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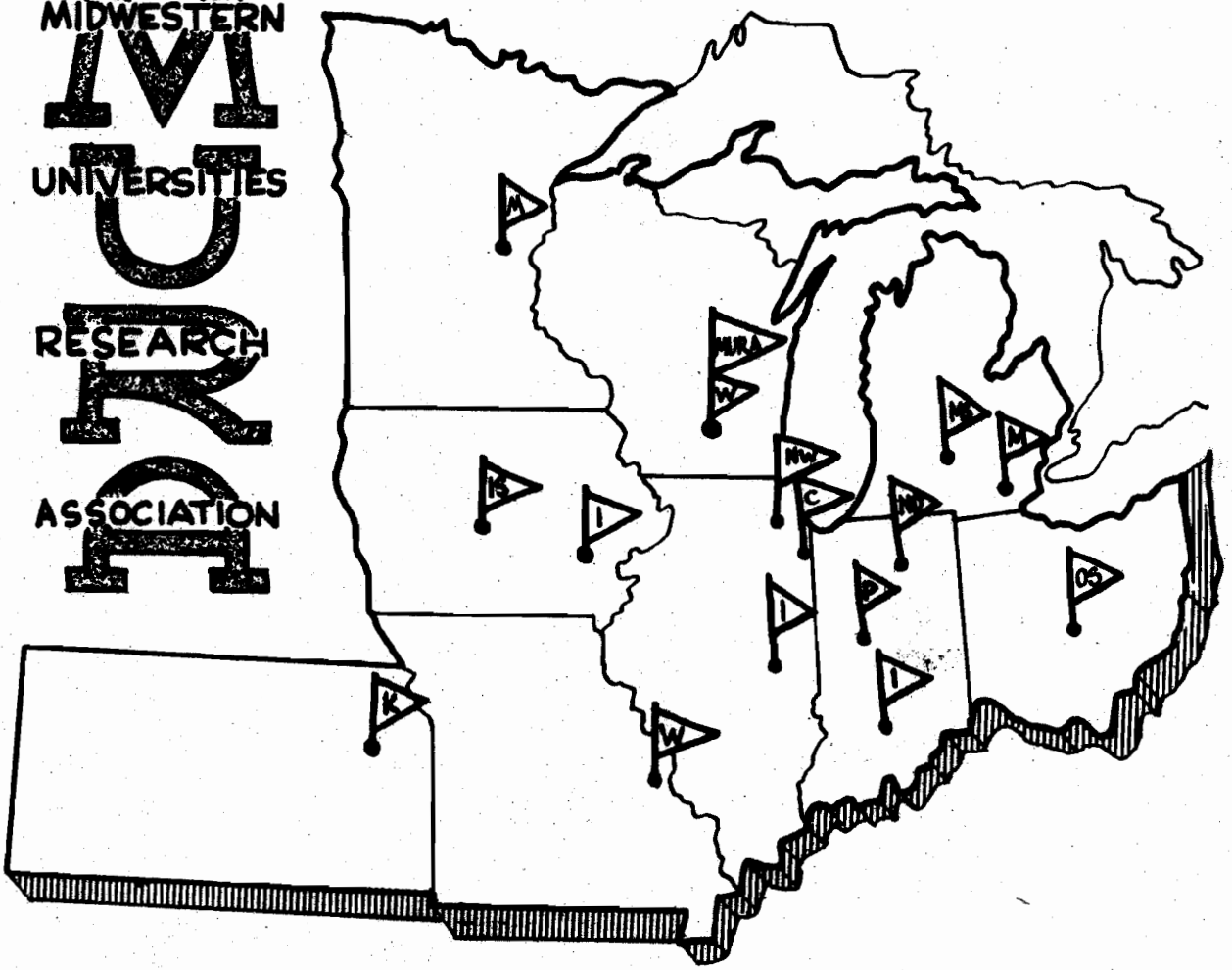

LONGITUDINAL SPACE CHARGE EFFECTS IN FIXED FIELD PARTICLE ACCELERATORS

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LONGITUDINAL SPACE CHARGE EFFECTS IN FIXED FIELD PARTICLE ACCELERATORS

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ABSTRACT

The modification of the single particle theory of particles subject to radio frequency acceleration caused by electrostatic repulsion between particles is calculated. With the aid of an appropriate Hamiltonian the Boltzmann Equation is reduced to a one-dimensional integral equation for the case of a uniform particle density in a limited region of phase space. For geometries typical of large scale accelerators operating below the transition energy this equation is approximated by an algebraic equation, and thus solved analytically. Computations are given of the "longitudinal space charge limit," as well as the effect of space charge on the azimuthal bunching of a beam by an accelerating cavity. Numerical examples of these two effects are given for parameters typical of a 15 Bev symmetric radial sector fixed field accelerator, where the effects are found to be small.

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1. INTRODUCTION

If a large number of particles are accumulated in a particle accelerator, then it is well known that the electrostatic and magnetic forces between the particles can have an important effect on the trajectories of particles. In particular the betatron oscillation frequencies will be modified in either a weak focusing[1] or a strong focusing[2] accelerator, leading to a limit on the number of particles which can be accommodated in the accelerator. This effect will be referred to as the "transverse" space charge limit. It is the purpose of this paper to discuss the "longitudinal" space charge effect, namely the effect of space charge on the azimuthal distribution of particles in an accelerator as differentiated from the modifications space charge imposes upon the radial and axial motion of particles. The azimuthal position of particles becomes important when the particles are under the influence of radio frequency accelerating cavities such that phase stability[3] is an important consideration. The electrostatic repulsion between particles which dominates at low energies, leads in most accelerators[4] to a decrease of phase oscillation frequency which for sufficiently numerous particles will cause a lack of phase stability and hence a longitudinal space charge limit.

In Section II the problem is stated rather generally with the aid of the Hamiltonian formulation[5, 6] of the acceleration process. With the approximation that particle density is uniform in a limited region of phase space, the Boltzmann Equation is reduced to an integral equation for the particle density as a function of azimuth. This equation may be simply interpreted physically, and furthermore may be approximately reduced to an algebraic equation.

In Section III general results for the longitudinal space charge limit are derived, and expressed in convenient graphical form. A numerical example is given which shows that for a particular 15 Bev symmetric radial sector fixed field accelerator[7] the longitudinal space charge limit is unimportant in comparison with the transverse space charge limit. In Section IV the effect of space charge on the bunching of particles in azimuth is investigated for the situation in which the number of particles is far less than the longitudinal space charge limit, but the electrostatic repulsion still affects the azimuthal bunching. By including the transverse space charge limit, a voltage modulation program is derived that allows the maximum accelerating voltage consistent with a given amount of accelerated charge. A numerical example is given for the same accelerator discussed in Section III, and it is again found that for the contemplated parameters the space charge effect is only a small correction.

II. GENERAL FORMULATION

a. Statement of the Problem

We consider an assembly of particles of charge e, rest energy $E_0 = mc^2$, with some spatial density distribution in coordinates R, θ, z. Since we ignore both the coupling between betatron and synchrotron oscillations and the influence of synchrotron oscillations upon radius, only the distribution in θ is relevant.[8] We assume that a stationary density distribution is possible in the presence of space charge forces, and we look for such a distribution. For present purposes a density distribution is stationary in θ if to a sequence of observations repeated at the synchronous particle frequency f_s it appears stationary. This is equivalent to describing it as stationary in the rotating coordinate system of frequency f_s used

in Reference 5. Within the density distribution the particles must individually execute phase oscillations in such a manner as to preserve the distribution in density.

