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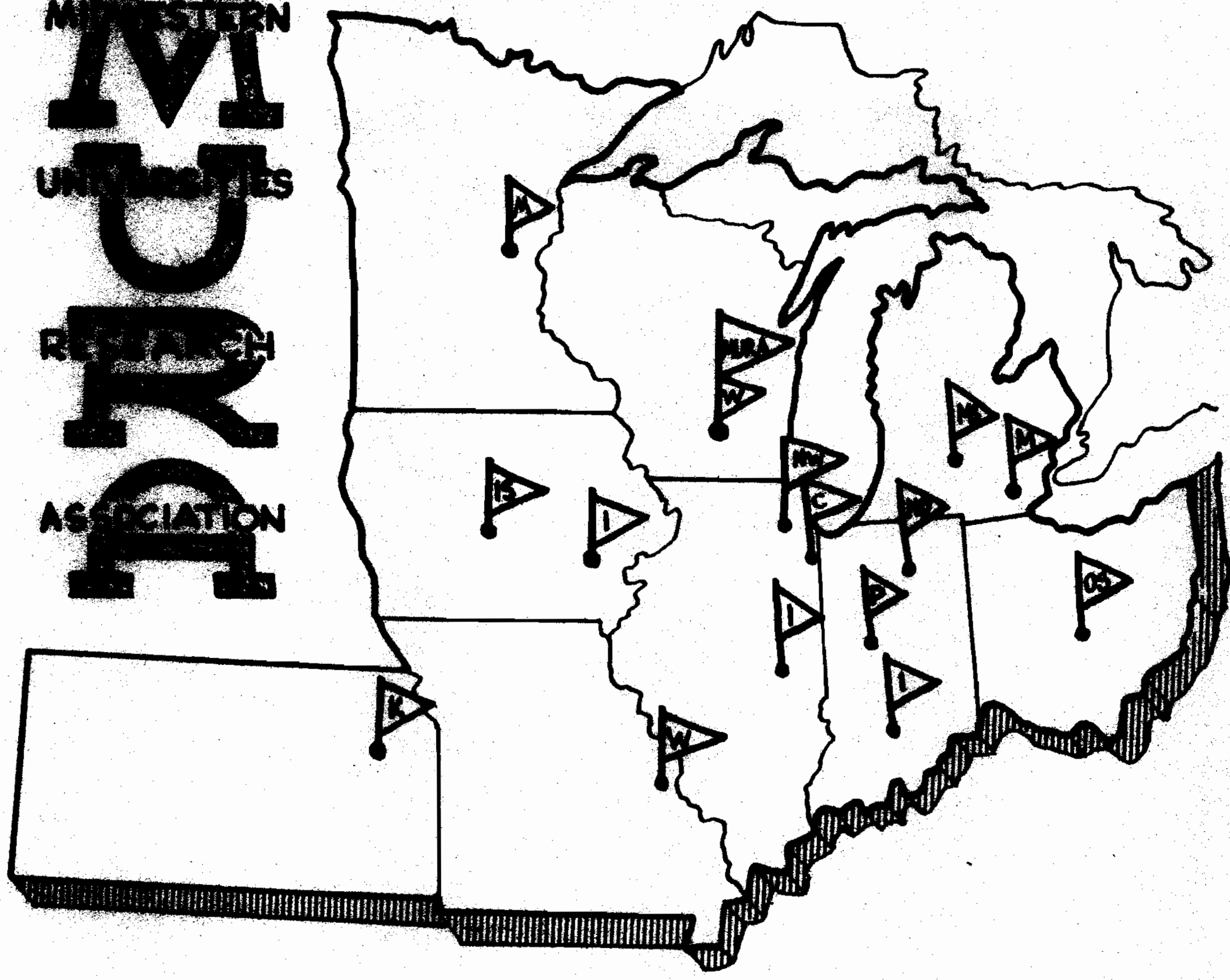

MOMENTUM KICKS FOR PARTICLES CROSSING A GAP AT AN ANGLE

J. Van Bladel

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MOMENTUM KICKS FOR PARTICLES CROSSING A GAP AT AN ANGLE

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ABSTRACT

Formulas are derived for the longitudinal and lateral momentum kicks to which a particle is subjected upon crossing an accelerating gap. The angle ε between the particle's trajectory and the axis of the accelerator tube is assumed to be small, and terms of order higher than the first in ε are neglected in the formulas.

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Consider an infinitely long rectangular tube cut in two by a narrow gap perpendicular to the tube's axis O Z (Fig. 1). A voltage $V \cos \omega t$, constant in phase and amplitude all along the gap, is assumed to be impressed across the lips of the latter. A particle, moving in the O X Z plane with a velocity $\vec{v}$, is crossing the plane of the gap at an angle ε, and at a point X_0. It is desired to know how large a momentum kick the electromagnetic field will apply to the particle. The problem is of importance for the design of particle accelerators.

The solution is immediate if one assumes the electric field to have a δ function behavior throughout the tube, i.e., to be of the form $V \delta(z) \cos \omega t \, \vec{u}_z$, and if one neglects the effects of the magnetic field. If φ is the phase of the voltage at the moment the particle crosses the plane of the gap (i.e., the plane Z = 0), the momentum kick turns out to be

$$\Delta \vec{p} = \frac{qV}{c} \frac{\cos \varphi}{\cos \varepsilon} \frac{1}{\beta} \vec{u}_z \qquad (1)$$

The electromagnetic field, however, does not follow the assumed δ- function behavior, and penetrates on both sides of the gap. The momentum kick will consequently depend on the nature of the particle's trajectory throughout the gap region[1]. In many situations of practical interest, this trajectory will be practically linear. An accelerating gap in the straight section of an accelerator doughnut, where no focussing magnetic fields are present, constitutes an example of such a situation, provided the gap voltage is not of such magnitude as to materially alter

1. Data on the penetration of the fields in the gap region can be found in Journal of Applied Physics 28, 12, 1479 - 1483, 1957.

the particle's trajectory. It will be explicitly assumed, in the present analysis, that the trajectory possesses a linear portion CE, beyond which the gap region does not extend[1]. The momentum kick, can then be expressed as

$$\Delta\vec{p} = \int_C^E \vec{F}\,dt = \int_A^B \vec{F}\,dt$$

This we now proceed to compute.

In the OXZ plane, the electric and magnetic fields reduce to the following components[1]:

$$E_x = \mp \frac{8V}{\pi a} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} (-1)^{\frac{m+n}{2}} \cdot \frac{\sin \frac{m\pi X}{a}}{n} \cdot e^{-\frac{\pi}{a}\sqrt{m^2+n^2\tau^2-\nu^2}\cdot Z} \cdot \cos \omega t \tag{2}$$

$$E_z = -\frac{8V}{\pi a} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} (-1)^{\frac{m+n}{2}} \cdot \frac{\cos \frac{m\pi X}{a}}{mn} \cdot \frac{m^2 + n^2\tau^2}{\sqrt{m^2+n^2\tau^2-\nu^2}} \cdot e^{-\frac{\pi}{a}\sqrt{m^2+n^2\tau^2-\nu^2}\cdot Z} \cdot \cos \omega t \tag{3}$$

$$H_y = \frac{8\omega\varepsilon_0 V}{\pi^2} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} (-1)^{\frac{m+n}{2}} \cdot \frac{\sin \frac{m\pi X}{a}}{n\sqrt{m^2+n^2\tau^2-\nu^2}} \cdot e^{-\frac{\pi}{a}\sqrt{m^2+n^2\tau^2-\nu^2}\cdot Z} \sin \omega t \tag{4}$$

where $\tau = a/b$ is the aspect ratio of the cross section, and ν the frequency expressed in units of $c/2a$, the cut-off frequency of the tube. The upper and lower signs correspond to positive and negative values of Z respectively.

