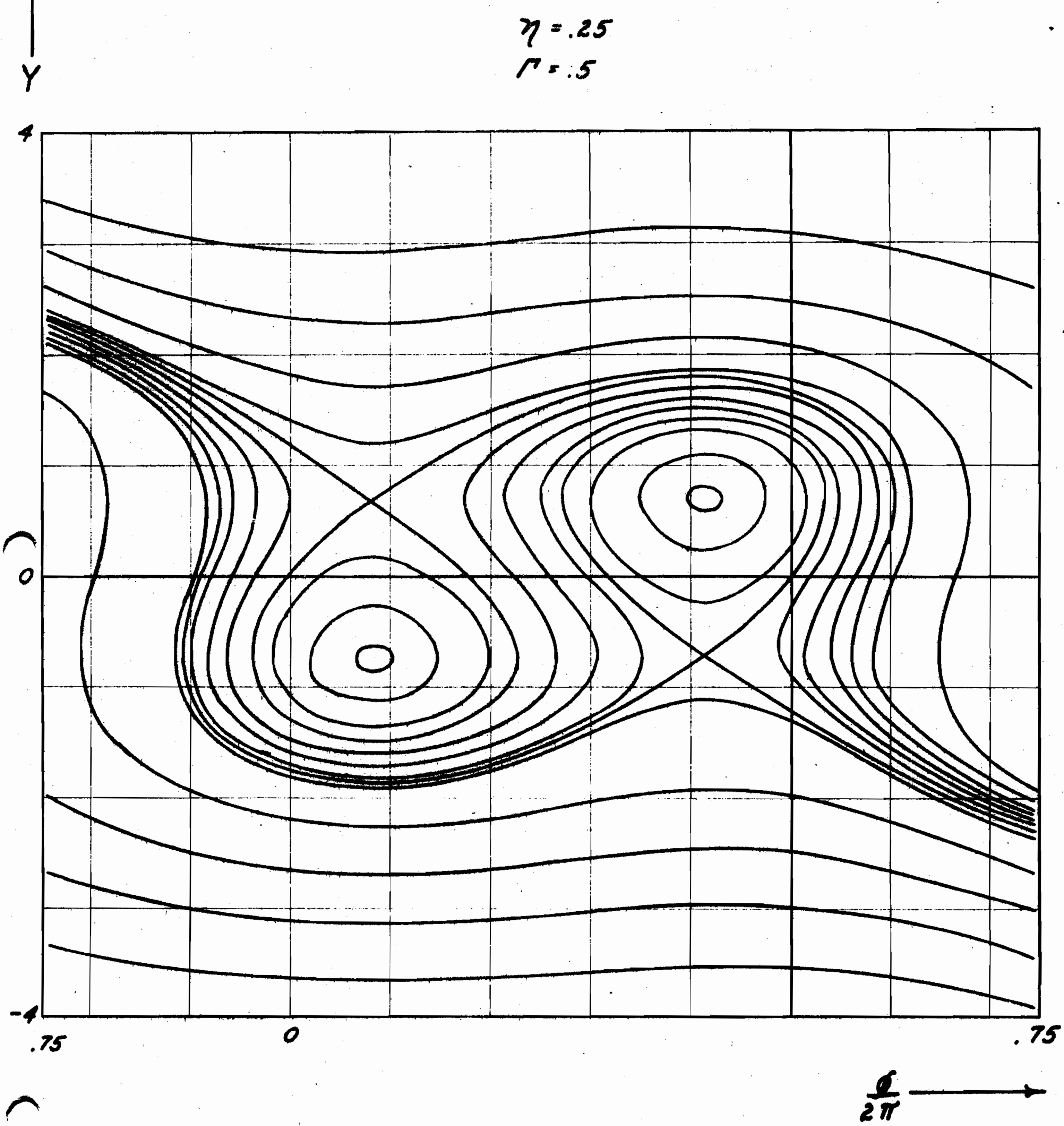


Fig. 6

Phase Plane at Transition - Moving Buckets