Particle acceleration may result from particle transit across a gap through a time varying potential difference $V \sin 2\pi f t$. Successive gap transits in general occur at different phases φ, except for synchronous particles of frequency $f_s = f/h$, h being an integer which we call the harmonic number. If f is a function of time, synchronous particles must cross the gap at some phase φ_s different from zero such that their synchronous energy gain per revolution $e V \sin \varphi_s$ maintains them at synchronous frequency; other particles oscillate in phase and frequency about the synchronous values.

In order to write a differential equation for rate of change of particle energy, it is convenient to resolve the gap field into a superposition of travelling waves and neglect all components of frequency different from f which contribute nothing on the average; this procedure is simply the replacement of energy jumps by the continuous function with the same integral, and gap transit phase is then replaced by travelling wave phase. At azimuth θ (positive in the sense of particle revolution) the wave phase at $t = 0$ in the laboratory system, and at any time in the rotating coordinate system, is $\varphi = h\theta$, since there are h wavelengths in the circumference. The corresponding instantaneous rate of energy gain from the wave is $e V \sin \varphi$ per revolution.

To include in the equation of motion the effect on each particle of the electric field due to the other particles we write $\mathcal{E}(\theta)$ as the instantaneous space charge electric field; alternatively we may write $\mathcal{E}(\varphi/h)$. If $\mathcal{E}$ is taken as positive in the sense of decreasing θ, it removes energy from a particle at the rate $2\pi R e \mathcal{E}$ per revolution, and the net rate of energy gain per turn becomes

$\frac{dE}{dn} = eV\sin\varphi - 2\pi R e\mathcal{E}$. Following Symon and Sessler[5] we define the new variable

$$w = \int_{E_0}^{E} \frac{dE}{f_p(E)} \tag{1}$$

where $f_p(E)$ is the frequency of a particle of energy E , in terms of which $\frac{dE}{dn} = f_p \frac{dw}{dn} = \frac{dw}{dt} \equiv \dot{w}$, and

$$\dot{w} = eV\sin\varphi - 2\pi R e\mathcal{E} \tag{2}$$

The description of the motion is completed by the equation giving the time rate of change of phase of a non-synchronous particle of frequency f_p differing slightly from f_s , namely by

$$\dot{\varphi} = h\frac{d\theta}{dt} = 2\pi h(f_p - f_s)$$
$$\dot{\varphi} = 2\pi(h f_p - f) \tag{3}$$

The variables w and φ so defined are canonical since these two equations are derivable from the time dependent Hamiltonian

$$H(w,\varphi,t) = -2\pi f w + 2\pi h E(w) + eV\cos\varphi - 2\pi e h U(\varphi/h), \tag{4}$$

which differs from the corresponding single particle Hamiltonian without space charge only in the addition of the term containing the space charge potential $U(\varphi/h)$. (It of course remains to be shown that the assumed stationary distribution actually exists in the presence of this potential.)

It turns out that a canonical transformation to new variables W , φ defined by the equations

$$W = w - w_s$$

$$\varphi = \varphi \, , \tag{5}$$

leads to a separation of the Hamiltonian into time dependent and time independent parts. This is discussed in detail, with certain differences in notation, in Reference 5. One may readily see that the result is, ignoring the arbitrary time functions that have no bearing on the equations of motion,

$$H(W, \varphi, t) = -2\pi f\,(W + w_s) + 2\pi h\, E\,(W + w_s) + eV\cos\varphi + \dot{w}_s\,\varphi - 2\pi e h\, U(\varphi/h).$$

Since W is a periodic function, the monotonic part of the increase in w with acceleration being contained in the w_s defined for the synchronous particle, we expand $E(W + w_s)$ about w_s, and obtain

$$E(W + w_s) = E_s + \left(\frac{dE}{dw}\right)_s W + \tfrac{1}{2}\left(\frac{d^2E}{dw^2}\right)_s W^2 + \cdots$$

$$= E_s + f_s W + \tfrac{1}{2}\left(f\,\frac{df}{dE}\right)_s W^2 + \cdots \, .$$

The first term E_s and the term $2\pi f w_s$ do not contain W and may be neglected, while the $2\pi f W$ and the $-2\pi h f_s W$ cancel, leaving

$$H(W, \varphi) = \pi h \left(f\,\frac{df}{dE}\right)_s W^2 + eV\cos\varphi + \dot{w}_s\,\varphi - 2\pi e h\, U(\varphi/h), \tag{6}$$

as the Hamiltonian for a single particle subject to space charge forces.

This is time independent, irrespective of the rate of energy gain defined by $\dot{w}_s$, if the coefficients $\left(f\,\frac{df}{dE}\right)_s$, V, and $\dot{w}_s$ are constant and the function $U(\varphi/h)$ remains unaltered during acceleration. This last is obviously impossible

with given total charge unless the radius remains constant; but in practical cases, especially if the magnetic field index k is large, R may be nearly enough constant. Even if all coefficients and U vary, the phase motion may be derived from this Hamiltonian if the variations are not too rapid.

The potential $U(\theta)$ involves an integral over the charge distribution; it may be written

$$U(\theta) = \int K(\theta, \theta') \lambda(\theta') d\theta', \tag{7}$$

in which λ is the charge density per unit length, written as a function of azimuth. The quantity K is a dimensionless kernel, which must take into account the influence of such important effects as electric shielding by the accelerator vacuum tank. Since λ is itself dependent upon the phase motion of the remaining particles, we look for a self-consistent solution.

b. Boltzmann Equation and Integral Equation

We have derived a Hamiltonian which defines time independent trajectories in $W-\varphi$ space if the potential function U is time independent, i.e., if the assumed stationary charge distribution exists. Let the corresponding density distribution in phase be $\psi(W, \varphi)$, which if stationary satisfies the reduced Boltzmann Equation:

$$\frac{\partial \psi}{\partial W} \dot{W} + \frac{\partial \psi}{\partial \varphi} \dot{\varphi} = 0 . \tag{8}$$

We may ask, for example, whether this equation is satisfied by the distribution ψ = const within the region bounded by $W = W_b(\varphi)$ and zero outside. (Actual density is determined by the particle injector since the system is Liouvillian in

W-φ space and density is consequently a constant of the motion. Uniform density is probably not too far from correct for most contemplated injectors.) This case may be represented by

$$\rho(W, \varphi) = e\sigma S(|W| - |W_b|), \tag{9}$$

in which $e\sigma$ is a constant equal to the charge density in W-φ space, and S is a step function which is unity for negative argument and zero otherwise. Use of this in Eq. (8) gives

$$\delta(W - W_b)\dot{W} - \delta(W - W_b)\frac{dW_b}{d\varphi}\dot{\varphi} = 0$$

from which, since W and φ are canonical variables,

$$\delta(W - W_b)\left[-\frac{\partial H}{\partial \varphi} - \frac{\partial H}{\partial W}\frac{dW_b}{d\varphi}\right] = 0 . \tag{10}$$

This equation is identically satisfied for $W \neq W_b$; and for $W = W_b$ we must equate the second factor to zero. It integrates immediately to give

$$H(W_b, \varphi) = \text{const} \tag{11}$$

as the equation for the boundary curve, which is not trivial since it is only valid for the boundary curve W_b and not for all values of W .[9]

Although our subsequent discussion will be limited to the case of uniform charge density within the phase boundary, it is interesting to note in passing that Eq. (8) may be written

$$\text{grad}\ \rho \cdot \mathbf{v} = 0 \tag{12}$$

in which $\mathbf{v}$ is the phase velocity vector directed along a phase trajectory. It

follows that any density function ψ with zero gradient along trajectories and arbitrary variation perpendicular to trajectories leads to the required stationary solution.