The momentum kick is equal to

$$\vec{\Delta p} = q\int_A^B \vec{E}\,dt + q\mu_0 \int_A^B (\vec{v} \times \vec{H})\,dt$$

$$= \vec{u}_x \left[q\int_A^B E_x\,dt - q\mu_0 \int_A^B v_z H_y\,dt \right] + \vec{u}_z \left[q\int_A^B E_z\,dt + q\mu_0 \int_A^B v_x H_y\,dt \right] \tag{5}$$

Its explicit value can be calculated by substitution of the expressions for E_X, E_Z, H_Y. The resulting formulas are extremely involved. We shall restrict ourselves to the situation where $\mathcal{E}$ is a small angle, and compute the zero and first order terms in Eq. (5) only. The formulas become much more tractable in these circumstances. The lateral momentum kick, for instance, turns out to be

$$\vec{\Delta p}_{lat} = \left[\frac{qV}{c} \sin\varphi\, p_1 + \mathcal{E}\, \frac{qV}{c} \cos\varphi\, p_2 \right] \vec{u}_x$$

The physical interpretation of p_1 is as follows. It is the momentum kick,, expressed in units of qV/c, to which a particle traveling parallel to the tube's axis is subjected when crossing in quadrature ($\varphi = \pi/2$). This zero order component in $\mathcal{E}$ vanishes when the particle crosses in phase ($\varphi = 0$), and the only source of lateral kick, in those circumstances, is the term $\mathcal{E} p_2$ due to the obliquity of the trajectory. The explicit expressions for p_1 and p_2 are as follows:

$$p_1 = \left(\frac{1}{\beta^2} - 1\right) \frac{16\nu}{\pi^2} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{m+n}{2}}}{n} \cdot \frac{\sin \frac{m\pi X_0}{a}}{m^2 + n^2\tau^2 + \left(\frac{1}{\beta^2}-1\right)\nu^2} \tag{6}$$

$$p_2 = \frac{16}{\beta\pi^2} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \frac{m(-1)^{\frac{m+n}{2}}}{n} \cos \frac{m\pi X_0}{a} \cdot \frac{m^2 + n^2\tau^2 - \nu^2\left(\frac{1}{\beta^2}-1\right)}{\left[m^2 + n^2\tau^2 + \nu^2\left(\frac{1}{\beta^2}-1\right)\right]^2} \tag{7}$$

for a point in the OXZ plane. There, $\beta = v/c$ is the velocity measured in terms of the velocity of light.

The formula for the longitudinal momentum kick is obtained as

$$\Delta \vec{p}_{long} = \left[\frac{qV}{c} \cdot \cos\varphi \cdot p_3 + \varepsilon \cdot \frac{qV}{c} \cdot \sin\varphi \cdot p_4 \right] \vec{u}_z$$

There the obliquity of the trajectory introduces a quadrature kick p_4. The expressions for p_3 and p_4 are

$$p_3 = -\frac{16}{\beta \pi^2} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{m+n}{2}}}{mn} \cdot \cos\frac{m\pi X_0}{a} \cdot \frac{m^2 + n^2\tau^2}{m^2 + n^2\tau^2 + \nu^2\left(\frac{1}{\beta^2} - 1\right)} \tag{8}$$

$$p_4 = \frac{16\,\nu}{\pi^2} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{m+n}{2}}}{n} \cdot \sin\frac{m\pi X_0}{a} \cdot \frac{(m^2 + n^2\tau^2)\left(\frac{2}{\beta^2} - 1\right) - \nu^2\left(\frac{1}{\beta^2} - 1\right)}{\left[m^2 + n^2\tau^2 + \nu^2\left(\frac{1}{\beta^2} - 1\right)\right]^2} \tag{9}$$

Curves for p_1, p_2, p_3, p_4 are displayed in Figs. 2 to 5 for a fairly wide range of parameters. The various figures are drawn with identical scales in order to facilitate quick estimates of the relative importance of the various kicks. Curves relative to aspect ratios larger than one refer to points in the OXZ plane, and are drawn as a direct application of Eqs. (6) to (9). The lateral momentum kick is parallel to the X axis in those conditions. Curves relative to aspect ratios smaller than one imply that the O X Z plane is now parallel to the narrow wall of the tube, because $a < b$. In the parlance of Fig. 1, however, they can be interpreted as yielding data on points in the OYZ plane, and these data are still expressed by Eqs. (6) to (9), provided the roles of X and Y (and associated quantities) are exchanged in these formulas. The lateral momentum kick is now parallel to the Y axis. To give an example, in a 2 x 1 cross section, data for the O Y Z plane will be found from curves labeled $\tau = 1/2$.

It will be noticed from Eq. (1) that the δ– function approximation does not account for the p_1, p_2 and p_4 terms, and that it predicts a value $\frac{1}{\beta}$ for p_3. This value is excellent at low frequencies, as shown by Fig. 4. How bad an error is introduced by neglecting p_1, p_2, p_4 will be illustrated in a sample calculation to be given shortly.

The lateral kick due to the slanted character of the particle's trajectory is very often a nuisance, and one might conceivably try to offset it, at least locally, by a counter-kick obtained by slanting the gap by an angle ε' (Fig. 6). The total lateral kick is then equal to

$$\Delta \vec{p}_{lat} = \left[\frac{qV}{c} \cdot \sin\varphi \cdot p_1 + \varepsilon \frac{qV}{c} \cos\varphi \cdot p_2 + \varepsilon' \frac{qV}{c} \cdot \cos\varphi \cdot p_5 \right] \vec{u}_x$$

with the following expression[2] for p_5:

$$p_5 = \frac{8}{\beta\pi} \sum_{m=0,2}^{\infty} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{m+n-1}{2}} \cos\frac{m\pi X}{a}}{\varepsilon_m \cdot n \cdot \left[m^2 + n^2\tau^2 + \nu^2\left(\frac{1}{\beta^2} - 1\right) \right]} \cdot \left[\nu^2\left(\frac{1}{\beta^2} - 1\right) + (\varepsilon_m - 1) n^2 \tau^2 \right] \tag{10}$$

where ε_m is equal to 2 for m = 0, to 1 for other values of m. Curves for p_5 are displayed in Fig. 7.

