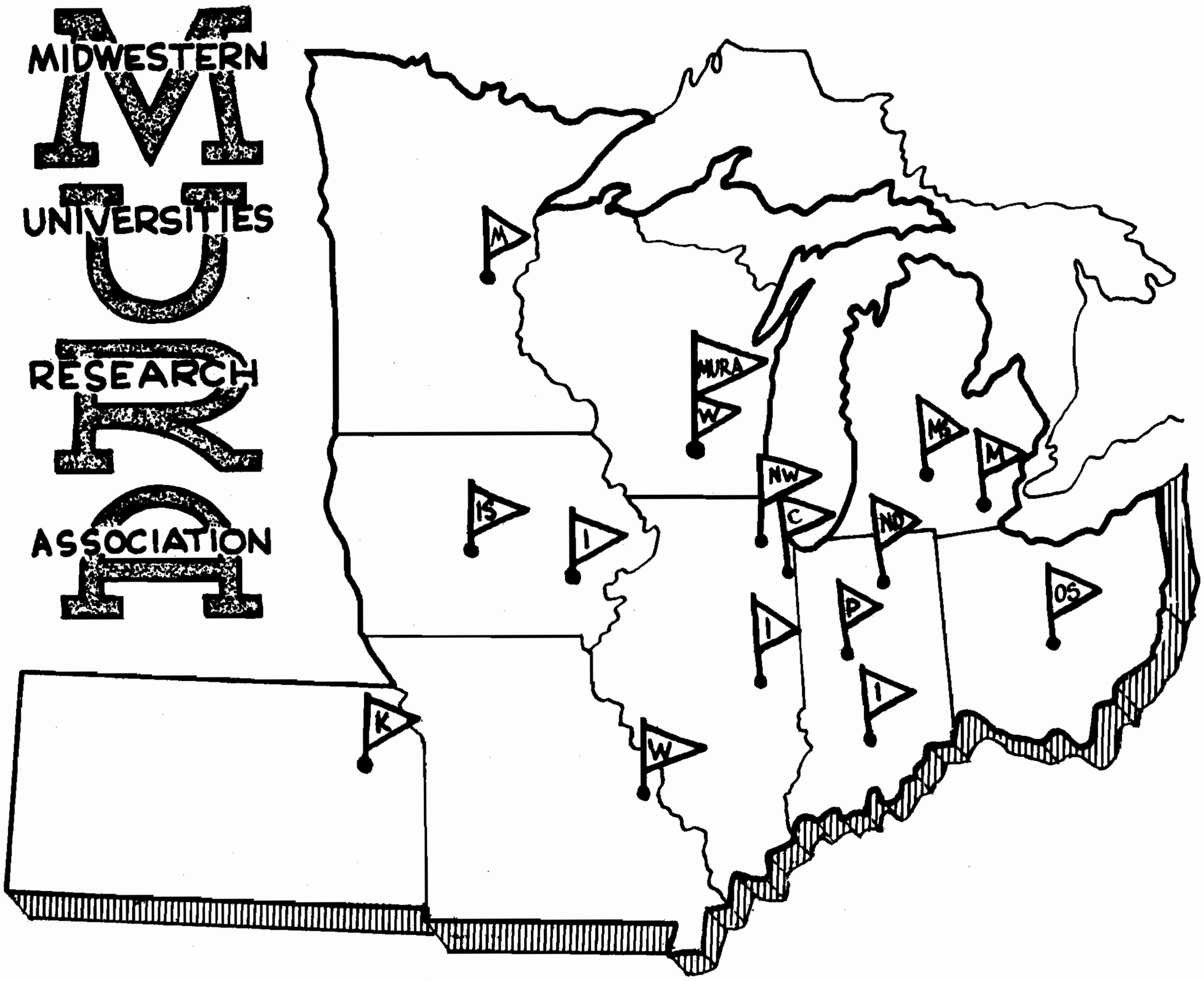

EXPANSION OF A HAMILTONIAN USING THE MOSER METHOD

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ABSTRACT

The terms through fifth degree are given which occur when a Hamiltonian is subjected to a Moser type canonical transformation. Also, an expansion of the original dynamical variables through fourth power terms in the new variables is calculated.