For the uniform distribution we have, since we are neglecting the influence of phase oscillations upon coordinates other than θ and the equivalent φ, the relation

$$\lambda(\varphi)\, d\ell = \lambda(\varphi)\,(R/h)\, d\varphi = 2\sigma e\, |W_b|\, d\varphi, \tag{13}$$

in which we have used the fact that $W = W(\varphi)$ is symmetric in W. This is valid unless E is near the transition energy. The boundary curve is therefore described by the integral equation obtained by putting the expression (6) for the Hamiltonian and the relation (13) together with the potential integral (7) into Eq. (11), namely by

$$\pi h \left(f \frac{df}{dE}\right)_s W_b^2 - \frac{4\pi e^2 \sigma h^2}{R} \int |W_b|\, K(\theta, \theta')\, d\theta' + e V_{\omega} \varphi + i v_s \varphi = \text{const.} \tag{14}$$

This equation is the condition for a stationary density distribution given an initially uniform distribution within the boundary. The potential determined as a function of φ by the values of W_b obtained from this equation may then be used in the Hamiltonian (6) to derive the self-consistent equations of motion for the particles in the many particle system under consideration. The integral with values of W_b substituted becomes a function of φ independent of the particular trajectory, and the W^2 term alone determines $\dot{\varphi}$ along any trajectory inside of the boundary.

Since in the present paper we shall be concerned solely with the total charge within the boundary we shall drop the subscript, it being understood hereafter that W always refers to the value on the boundary curve.

c. Reduction to Algebraic Equation

We may observe that in a large accelerator $K(\theta, \theta')$ will be essentially zero for $|\theta - \theta'| \to \infty$ since the accelerator vacuum tank shields the contribution of charges separated in angle by

$$|\theta' - \theta| \gtrsim G/R$$

where G is the vertical gap of the vacuum tank. Except near the ends of a bunch of particles localized in azimuth by the accelerating cavity, the variation of charge density within an angle G/R is negligible. Since the ends are of order G/R and also negligible compared to the extent of a bunch, for anything except operation on a harmonic of order $h > \pi R/G$ we may neglect the ends and consequently the dependence of $W(\theta')$ on θ' in the integral in Eq. (14). Formally:

$$\int K(\theta,\theta')\,|W(\theta')|\,d\theta' = |W(\theta)| \int K(\theta,\theta')\,d\theta' + \left|\frac{dW(\theta)}{d\theta}\right| \int K(\theta,\theta')\,|\theta'-\theta|\,d\theta' + \cdots \tag{15}$$

and we keep only the first term in the expansion.

Now the integral of the kernel in the first term on the RHS of Eq. (15) is simply proportional to the potential at an angle θ due to a uniform tube of charge. We may ignore the curvature of the tube since the turning is negligible in an angle of the order of G/R, and replace the problem by the linear two-dimensional

problem of finding the potential in the center of a uniform tube of charge of infinite extent midway between two parallel conducting grounded plates. By replacing the infinite set of image tubes with line charges this is solved in Appendix I where it is shown that

$$\int K(\theta,\theta')\,d\theta' \simeq \left[1 + 2\log_e \tfrac{2}{\pi}\,\tfrac{G}{a}\right] = g \tag{16}$$

where a is the radius of the tube of charge.

We now obtain in place of the integral equation (Eq. (14), the algebraic equation:

$$\pi h \left(f\frac{df}{dE}\right)_s W^2 + \dot{w}_s \varphi + eV\cos\varphi - \frac{4\pi e^2 \sigma h^2 g}{R}|W| = \text{const.} \tag{17}$$

This may be written in the form

$$W^2(\varphi) + a|W| + b\cos\varphi + c\varphi = \text{const.}, \tag{18}$$

where:

$$a = \frac{4\pi e^2 \sigma g h}{R\pi\left(f\frac{df}{dE}\right)_s}$$

$$b = \frac{eV}{\pi h\left(f\frac{df}{dE}\right)_s}$$

$$c = \frac{\dot{w}_s}{\pi h\left(f\frac{df}{dE}\right)_s} \tag{19}$$

A relativistic treatment would yield this same result with the quantity R in Eq. (19) replaced with $\gamma^2 R$, where $\gamma = E/E_0$.

III. LONGITUDINAL SPACE CHARGE LIMIT

In this section we first show that the approximate linear relation between potential and charge density leads very simply to an upper limit on the total charge. We then return to the more general formulation, still restricting ourselves to a range of parameters of most interest (e.g. to the non-relativistic limit where longitudinal space charge effects are largest), and develop relations for the space charge limit as a function of accelerator dimensions, rate of frequency modulation, etc.

The confinement of charge in a phase-stable region involves the balance of the electrostatic repulsion between particles and the bunching effect of the accelerating gap. Evidently the maximum charge is contained when there is neither phase motion nor frequency modulation, i.e., when the particles are all moving synchronously at constant frequency. The problem is then merely one of static balance, the condition for which is Eq. (2), with energy gain per turn set equal to zero. This, together with the value of $\mathcal{E}$ derived from linear charge density, gives

$$eV\sin\varphi + 2\pi R e g \frac{d\lambda}{dl} = 0$$

$$V\sin\varphi + 2\pi g h \frac{d\lambda}{d\varphi} = 0,$$

which immediately integrates to

$$V(\cos\varphi + 1) - 2\pi g h \lambda = 0 \tag{20}$$

in which the integration constant is evaluated to make $\lambda = 0$ at $\varphi = \pm \pi$. (Strictly speaking, this charge is not bound in a phase-stable region, and any arbitrary constant density could be added; but we shall see later from the general formulation that the distribution specified is obtained for phase-stable charge in the limit of phase motion approaching zero.) Total charge in one bunch is then merely the integral of λ, i.e.,

$$q = \frac{VR}{g h^2} \tag{21}$$

If at harmonic number h all phase-stable regions are filled, the maximum total charge in the machine is $Q = hq$.