Sample Calculation

Consider an accelerator tube with dimensions a = 1 m, b = 0.1 m. The cut-off frequency c/2a is 150 Mc/s. Assume the gap to be in a straight section, and

2. J. Van Bladel: Paper accepted for publication in "Nuclear Instruments."

the average radius of the machine to be 8 m. Consider particles with velocity $2c/3$: Their frequency of rotation is 4 Mc/s. If the accelerating voltage is tuned to the 5th harmonic, the R-F will be 20 Mc/s, yielding a value $\frac{20 \text{ Mc/s}}{150 \text{ Mc/s}} = 0.133$ for the design parameter ν, while $\tau = \frac{a}{b} = 10$. Let the particles cross the gap at a phase angle $\varphi = 60^{\circ}$. Then:

1. A particle crossing at $X = Y = 0$, and traveling parallel to the Z axis will experience a longitudinal kick $0.75\ qV/c$, and no lateral kick.
2. If this particle travels in the OXZ plane, but crosses at an angle 3°, the kicks are practically the same as in 1.
3. If this particle travels in the OYZ plane, but crosses at an angle 3°, its longitudinal kick will remain as in 1, and a sizable side kick $0.036\ qV/c$, i.e., 4.8% of the longitudinal kick, will appear.

If one considers a particle in the OXZ plane, but crossing at $X = a/4$, the results would be practically the same as in 2. For a particle in the OYZ plane, however, crossing at $Y = b/4$, the results are somewhat different:

1. If this particle travels parallel to the Z axis, the longitudinal kick will be $0.75\ qV/c$, the lateral kick $-0.103\ qV/c$.
2. If this particle crosses at an angle 3°, the longitudinal kick is reduced to $0.735\ qV/c$, the lateral kick increased to $-0.140\ qV/c$.

The δ-function approximation would have predicted a longitudinal kick $0.75\ qV/c$ in both situations, and no lateral kick.

LIST OF CAPTIONS

Fig. 1. Rectangular duct and parameters of the particle's trajectory.

Fig. 2. Plot of p_1 as a function of frequency for two values of the particle's velocity, and a point halfway between the main axis and the wall ($X = a/4$, $Y = 0$ for $\tau > 1$; $X = 0$, $Y = b/4$ for $\tau < 1$). The three curves relative to each velocity corrspond to (reading from the uppermost) aspect ratios of 2, 1, and 0. For $\tau = 5$ and above, the kick is too small to be clearly plotted on the figure. It will be noticed from Eq. (6) that p_1 vanishes at the velocity of light.

Fig. 3. Plot of p_2 as a function of frequency for three values of the particle's velocity. The four curves relative to each velocity correspond to (starting from the uppermost at low frequencies) aspect ratios of 2, 1, 1/2 and 0. For $\tau = 5$ and above the kick is too small to be accurately plotted on the figure. Considered are: (a) a point on the main axis ($X = Y = 0$), (b) a point halfway between the main axis and the wall ($X = a/4$, $Y = 0$ for $\tau > 1$; $X = 0$, $Y = b/4$ for $\tau < 1$).

Fig. 4. Plot of p_3 as a function of frequency for three values of the particle's velocity. The six curves relative to $2c/3$ and $c/3$ correspond to (reading from the uppermost) aspect ratios of 25, 5, 2, 1, 1/2 and 0. Considered are: (a) a point on the main axis ($X = Y = 0$), (b) a point halfway between the main axis and the wall ($X = a/4$, $Y = 0$ for $\tau > 1$; $X = 0$, $Y = b/4$ for $\tau < 1$).

Fig. 5. Plot of p_4 as a function of frequency for three values of the particle's velocity, and a point halfway between the main axis and the wall ($X = a/4$, $Y = 0$ for $\tau > 1$; $X = 0$, $Y = b/4$ for $\tau < 1$). The four curves relative to

each velocity correspond to (starting from the uppermost at low frequencies) aspect ratios of 2, 1, 1/2 and 0. For $\tau = 5$ and above, the kick is too small to be accurately plotted on the figure.

<u>Fig. 6.</u> Rectangular duct with slanted gap.

<u>Fig. 7.</u> Plot of p_5 as a function of frequency for three values of the particle's velocity. The five curves relative to each velocity correspond to (reading from the uppermost) aspect ratios of 25, 5, 2, 1 and 0. Considered are: (a) a point on the main axis ($X = Y = 0$), (b) a point halfway between the main axis and the wall ($X = a/4$, $Y = 0$ for $\tau > 1$; $X = 0$, $Y = b/4$ for $\tau < 1$).