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INTRODUCTION

The equations of motion for an accelerator can be derived from a Hamiltonian of the form

$$H(x,y,p_x,p_y,\theta) = a_x(\theta)p_x^2 + b_x(\theta)x p_x + c_x(\theta)x^2 + a_y(\theta)p_y^2 + b_y(\theta)y p_y + c_y(\theta)y^2 + H^{(3)} + H^{(4)} + H^{(5)} + \cdots ,$$

where

$$H^{(m)} = \sum_{k l k' l' m} H^{(m)}_{k l k' l' m}\, e^{i m N \theta} x^k p_x^l y^{k'} p_y^{l'} ,$$

and

$$k + l + k' + l' = m = 3, 4, 5, \cdots ,$$
$$m = 0, \pm 1, \pm 2, \pm 3, \cdots .$$

H is periodic in θ with a period of $\frac{2\pi}{N}$. N is the number of sectors of the machine.

Moser, Sturrock, and Hagedorn[1] have devised a method of determining the effects of resonances by simplifying the equations of motion through a series of canonical transformations. The first transformation eliminates θ from the quadratic terms, and introduces complex variables.[2] This transformation has the form

$$x = \frac{1}{2}(\phi_x \zeta_0 + \phi_x^* \bar{\zeta}_0) \qquad y = \frac{1}{2}(\phi_Y \eta_0 + \phi_Y^* \bar{\eta}_0)$$

$$p_x = \frac{1}{2i}(\gamma_x \zeta_0 - \gamma_x^* \bar{\zeta}_0) \qquad p_Y = \frac{1}{2i}(\gamma_Y \eta_0 - \gamma_Y^* \bar{\eta}_0)$$

ϕ_X, ϕ_Y, ψ_X, and ψ_Y are periodic functions in θ with period $\frac{2\pi}{N}$, and are obtained by finding the particular linear solutions x_1, p_{x_1}, y_1, and p_{y_1} of the Hamiltonian where

$$x_1 = \phi_x e^{-i\nu_x\theta}$$
$$p_{x_1} = -i\psi_x e^{-i\nu_x\theta}$$
$$y_1 = \phi_Y e^{-i\nu_Y\theta}$$
$$p_{Y_1} = -i\psi_Y e^{-i\nu_Y\theta}$$

Symon has shown that the real parts of $\psi_x \phi_x^*$ and $\psi_Y \phi_Y^*$ are constants. ϕ_x, ψ_x, ϕ_Y, and ψ_Y are normalized by requiring that

$$\text{Real}\left[\psi_x \phi_x^*\right] = 1.$$

$$\text{Real}\left[\psi_Y \phi_Y^*\right] = 1.$$

This fixes ϕ_x, ψ_x, ϕ_Y, and ψ_Y with the exception of an arbitrary phase factor which is of no importance.

The new Hamiltonian is

$$\begin{aligned}\Omega_0(\zeta_0, \bar{\zeta}_0, \eta_0, \bar{\eta}_0, \theta) = {} & -i\nu_x \zeta_0\bar{\zeta}_0 - i\nu_y \eta_0\bar{\eta}_0 \\ & -2i\sum_m H^{(m)}\Big(\tfrac{1}{2}(\phi_x\zeta_0 + \phi_x^*\bar{\zeta}_0), \tfrac{1}{2}(\phi_Y\eta_0 + \phi_Y^*\bar{\eta}_0), \\ & \qquad \tfrac{1}{2i}(\psi_x\zeta_0 - \psi_x^*\bar{\zeta}_0), \tfrac{1}{2i}(\psi_Y\eta_0 - \psi_Y^*\bar{\eta}_0), \theta\Big)\end{aligned}$$

$$\Omega_0 = -i\nu_x\zeta_0\bar{\zeta}_0 - i\nu_y\eta_0\bar{\eta}_0 + \Omega_0^{(3)} + \Omega_0^{(4)} + \Omega_0^{(5)} + \dots$$

where

$$\Omega_0^{(m)}(\zeta_0, \bar{\zeta}_0, \eta_0, \bar{\eta}_0, \theta) = \sum \Omega_{0,klk'l'm}^{(n)} e^{imN\theta} \zeta_0^{k} \bar{\zeta}_0^{l} \eta_0^{k'} \bar{\eta}_0^{l'} ,$$

and

$$k+l+k'+l' = n = 3, 4, 5, \ldots ,$$
$$m = 0, \pm 1, \pm 2, \pm 3, \ldots .$$

It is apparent that ζ_0, $\bar{\zeta}_0$ and η_0, $\bar{\eta}_0$ are complex conjugates, and that Ω_0 is pure imaginary since H is real.