To develop the general case, we first note that in the non-relativistic limit and in the approximation that variations of R are negligible, the coefficient of W^2 in Eq. (17) is

$$\pi h \left(f \frac{df}{dE} \right)_s = \frac{h}{4\pi R^2 m}$$

and the coefficients in Eq. (18) consequently have the values;

$$a = 16 \pi^2 R m e^2 \sigma h g$$

$$b = 4\pi R^2 m e V/h$$

$$c = 4\pi R^2 m \dot{w}_s/h .$$

If we introduce the new variable $y = KW$, with

$$K^2 = \frac{h}{2\pi R^2 m e V} , \tag{22}$$

Eq. (18) becomes

$$\frac{y^2}{2} + \Lambda |y| + \cos\varphi + \Gamma\varphi = \text{const.} \tag{23}$$

in which

$$\Lambda = \left[32\pi^3 \hbar^3 e^3 m/V\right]^{1/2} \sigma g$$

$$\Gamma = \frac{\dot{w}_s}{eV} = \sin\varphi_s \quad . \tag{24}$$

This form of equation is convenient for computation since the whole effect of space charge is contained in the one parameter Λ, while the remaining parameter Γ contains the effect of frequency modulation. It must however be remembered that phase density σ refers to $W-\varphi$ space; and in the transformation to y - variables phase density is altered by the factor K^{-1}, which is the function of operating parameters defined by Eq. (22). For a given Γ and <u>independent</u> of Λ, the stable regions of phase space will have maximum extent in φ for appropriate choice of the constant on the R. H. S. of Eq. (23). Let the limits of φ be called φ_1 and φ_2. The maximum area A, in $y-\varphi$ space encompassed by the stable region is given by solving Eq. (23) for $y(\varphi)$ and then integrating:

$$A(\Lambda, \Gamma) = 2\int_{\varphi_1}^{\varphi_2} y \, d\varphi \quad . \tag{25}$$

The total charge in the accelerator Q is then given by:

$$Q = \frac{e h \sigma}{K} A(\Lambda, \Gamma) = \frac{e h \sigma}{K} A(\Lambda, \Gamma), \tag{26}$$

since area in $W-\varphi$ space is related to area in $y-\varphi$ space by the factor of K^{-1}.

In Fig. 1 we have plotted the quantity $A(\Lambda, \Gamma)$ as a function of Λ for various Γ. To use these curves one obtains σ from the known properties of the injector; then calculates Λ, K, and Γ knowing the properties of the accelerator. The value of Q given by Eq. (26) and Fig. 1 is then the longitudinal space charge limit.

In Figs. 2 and 3 we have plotted some of the parameters characterizing the maximum stable phase region as a function of Λ and Γ. The quantities φ_1 and φ_2 are the limits of the stable region and independent of Λ, while y_{Max} is the maximum value of y which occurs when $\varphi = \varphi_s = \sin^{-1}\Gamma$. The energy spread of the particles is given then by:

$$\Delta E = \frac{f_s}{K} y_{Max}(\Lambda, \Gamma)$$

$$\Delta E = \frac{1}{2\pi R K} \left[\frac{2\ K.E.}{m}\right]^{1/2} y_{Max}(\Lambda, \Gamma). \tag{27}$$

In the case that $\Lambda \gg 1$ we may obtain an approximate solution to Eq. (23) without recourse to numerical integration, since the term in y^2 becomes negligible compared to the linear term in y, and Eq. (23) becomes:

$$\Lambda |y| + \cos\varphi + \Gamma \varphi = D \tag{28}$$

The constant D is determined by maximizing the azimuthal spread, i. e., by solving

$$\Gamma - \sin\varphi = 0, \tag{29}$$

to obtain $\varphi_s = \sin^{-1}\Gamma$, $\varphi_1 = \pi - \varphi_s$. Thus

$$D = \cos(\pi - \varphi_s) + \Gamma(\pi - \varphi_s), \tag{30}$$

and

$$y = \frac{-\cos\varphi_s + \Gamma(\pi - \varphi_s) - \Gamma\varphi - \cos\varphi}{\Lambda}. \tag{31}$$

Now:

$$A(\Lambda, \Gamma) = \frac{2}{\Lambda}\int_{\varphi_1}^{\varphi_2}\left[-\cos\varphi_s + \Gamma(\pi - \varphi_s) - \Gamma\varphi - \cos\varphi\right] d\varphi$$

$$A(\Lambda, \Gamma) = \frac{2}{\Lambda}\left[\left(\Gamma(\pi - \varphi_s) - \cos\varphi_s\right)(\varphi_2 - \varphi_1) - \frac{\Gamma}{2}\left(\varphi_2^2 - \varphi_1^2\right) + \right.$$
$$\left. + \sin\varphi_1 - \sin\varphi_2\right] \tag{32}$$

This yields, using Eq. (26),

$$Q = \frac{e v h^2}{K \Lambda}\left[\left(\Gamma(\pi-\varphi_s)-\cos\varphi_s\right)(\varphi_2-\varphi_1)-\frac{\Gamma}{2}\left(\varphi_2^2-\varphi_1^2\right)+ \right.$$
$$\left. +\sin\varphi_1-\sin\varphi_2\right] \tag{33}$$

$$Q = \frac{VR}{2\pi g h}\left[\left(\Gamma(\pi-\varphi_s)-\cos\varphi_s\right)(\varphi_2-\varphi_1)-\frac{\Gamma}{2}\left(\varphi_2^2-\varphi_1^2\right)+\sin\varphi_1-\sin\varphi_2\right]$$

$$Q = \frac{VR}{2\pi g h}\, B(\Gamma) \qquad (\Lambda \gg 1) \tag{34}$$

From Fig. 2 one can obtain φ_1, φ_2, φ_s for any particular Γ. For convenience we have plotted the function $B(\Gamma)$ in Fig. 4. This then serves to augment Fig. 1 for large values of Λ. It is clear that

$$y_{Max}(\Lambda,\Gamma) = \frac{1}{\Lambda}\left[-\cos\varphi_s+\Gamma(\pi-\varphi_s)-\Gamma\varphi_s-\cos\varphi_s\right] \tag{35}$$

$$y_{Max}(\Lambda,\Gamma) = \frac{1}{\Lambda}\, C(\Gamma), \qquad (\Lambda \gg 1) \tag{36}$$

$$\Delta E = \frac{V}{8\pi^2 g h^2 \sigma e}\left(\frac{2\,K.E.}{m}\right)^{1/2} C(\Gamma), \qquad (\Lambda \gg 1) \tag{37}$$

and we have plotted $C(\Gamma)$ also in Fig. 4, thus augmenting Fig. 3 for large values of Λ.

If the cavity is unmodulated, $\Gamma = 0$, $\varphi_s = \pi$, $\varphi_1 = 0$, $\varphi_2 = 2\pi$ and Eq. (34) becomes:

$$q \simeq \frac{VR}{g h^2} \qquad (\Lambda \gg 1,\ \Gamma = 0) \tag{38}$$

in agreement with the static computation given at the beginning of this section. For a typical radial sector symmetric fixed field accelerator, $R = 10^4$ cm, $h = 10$, $V = 50$ kev = 113 esu, $G = 30$ cm, $a = 5$ cm; and Eq. (38) yields $Q = 3.52 \times 10^4$ esu, which corresponds to 7.3×10^{13} protons. Of course for $\Gamma \neq 0$ and Λ closer to zero this number will be reduced as can be seen from Fig. 1. It should be noted that in this case the transverse space charge limit is more severe than the above limit. It should be noted also that the numerical computation for any likely values of parameters (readily performed with the aid of the curves here given) will in general lead to a space charge limit not more than an order of magnitude smaller than the static upper limit of Eq. (38), assuming of course that the entire phase stable region or bucket is full of particles.

IV. VOLTAGE MODULATION PROGRAM

In this section a program for the accelerating cavity voltage is derived, which allows the maximum accelerating voltage consistent with a given amount of accelerated charge. For the fixed field accelerator discussed in the previous section in order to accelerate the charge requisite for colliding beams, the initial accelerating voltage turns out to be rather low, but as will be shown here can rapidly be amplitude modulated so that in fact the effects of gas scattering are not too serious.