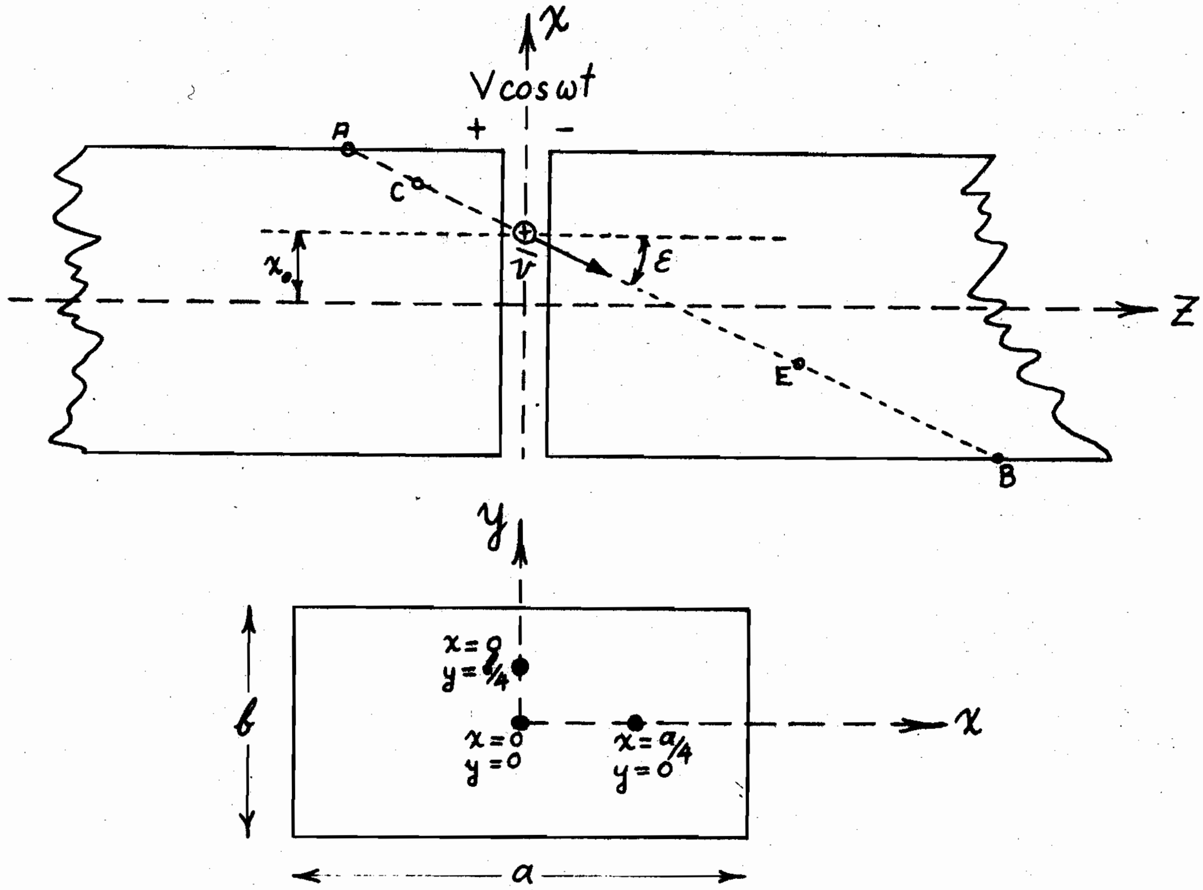


Fig. 1

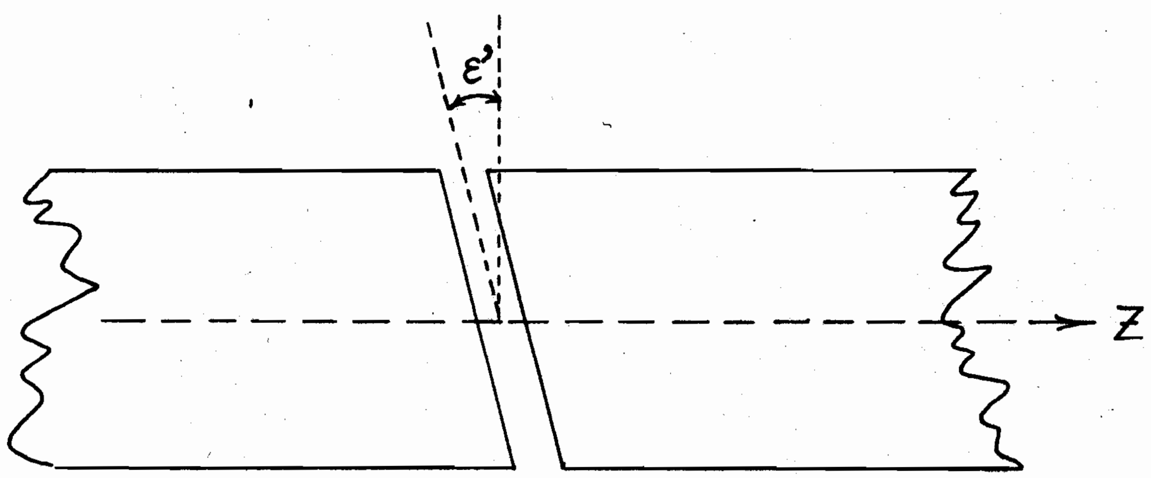


Fig. 6

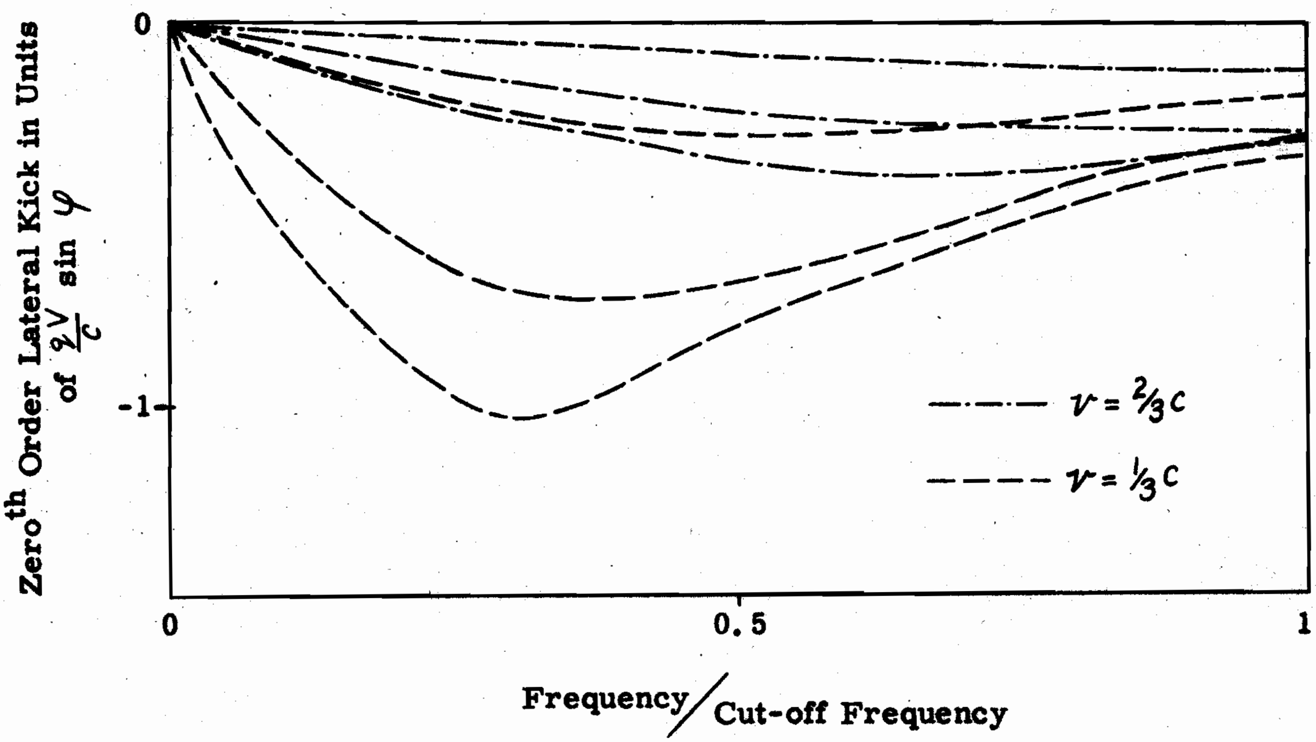

Zeroth Order Lateral Kick in Units of $\frac{2V}{c}$ sin φ
0
-1
0
0.5
1
$v = \frac{2}{3}c$
$v = \frac{1}{3}c$
Frequency/Cut-off Frequency