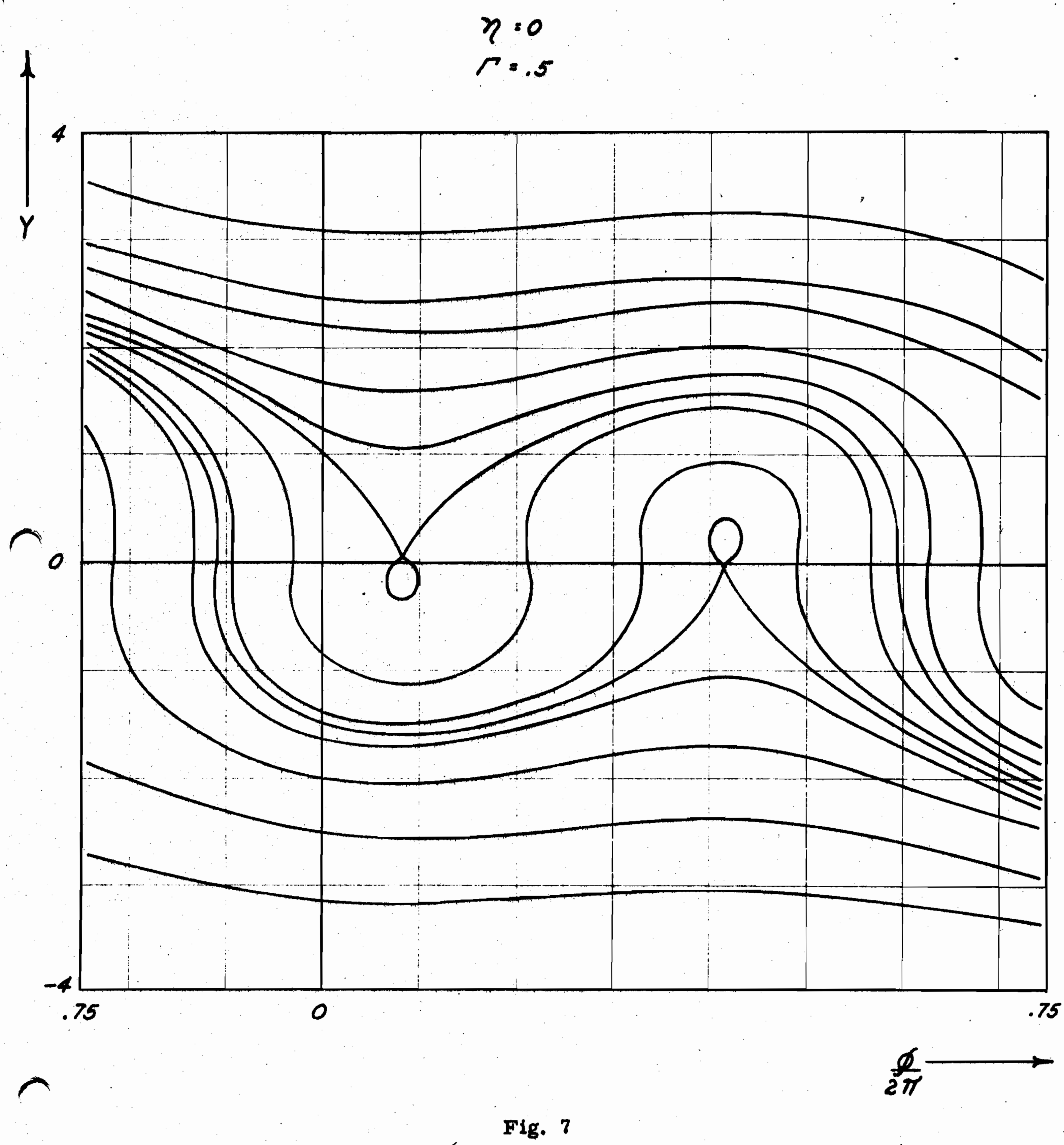


Fig. 7

Phase Plane at Transition

$\eta = -1$