The next transformation removes all cubic terms except those driving the resonances under study. The only restriction is that all of the resonant lines whose driving terms are retained must intersect at a single point in the ν_x, ν_y diagram. This transformation has no effect on the quadratic terms or the cubic terms which are retained. New terms are added to the higher powers, but they are small compared to the original cubic terms if ζ_0, $\bar{\zeta}_0$, η_0, and $\bar{\eta}_0$ are small. Part I of this paper performs this transformation.

Next all fourth and fifth power terms which do not contribute to the resonances under consideration are transformed away. Again the quadratic, cubic, and nontransformed fourth power terms are not affected, but new terms are added to the higher powers. Part II of this paper gives this transformation.

This process can be repeated any number of times, although the transformations become very complicated. Eventually the process is stopped, and the terms of all orders above the last which has been transformed are truncated.

PART I.

The first task is to eliminate the non-resonant terms from the cubic part of the Hamiltonian

$$\Omega_0 = -i\nu_x \zeta_0 \bar{\zeta}_0 - i\nu_y \eta_0 \bar{\eta}_0 + \Omega_0^{(3)} + \Omega_0^{(4)} + \Omega_0^{(5)} + \cdots \quad .$$

The following notation will be used

$$\Omega^{(m)}_{0,k_1 l_1 k_1' l_1'} = \Omega^{(m)}_{0,1} \quad ,$$

$$\sum_{k_1 l_1 k_1' l_1'} \Omega^{(m)}_{0,k_1 l_1 k_1' l_1'} = \sum_{1} \Omega^{(m)}_{0,1} \quad ,$$

$$\sum_{12} \Omega^{(m)}_{0,1} \Omega^{(m)}_{0,2} = \sum_{\substack{k_1 l_1 k_1' l_1' \\ k_2 l_2 k_2' l_2'}} \Omega^{(m)}_{0,k_1 l_1 k_1' l_1'} \Omega^{(m)}_{0\, k_2 l_2 k_2' l_2'} \quad ,$$

$$\Omega^{(m)}_{0,1} = \sum_{m_1=-\infty}^{\infty} \Omega_{0,1\,m_1} e^{i m_1 N\theta} , \qquad m_1 = 0, \pm 1, \pm 2, \pm 3, \cdots ,$$

$$k_1 + l_1 + k_1' + l_1' = k_2 + l_2 + k_2' + l_2' = k_3 + l_3 + k_3' + l_3' = 3 ,$$

$$k_4 + l_4 + k_4' + l_4' = 4 ,$$

$$k_5 + l_5 + k_5' + l_5' = 5 , \qquad \text{AND} \qquad k \geq 0, l \geq 0, k' \geq 0, l' \geq 0 \; .$$

The first canonical transformation has the generating function

$$S_0^{(3)} = \zeta_0 \bar{\zeta}_0' + \eta_0 \bar{\eta}_0' + \sum_1 S_{0,1}^{(3)} \zeta_0^{k_1} \bar{\zeta}_0^{l_1} \eta_0^{k_1'} \bar{\eta}_0^{l_1'}$$

where

$$S_{0,1}^{(3)} = \sum_{m_1=-\infty}^{\infty} S_{0,1\,m_1}^{(3)} e^{i m_1 \theta}$$

and

$$S_{0,1m_1}^{(3)} = \frac{i\,\Omega^{(3)}_{0,1\,m_1}}{m_1 - (k_1 - l_1)\nu_x - (k_1' - l_1')\nu_y}$$

for non-resonant terms. For resonant terms

$$m_1 = (k_1 - l_1)\nu_x + (k_1' - l_1')\nu_y$$

and on the resonance line $S^{(3)}_{0,1m_1}$ would go to infinity. In this case take $S^{(3)}_{0,1m_1} = 0$. This will leave the resonant $\Omega_{0,1m_1}$ as a cubic term in the new Hamiltonian.