Fig. 2

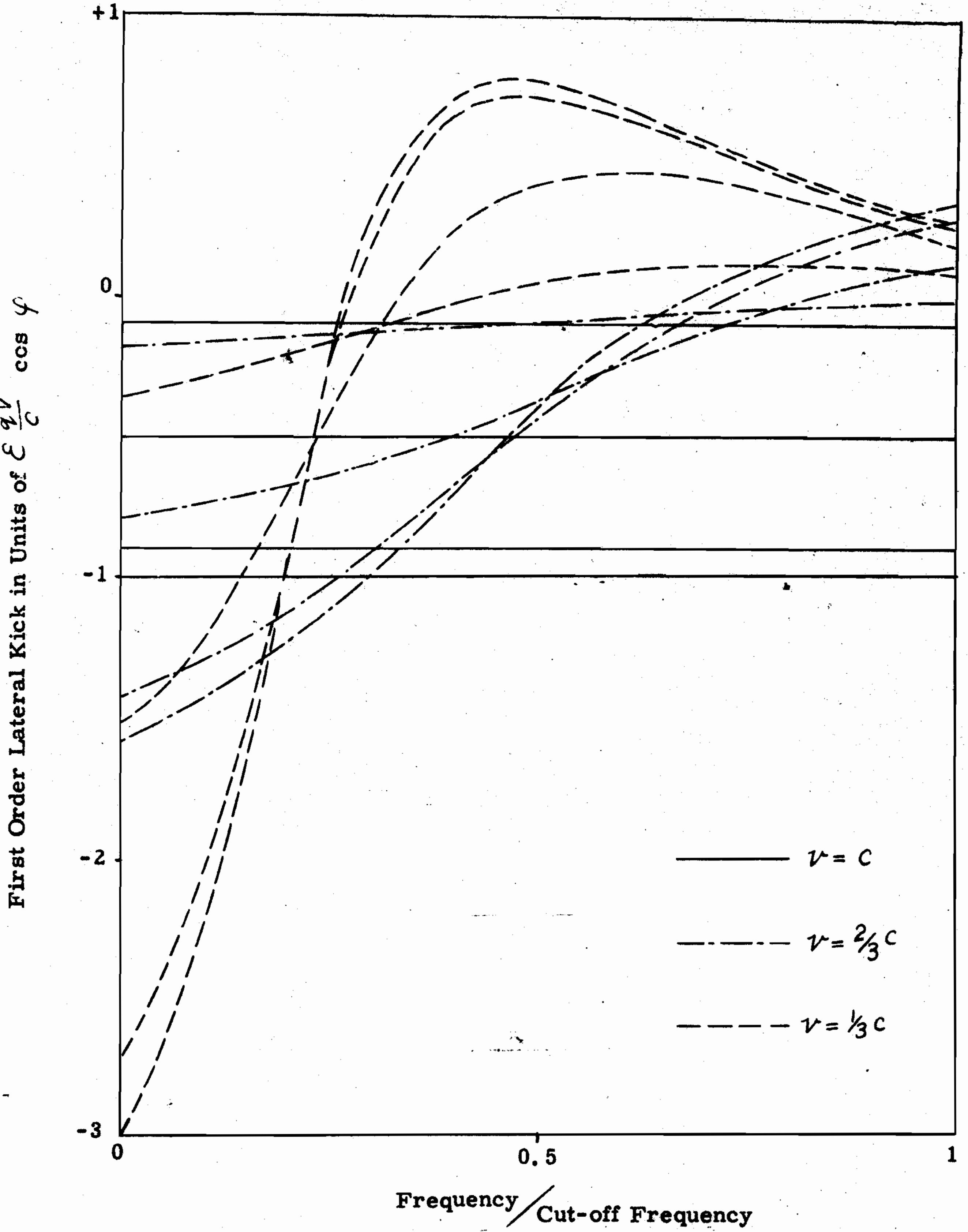

+1
0
-1
-2
-3
0
0.5
1
First Order Lateral Kick in Units of $\mathcal{E}\,\frac{qV}{c}$ cos φ
Frequency/Cut-off Frequency
$v = c$
$v = \frac{2}{3}c$
$v = \frac{1}{3}c$

Fig. 3 (a)

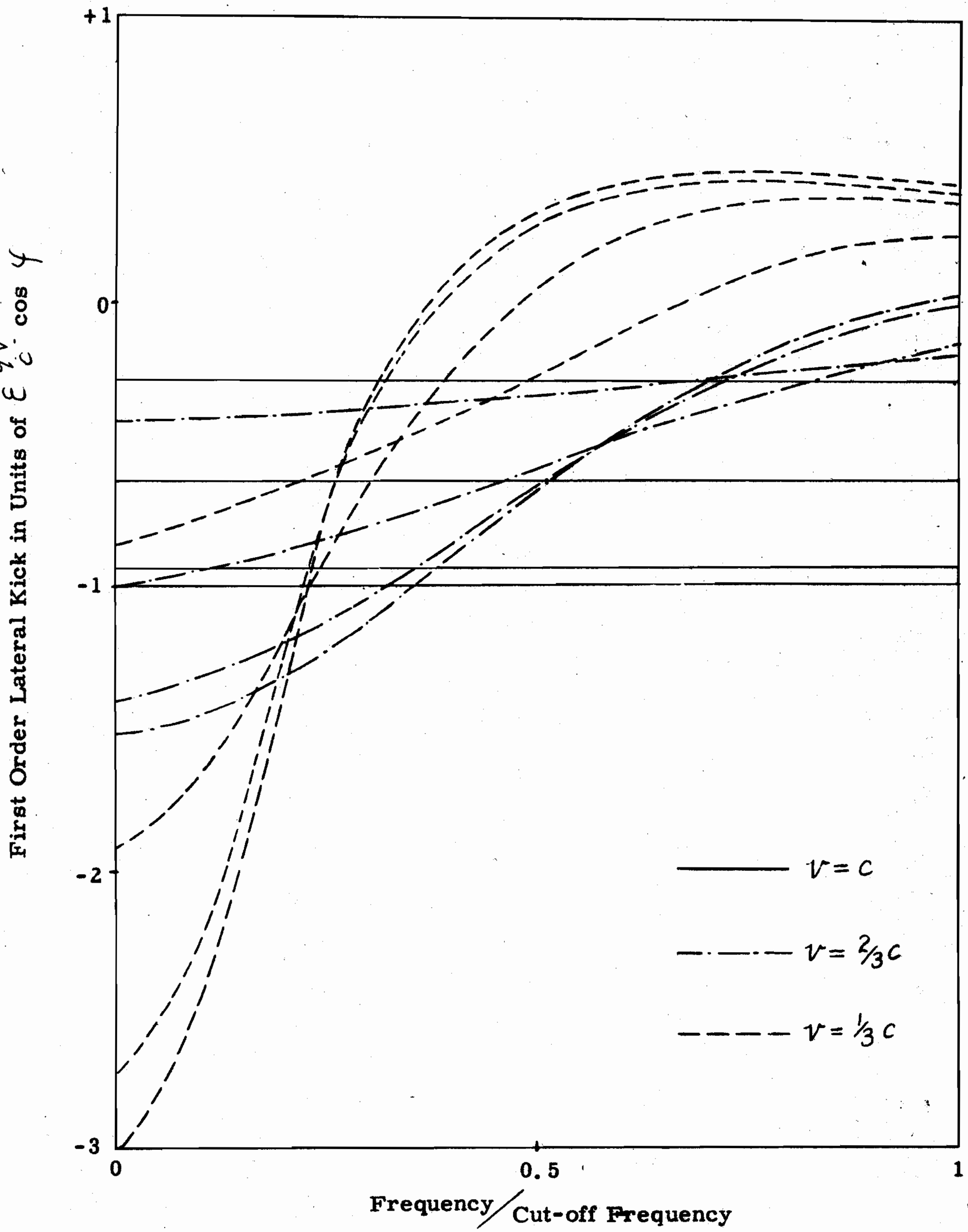

+1
0
-1
-2
-3
First Order Lateral Kick in Units of $\mathcal{E}\,\frac{qV}{c}\cos\varphi$
0
0.5
1
Frequency/Cut-off Frequency
$v = c$
$v = \frac{2}{3}c$
$v = \frac{1}{3}c$