Transverse space charge effects limit[1] the density of charge per unit length in a beam to less than

$$\rho = \frac{m}{2\pi e} \frac{\gamma^3 \beta^2 c^2}{R^2} \Delta(\nu^2) \tag{39}$$

where $\beta = v/c$, $\gamma = E/E_0$, and $\Delta(\nu^2)$ is the permissible change in the number of betatron wavelengths ν in one circumference of the accelerator. Taking $\Delta\nu = 1/4$, and evaluating Eq. (39) in the non-relativistic limit, we obtain:

$$\rho = \frac{m}{4\pi e} \frac{v^2 \nu}{R^2} = \frac{(K.E.)\nu}{2\pi e R^2} \tag{40}$$

For the cases we wish to study the accelerating voltage causes the accelerated beam to bunch into a very small phase angle $\Delta\varphi$. This, in fact, is the reason why the accelerating voltage cannot be too large, since the azimuthal bunching of the beam causes the local density to exceed the restriction of Eq. (40). Longitudinal space charge effects of course allow a larger voltage than would be estimated ignoring their helpful effects. The total charge in the beam is given by

$$Q = [\pi a^2 \rho R] \Delta\varphi \mu \tag{41}$$

where $\Delta\varphi$ is the spread in phase of the charge in one bunch, a is the radius of the beam and μ is a parameter expressing the distribution of charge in a bunch. If longitudinal space charge effects dominate, $\mu = 2/3$, whereas if the effects of longitudinal space charge can be neglected, $\mu = \pi/4$.

The radius of the beam a will damp as the particles are accelerated so that if we let a_0 be the radius at injection and $(K.E.)_0$ be the kinetic energy at injection, Eq. (41) becomes

$$\Delta\varphi = \frac{2QeR}{\mu\nu a_0^2 (K.E.)_0^{1/2} (K.E.)^{1/2}} . \tag{42}$$

The phase density $e\sigma$ is determined by the injector and injection mechanism and may be related to the beam energy spread before an accelerating voltage is turned on since σ is a constant as the voltage is varied. Let ΔE be the energy spread of the beam initially, and define P by $\Delta E = P(K.E.)_0$. Then $\Delta W = \Delta E / f_0$, where f_0 is the frequency of the particle at injection, and

$$Q = 2\pi e\sigma(\Delta W) \tag{43}$$

or

$$\sigma = \frac{Q f_0}{2\pi e P (K.E.)_0}$$

$$f_0 = \left[\frac{(K.E.)_0}{2\pi^2 R^2 m}\right]^{1/2} \tag{44}$$

Now starting with Eq. (23) we may evaluate the constant on the R. H. S. in terms of the phase spread $\Delta\varphi$, and then solve for the voltage V after having related y_{Max} to ρ. This relation between y_{Max} and ρ is clearly:

$$\pi a^2 \rho R = \frac{2e\sigma y_{Max}}{K} \tag{45}$$

which using Eq. (44) becomes:

$$y_{Max} = \frac{\pi}{2} \frac{P K V a_0^2 (K.E.)^{1/2} (K.E.)_0^{3/2}}{e Q f_0 R} \tag{46}$$

whereas Eq. (23) becomes:

$$\tfrac{1}{2} y^2 + \Lambda |y| + \cos\varphi + \Gamma y = \cos\left(\varphi_s + \frac{\Delta\varphi}{2}\right) + \Gamma\left(\varphi_s + \frac{\Delta\varphi}{2}\right). \tag{47}$$

Evaluating Eq. (47) at $\varphi = \varphi_s$, $y = y_{Max}$ we obtain

$$\tfrac{1}{2} y_{Max}^2 + \Lambda |y_{Max}| = -\cos\varphi_s \frac{(\Delta\varphi)^2}{8}, \tag{48}$$

which can now be solved for V after substituting Eqs. (46), (42), (22), (24), and (44):

$$V = \left(\frac{\mu^2 \pi^3}{4}\right) \frac{h P^2 a_0^8 \nu^4 (K.E.)_0^3 (K.E.)^2}{e^5 Q^4 R^4 (-\cos\varphi_s)} + \frac{2\pi\mu^2 g h \nu^3 a_0^6 (K.E.)_0^{3/2} (K.E.)^{3/2}}{e^3 Q^2 R^4 (-\cos\varphi_s)} \tag{49}$$

As an example we evaluate this for $R = 10^4$ cm, $h = 10$, $G = 30$cm, $a_0 = 5$ cm, $\nu = 4$, $\varphi_s = \pi$, $(K.E.)_0 = K.E. = 50$ Mev, $P = .001$, $Q = 160$ esu, and finds that the second term which is the effect of longitudinal space charge is negligible, while $V = 2$ kilovolts. Because of the dependence

of V on the kinetic energy, it is possible to increase V rapidly as a function of time, and thus avoid serious loss of beam due to gas scattering.

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APPENDIX

We wish to find the potential at the center of a uniform tube of charge of radius a, placed midway between and parallel to two parallel conducting planes separated by a distance G, as indicated in Fig. 5.

The problem of a similarly oriented line charge has been treated by Smyth,[10] who obtains for the potential (in electrostatic units):

$$\tanh \frac{U}{2\xi} = \frac{\sin \pi y/G}{\cosh \pi x/G}, \tag{A1}$$

where ξ is the charge per unit length of the line charge. If we let $y = G/2 + y'$ we obtain

$$\tanh \frac{U}{2\xi} = \frac{\cos \pi y'/G}{\cosh \pi x/G}. \tag{A2}$$

We wish now to study the potential in the neighborhood of the tube of charge, so let us expand the above expression for small values of $r = \sqrt{x^2+y'^2}$:

$$\frac{U}{2\xi} = \tanh^{-1}\left[1 - \frac{\pi^2 y'^2}{2G^2} - \frac{\pi^2 x^2}{2G^2} + \cdots\right]$$

$$\frac{U}{2\xi} = \frac{1}{2}\left[\log\left(2 - \frac{\pi^2}{2G^2}(x^2+y'^2)\right) - \log\left(\frac{\pi^2}{2G^2}(x^2+y'^2)\right)\right]$$

$$U \simeq \xi \log \frac{4G^2}{\pi r^2}$$

$$U \simeq -2\xi \log r + \xi \log \frac{4G^2}{\pi^2}. \tag{A3}$$

This diverges as r goes to zero, which is because we have replaced the tube of charge of radius a, with a line charge. For a tube of charge in free space the potential is clearly:

$$U = -2\,\xi \log r \quad , \; r \geq a \tag{A4}$$

$$U = -\xi\left[\frac{r^2}{a^2} + 2\log a - 1\right], \; r \leq a \tag{A5}$$

Thus we can identify the effect of shielding in Eq. (A3), and replace the singular first term with Eq. (A5) to obtain

$$U = -\xi\left[\frac{r^2}{a^2} + 2\log a - 1\right] + \xi \log\frac{4G^2}{\pi^2} \quad , \; r \leq a \tag{A6}$$

Thus for $r = 0$,

$$U(0) = \xi\left[1 + 2\log\frac{2G}{\pi a}\right]. \tag{A7}$$

FOOTNOTES

1. D. W. Kerst, Phys. Rev. 60, 47 (1941)
   J. P. Blewett, Phys. Rev. 69, 87 (1946)

2. L. W. Jones, MURA Report 385 (1958) (unpublished)

3. V. Veksler, J. Phys. USSR 9, 153 (1945)
   E. M. McMillan, Phys. Rev. 68, 143 (1945)

4. This excludes operation above the transition energy where the particle frequency decreases with increasing particle energy. Here space charge repulsion leads to increased phase stability. We will only consider operation below the transition energy.