$\Gamma = .5$

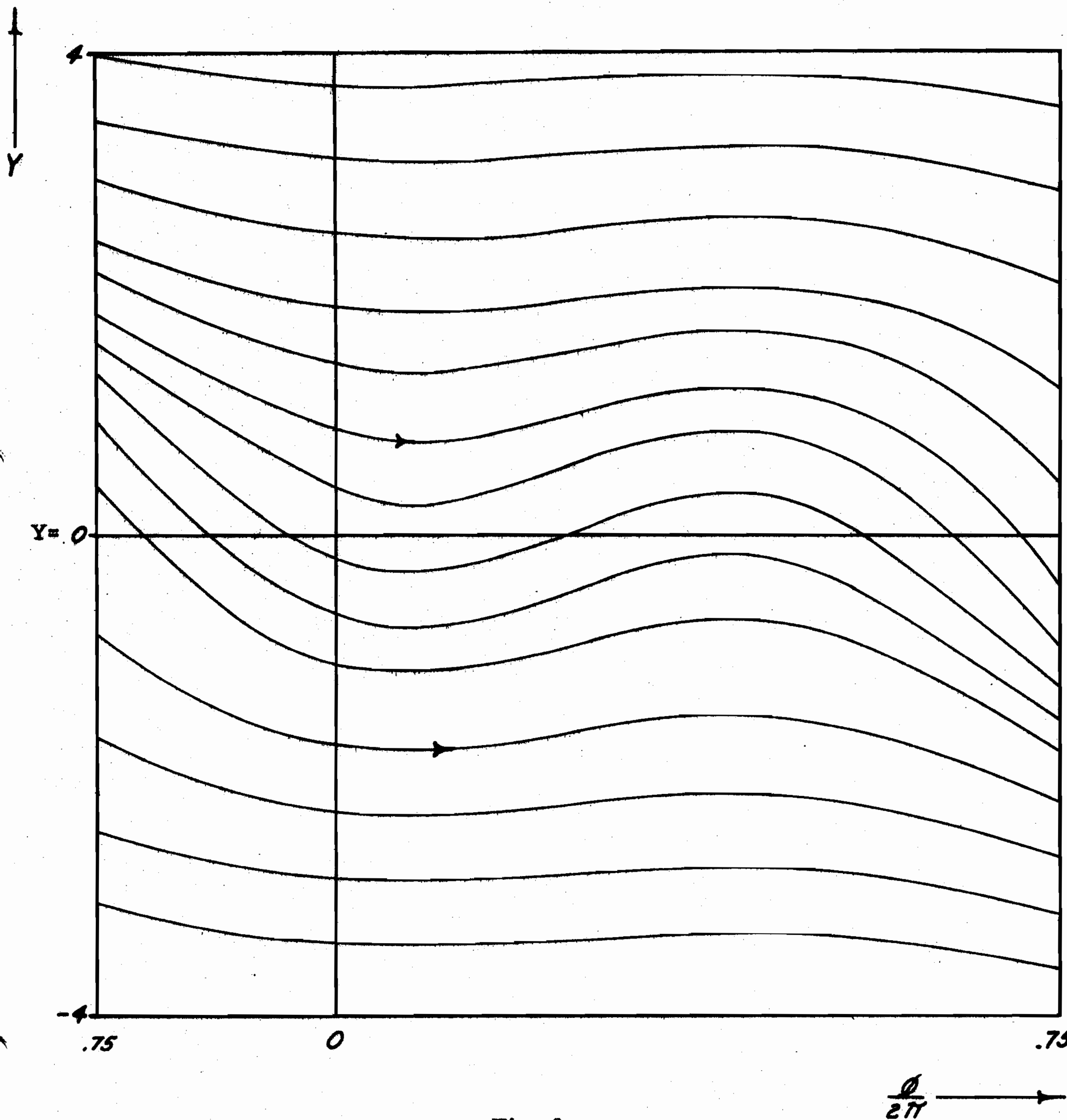


Fig. 8

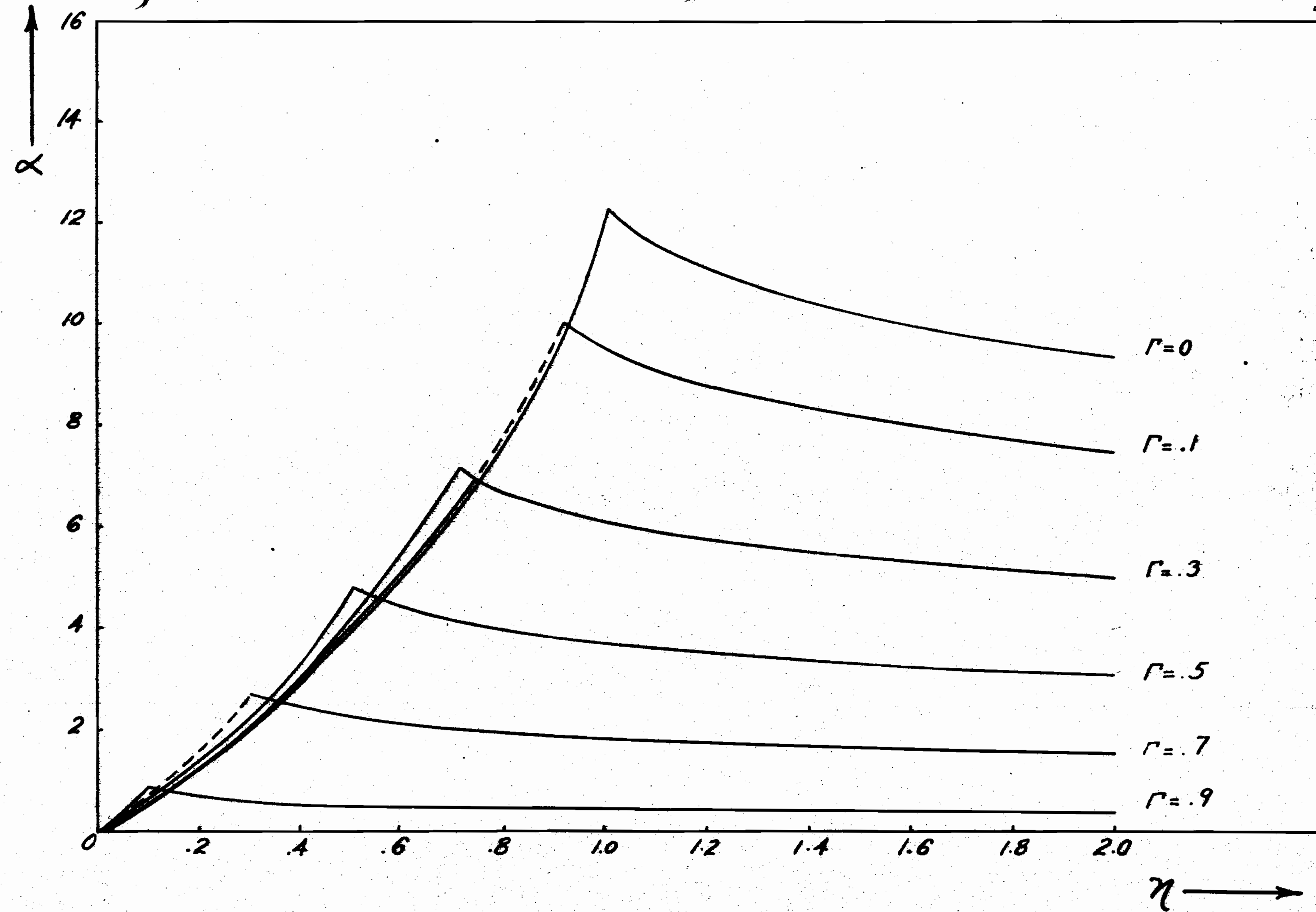