Fig. 3 (b)

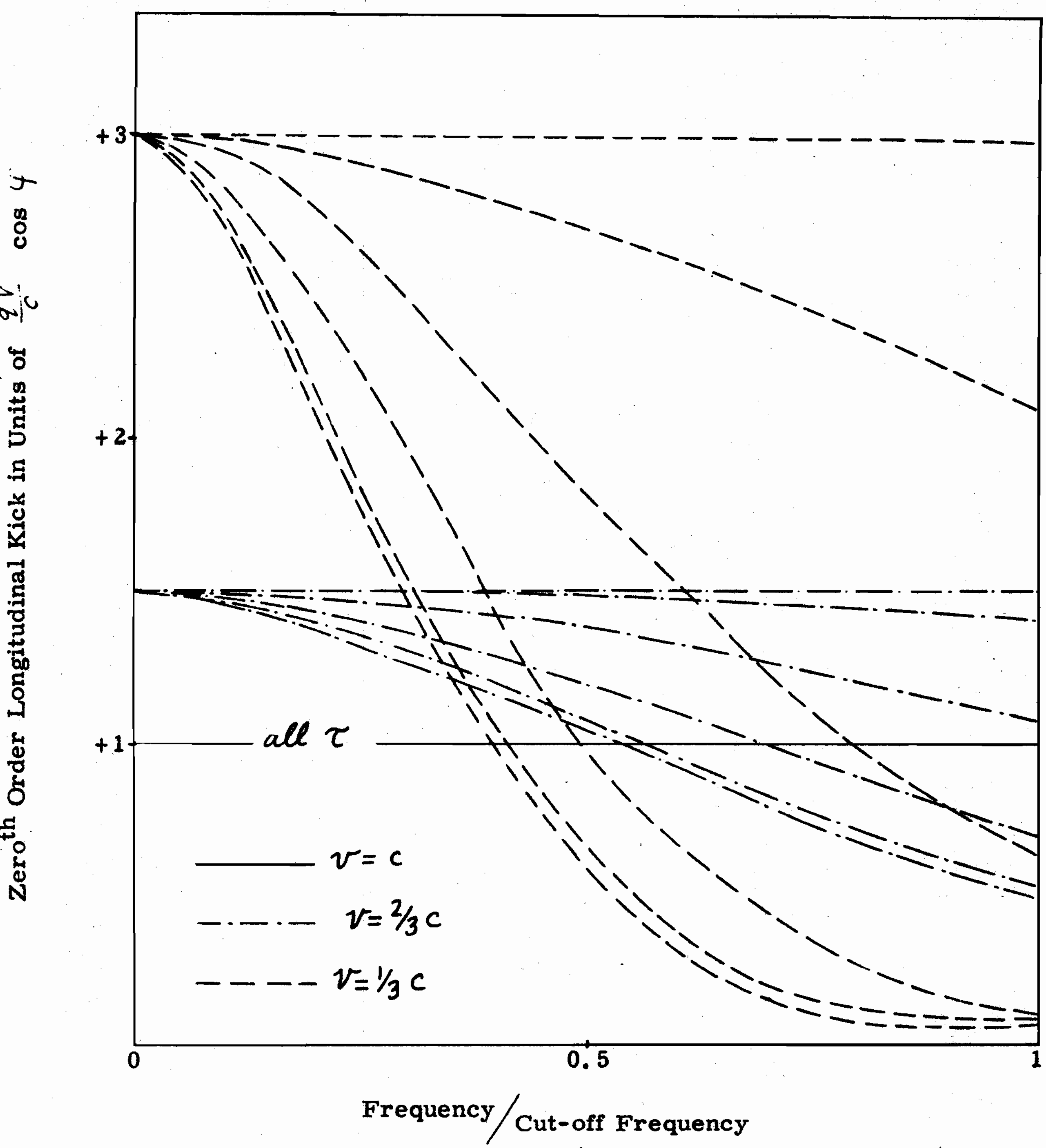