5. K. R. Symon and A. M. Sessler, Proceedings of the CERN Symposium on High Energy Accelerators and Pion Physics (CERN, Geneva, 1956) Vol. 1, 44.

6. A brief review of the results of Reference 5 is given in K. M. Terwilliger, L. W. Jones, C. H. Pruett, Rev. Sci. Inst., 28, 987 (1957).

7. K. R. Symon, D. W. Kerst, L. W. Jones, L. J. Laslett, and K. M. Terwilliger, Phys. Rev. 103, 1837 (1956)
   T. Ohkawa, Rev. Sci. Inst. 29, 108 (1958)

8. The range of radii associated with the energy spread in synchrotron oscillations may be ignored here both because we limit our discussion to accelerators in which R is slowly varying function of energy and because it turns out that the azimuthal space charge field has only a logarithmic dependence upon beam dimensions.

9. A similar result has been derived previously in connection with a seimi-classical treatment of the density matrix for a many-Fermion problem.
   P. A. M. Dirac, Proc. Camb. Phil. Soc. 26, 376 (1930)

10. Smythe, Static and Dynamic Electricity (McGraw-Hill, N. Y. 1950) p. 85

FIGURE CAPTIONS

Fig. 1. The area $A(\Lambda, \Gamma)$ of stable phase in $y-\varphi$ space as a function of the parameter Λ which characterizes the effect of space charge, for various values of the parameter Γ which characterizes the rate of modulation of the oscillator.

Fig. 2. The parameters characterizing the stable region of phase space, φ_1 and φ_2 which give the azimuthal extent of the region, and φ_s which is the stable angle, as a function of the modulation rate parameter Γ .

Fig. 3. The maximum half height of the stable region in phase space $y_{Max}(\Lambda, \Gamma)$ as a function of the space charge parameter Λ , for various values of the modulation rate parameter Γ .

Fig. 4. The functions which yield the longitudinal space charge limit (Eq. (34)), and the energy spread of particles (Eq. (37)) as a function of the modulation rate parameter, Γ for the case in which the charge density is high, i.e. $\Lambda \gg 1$.

Fig. 5. The coordinate systems to find the potential at the center of the centrally located beam of charge of radius a , placed between two conducting planes separated by a distance G .