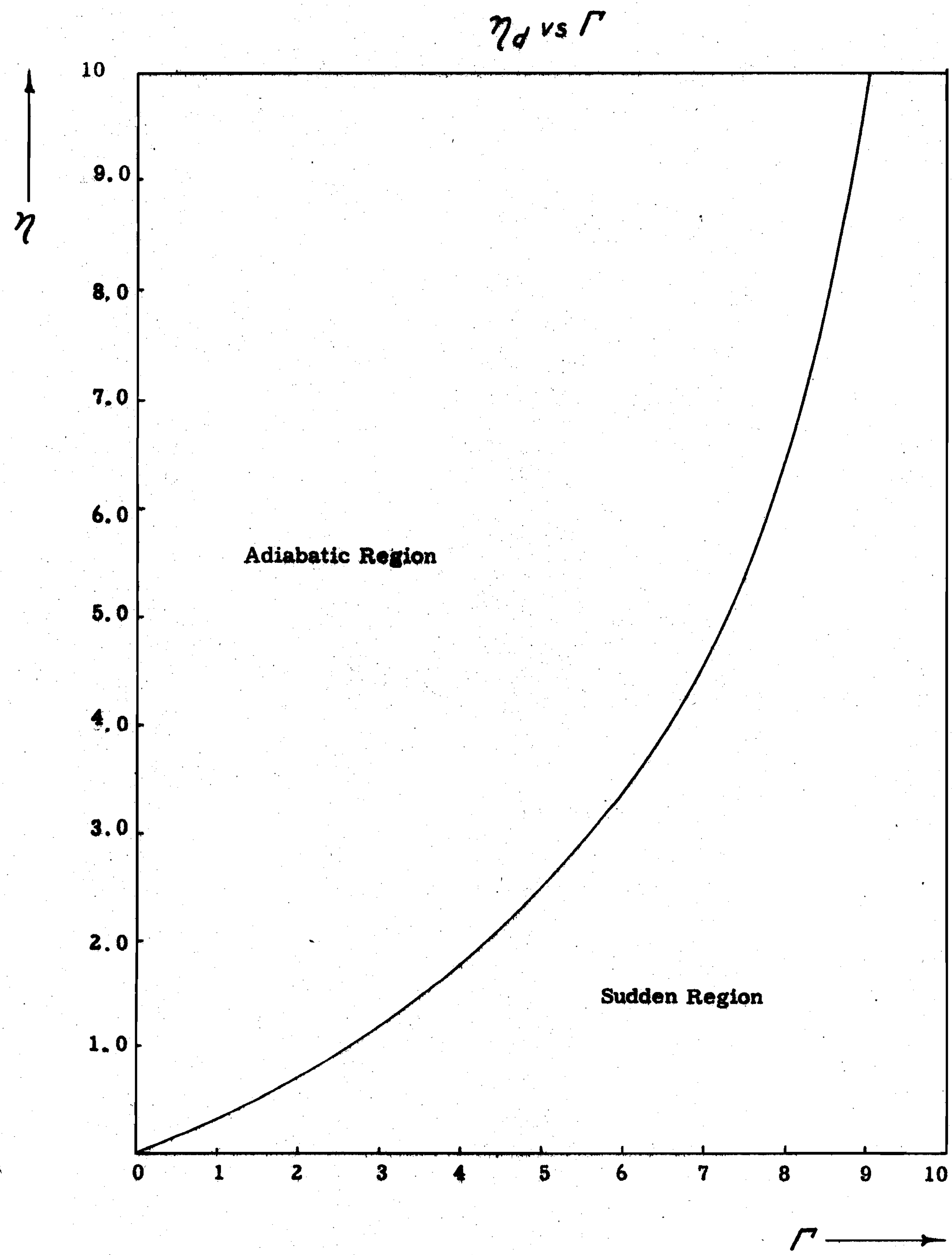


Fig. 9



η_d vs Γ
η
10
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
Adiabatic Region
Sudden Region
0
1
2
3
4
5
6
7
8
9
10
Γ

Fig. 10