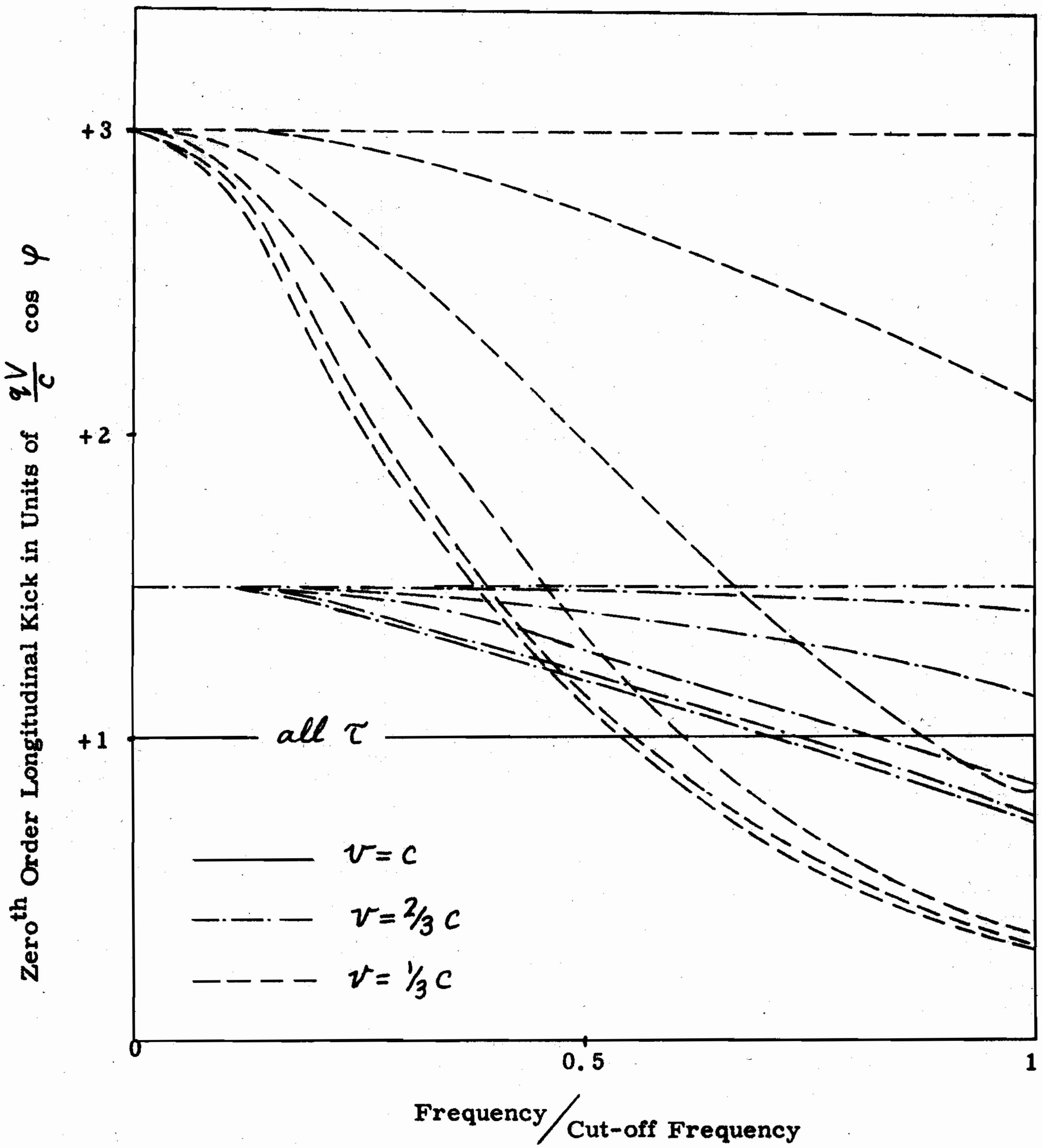
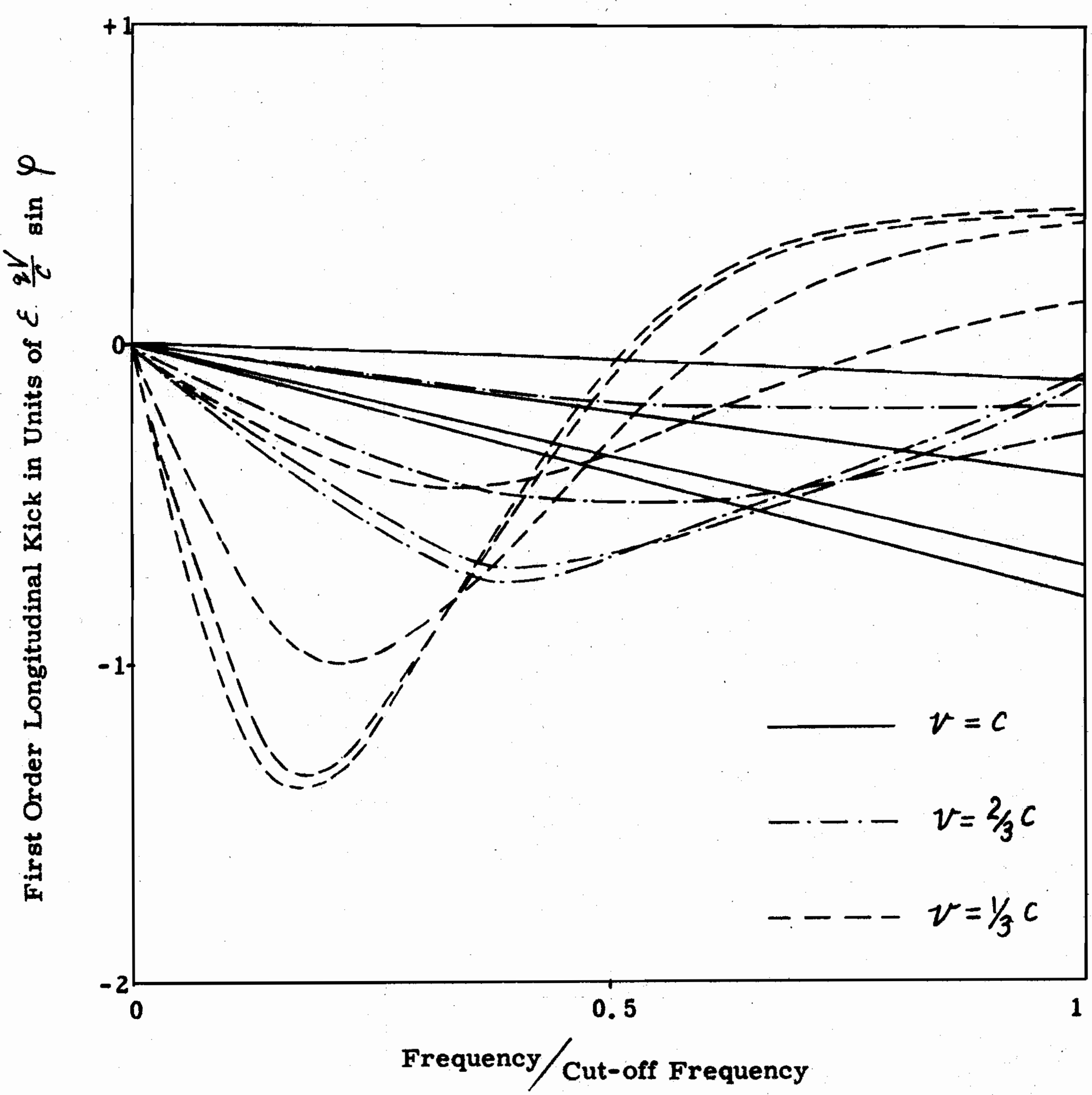