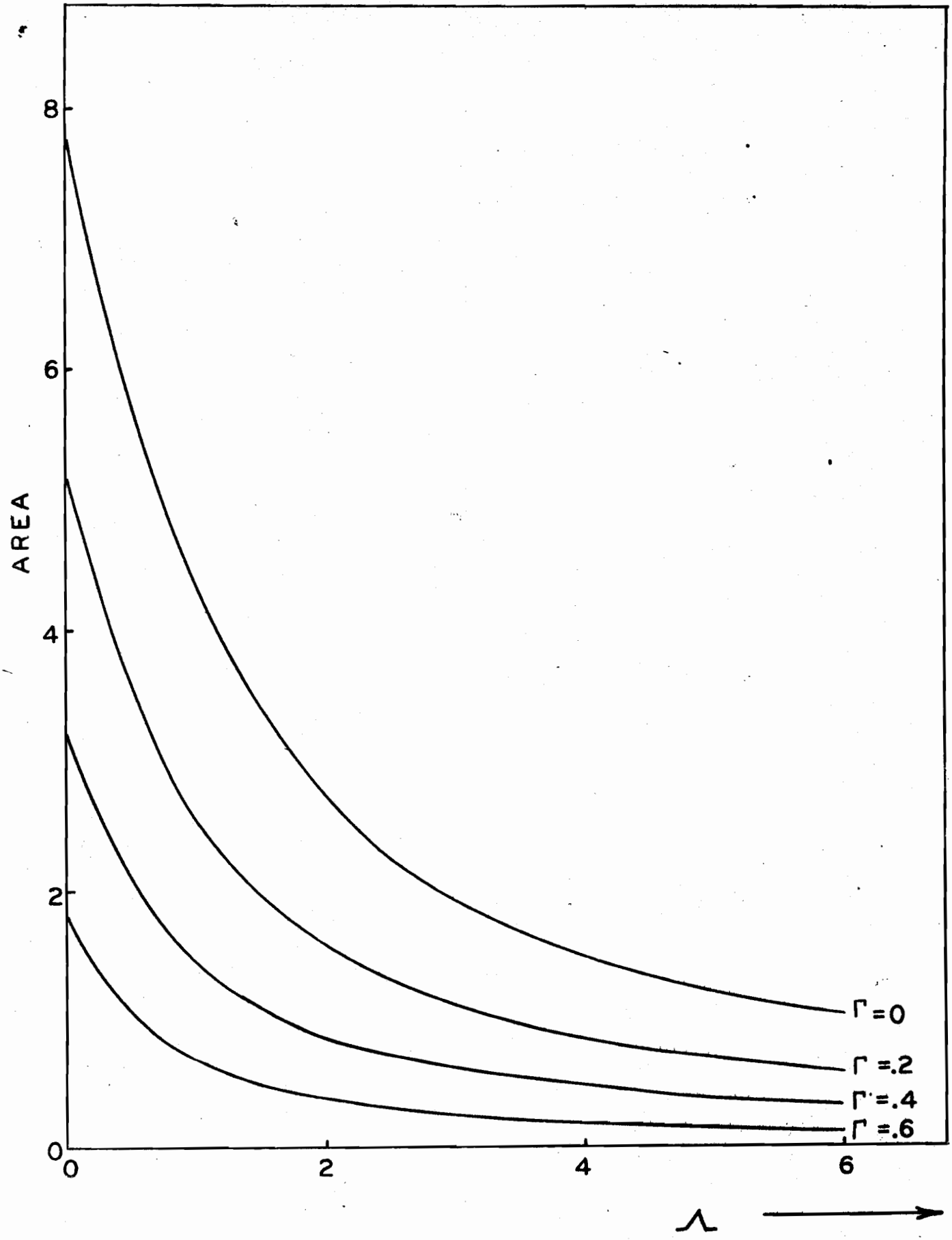

Fig. 1

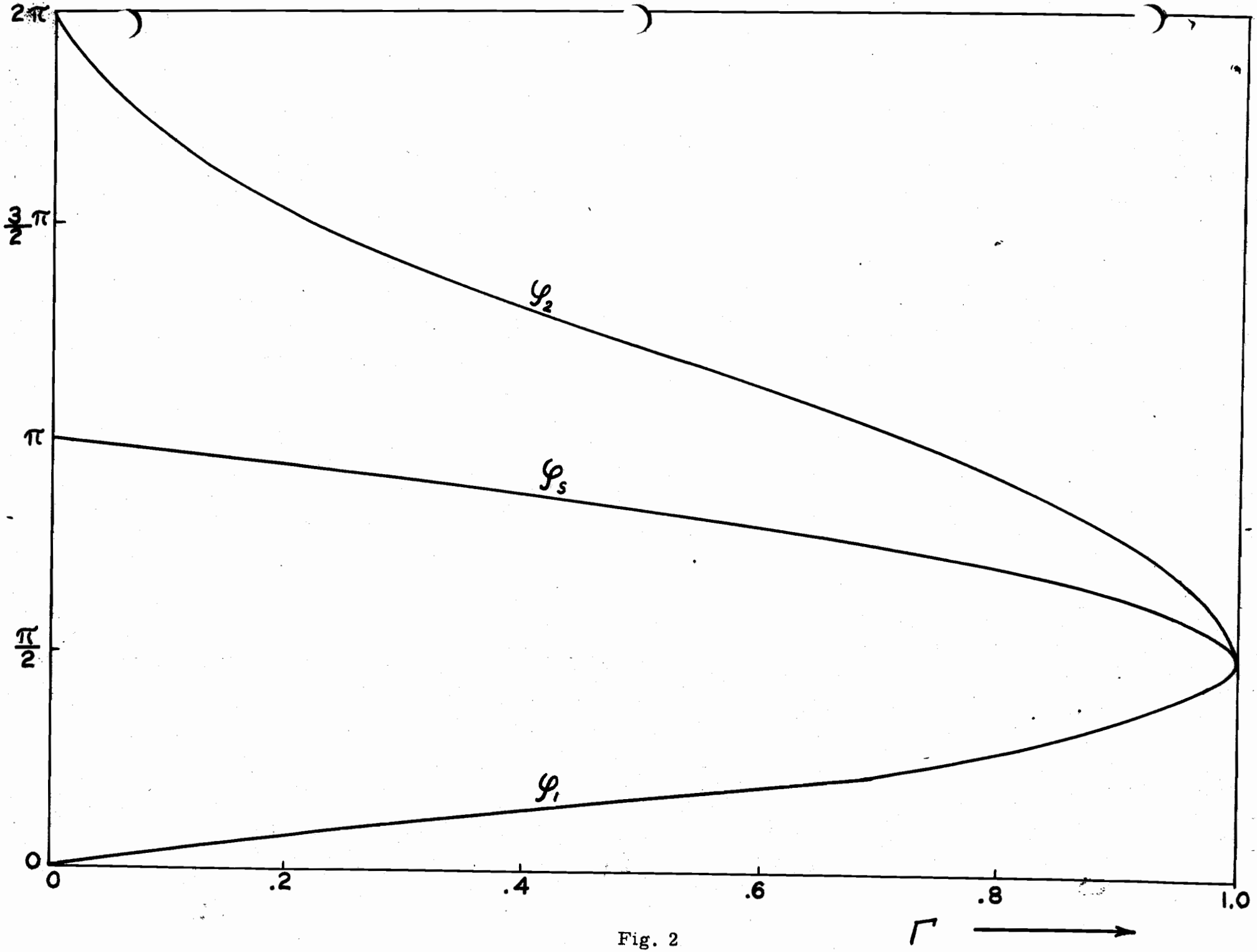

Fig. 2

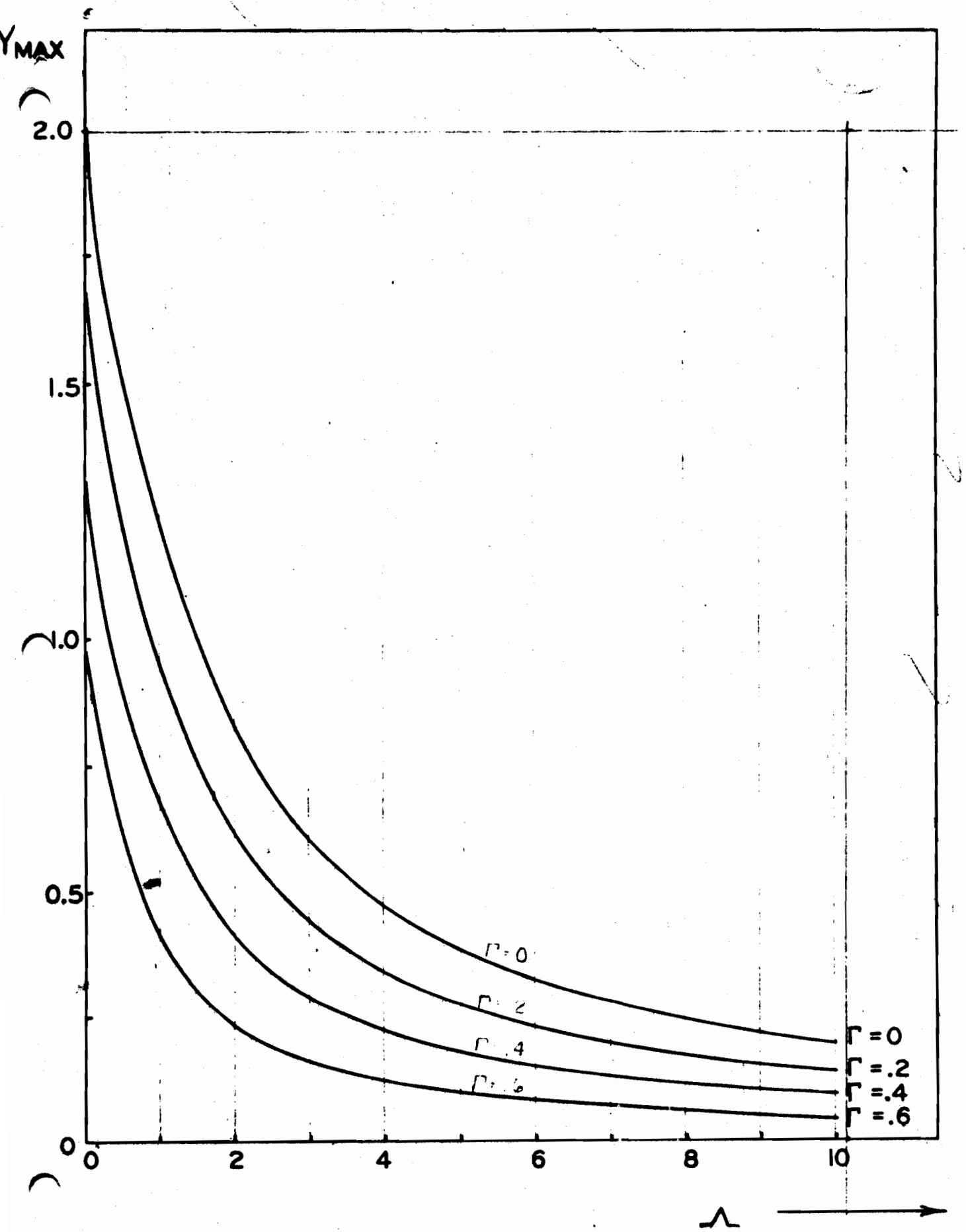