Zero^th Order Longitudinal Kick in Units of $\frac{qV}{c}$ cos ψ
+3
+2
+1
all τ
v = c
v = 2/3 c
v = 1/3 c
0
0.5
1
Frequency/Cut-off Frequency

Fig. 4 (a)



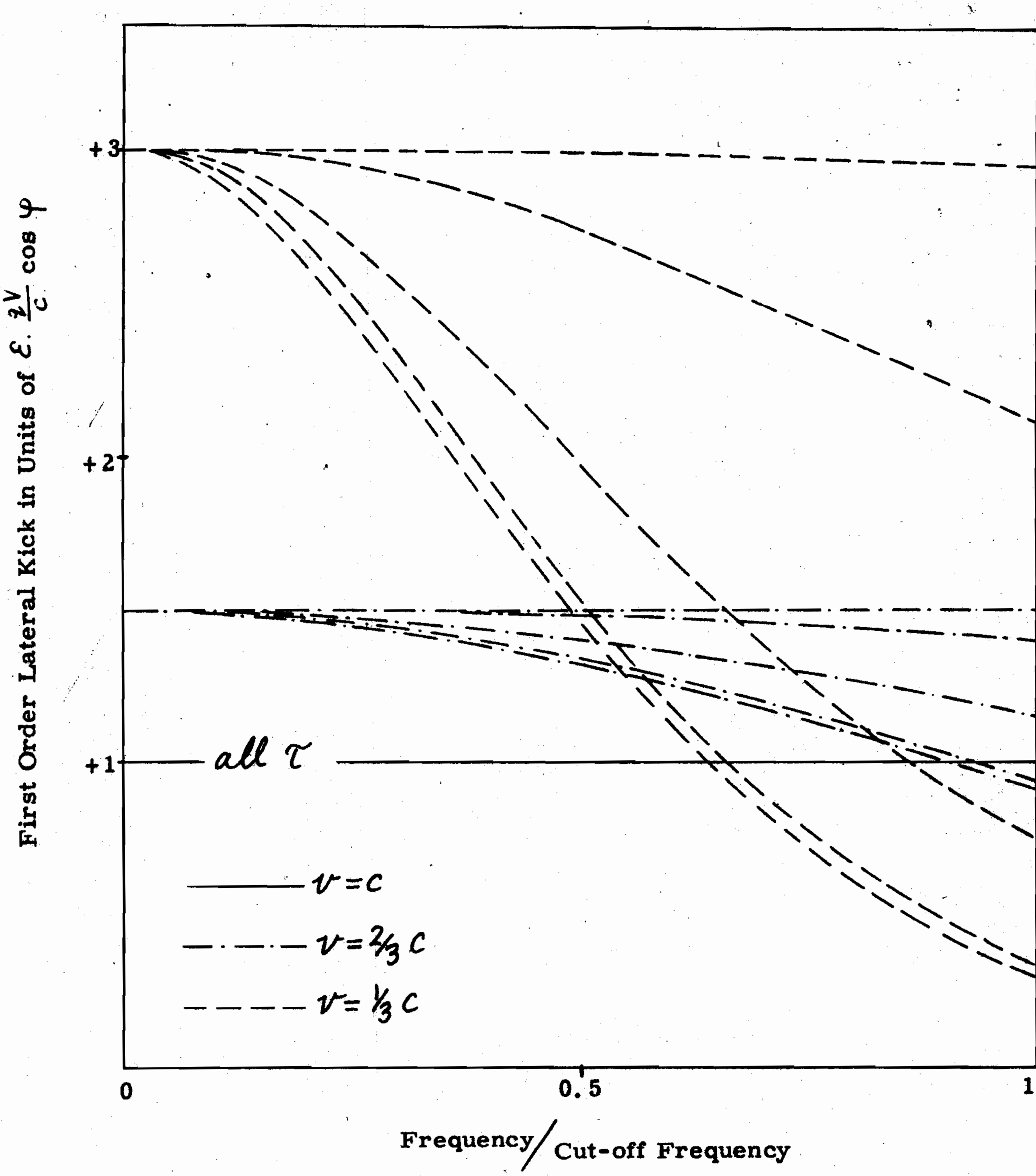
Zeroth Order Longitudinal Kick in Units of $\frac{qV}{c}$ cos φ
+3
+2
+1
all τ
$v = c$
$v = 2/3\, c$
$v = 1/3\, c$
0
0.5
1
Frequency/Cut-off Frequency

Fig. 4 (b)



+1
0
-1
-2
0
0.5
1
First Order Longitudinal Kick in Units of $\mathcal{E}\,\frac{qV}{c}\sin\varphi$
Frequency/Cut-off Frequency
$v = c$
$v = \frac{2}{3}c$
$v = \frac{1}{3}c$

Fig. 5



First Order Lateral Kick in Units of $\mathcal{E} \cdot \frac{2V}{c} \cos\varphi$
+3
+2
+1
all τ
$v = c$
$v = \frac{2}{3}c$
$v = \frac{1}{3}c$
0
0.5
1
Frequency/Cut-off Frequency

Fig. 7 (a)

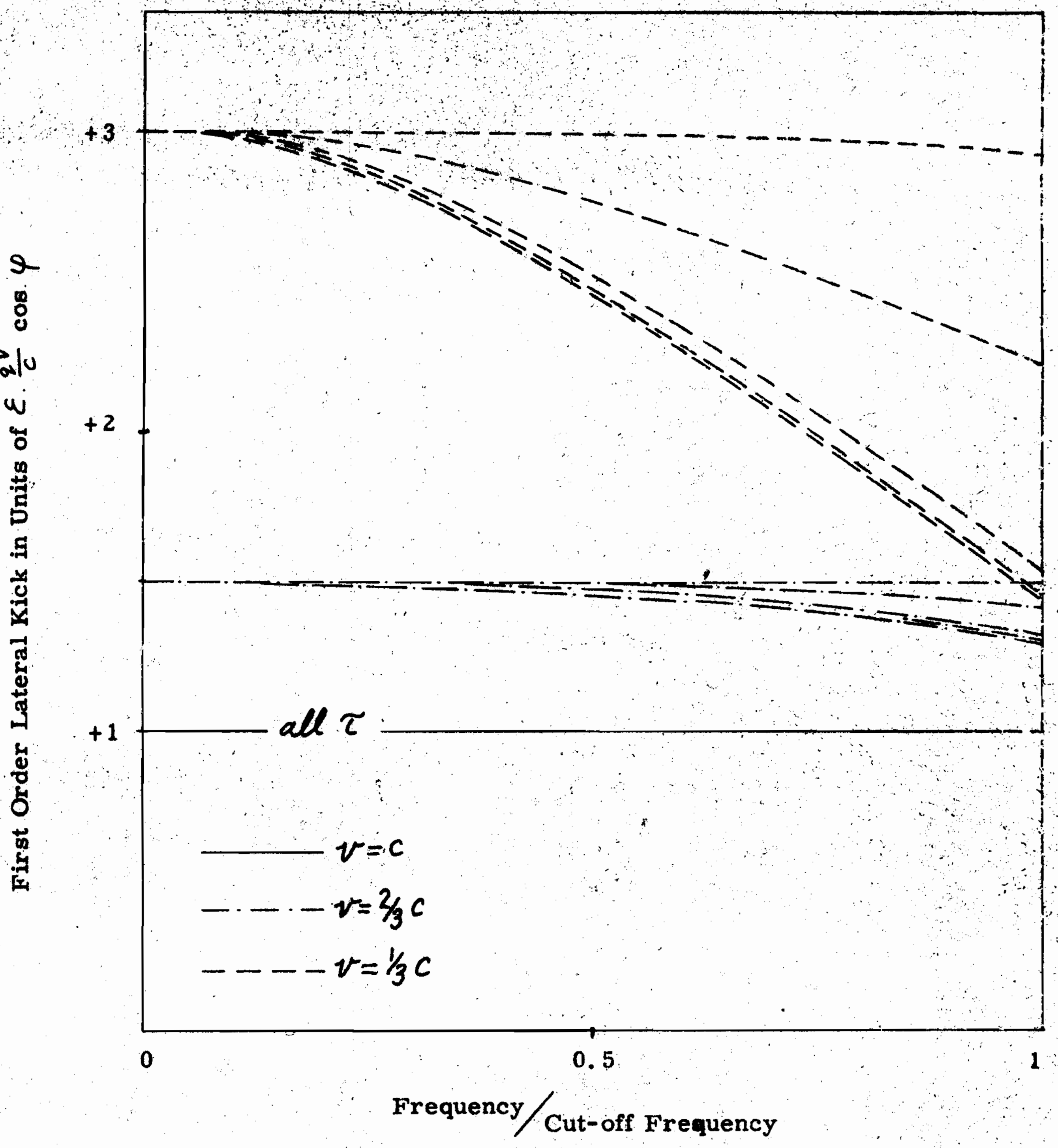

First Order Lateral Kick in Units of $\mathcal{E} \cdot 2\frac{V}{c} \cos\varphi$
+3
+2
+1
all τ
$v = c$
$v = \frac{2}{3}c$
$v = \frac{1}{3}c$
0
0.5
1
Frequency/Cut-off Frequency

Fig. 7 (b)