Fig. 3

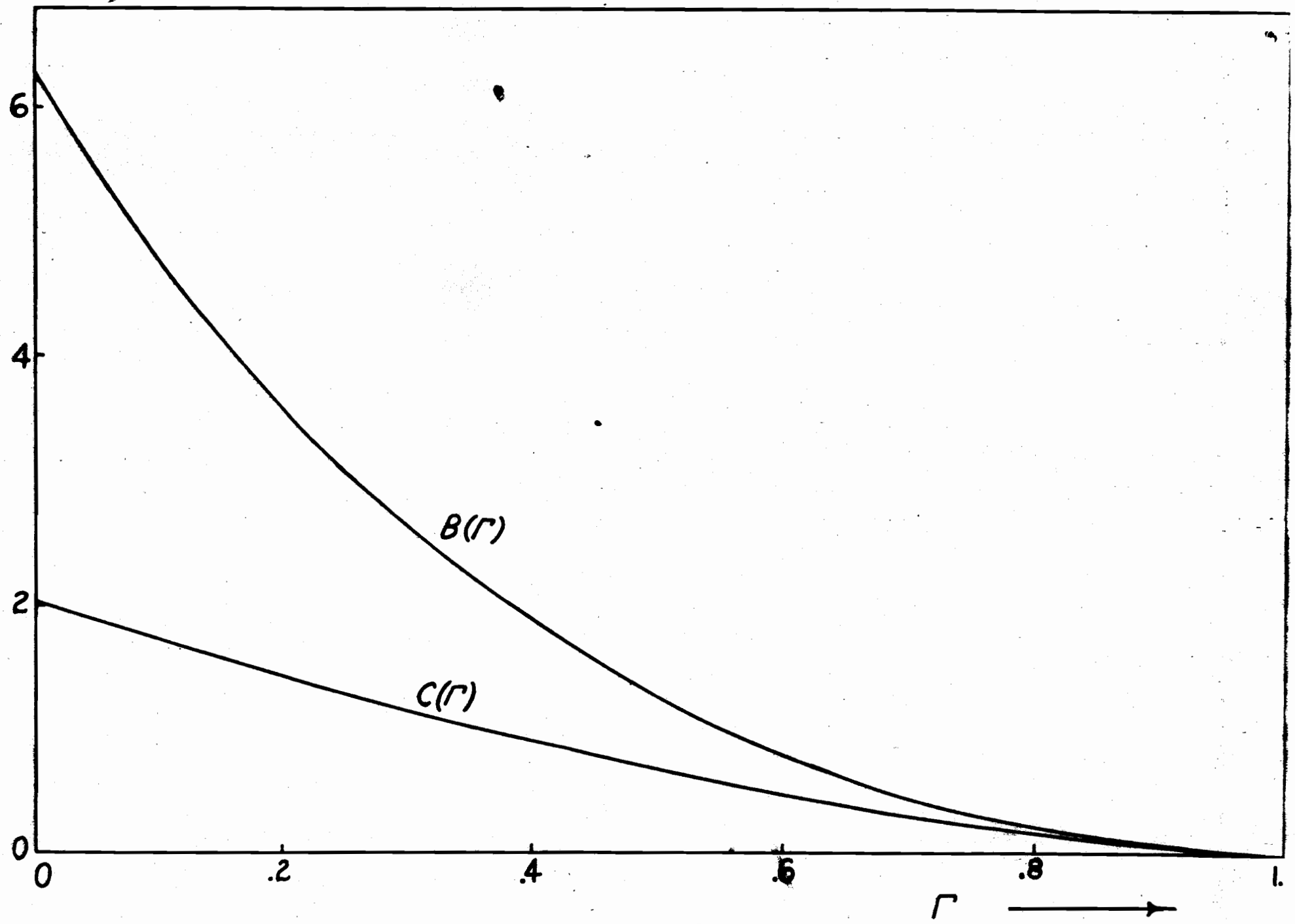

Fig. 4

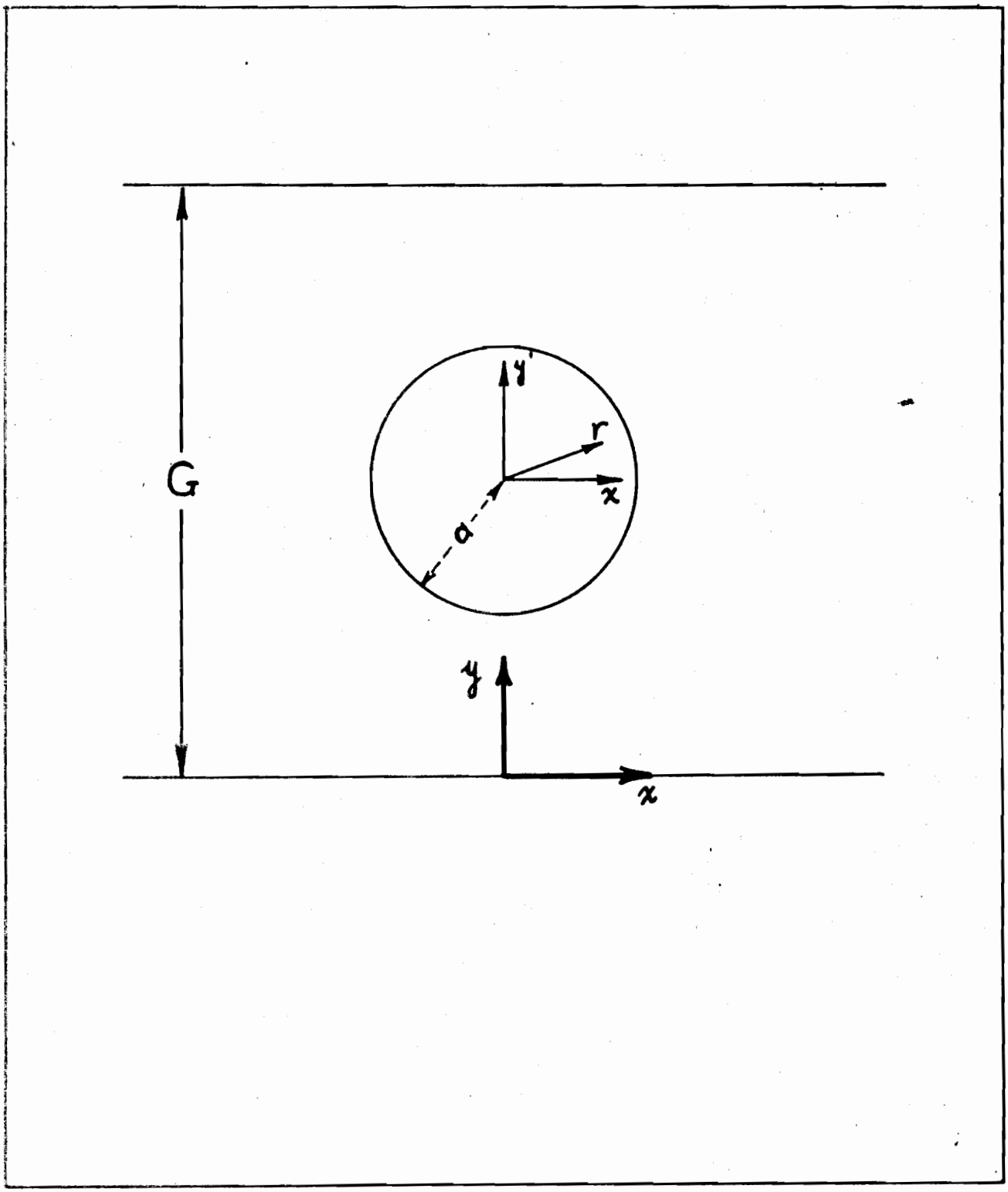

Fig. 5