ζ_0' and η_0' are the new space coordinates, while $\bar{\zeta}_0'$ and $\bar{\eta}_0'$ are the new momenta. For this type of generating function

$$\bar{\zeta}_0 = \frac{\partial S_0^{(3)}}{\partial \zeta_0}$$

$$\bar{\eta}_0 = \frac{\partial S_0^{(3)}}{\partial \eta_0}$$

$$\zeta_0' = \frac{\partial S_0^{(3)}}{\partial \bar{\zeta}_0'}$$

$$\eta_0' = \frac{\partial S_0^{(3)}}{\partial \bar{\eta}_0'}$$

ζ_0, $\bar{\zeta}_0$, η_0, and $\bar{\eta}_0$ are solved in terms of the new variables by an iterative process. As an example

$$\zeta_0' = \frac{\partial S_0^{(3)}}{\partial \bar{\zeta}_0'} = \zeta_0 + \sum_1 \ell_1 S_{0,1}^{(3)} \zeta_0^{k_1} \bar{\zeta}_0'^{\ell_1} \eta_0^{k_1'} \bar{\eta}_0'^{\ell_1'}$$

since in the notation being used $k_1 + \ell_1 + k_1' + \ell_1' = 3$ the second term on the right is a quadratic in the variables. If they are small $\zeta_0 \approx \zeta_0'$, and the quadratic can be considered as a small correction. So:

$$\begin{aligned}
\zeta_0 &= \zeta_0' - \sum_1 \ell_1 S_{0,1}^{(3)} \zeta_0^{k_1} \bar{\zeta}_0'^{\ell_1 - 1} \eta_0^{k_1'} \bar{\eta}_0'^{\ell_1'} \\
&= \zeta_0' - \sum_1 \ell_1 S_{0,1}^{(3)} \left[\zeta_0' - \sum_2 \ell_2 S_{0,2}^{(3)} \zeta_0^{k_2} \zeta_0'^{\ell_2 - 1} \eta_0^{k_2'} \bar{\eta}_0'^{\ell_2'}\right]^{k_1} \bar{\zeta}_0'^{\ell_1 - 1} \\
&\quad \cdot \left[\eta_0' - \sum_2 \ell_2' S_{0,2}^{(3)} \zeta_0^{k_2} \bar{\zeta}_0'^{\ell_2} \eta_0^{k_2'} \bar{\eta}_0^{\ell_2' - 1}\right]^{k_1'} \bar{\eta}_0'^{\ell_1'}
\end{aligned}$$

This expression can be expanded in a power series. If fifth power terms in the Hamiltonian are to be considered it will be necessary to develop ζ_0 to fourth degree terms in the new variables. There is one difficulty. ζ_0, $\bar{\zeta}_0$ and η_0, $\bar{\eta}_0$ were complex conjugate variables, and it would be convenient if the new variables were too. However when $\bar{\zeta}_0{}'$ is compared with $\zeta_0{}'^{*}$ it is found that the third and fourth degree terms are not equal. The generating function

$$
\begin{aligned}
F = {} & \zeta_0{}' \zeta_0{}'' + \eta_0{}' \bar{\eta}_0{}'' \\
& + \frac{1}{2} \sum_{12} k_1 l_2 S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_0{}'^{\,k_1+k_2-1} \bar{\zeta}_0{}''^{\,l_1+l_2-1} \eta_0{}'^{\,k_1'+k_2'} \bar{\eta}_0{}''^{\,l_1'+l_2'} \\
& + \frac{1}{2} \sum k_1' l_2' S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_0{}'^{\,k_1+k_2} \bar{\zeta}_0{}''^{\,l_1+l_2} \eta_0{}'^{\,k_1'+k_2'-1} \bar{\eta}_0{}''^{\,l_1'+l_2'-1}
\end{aligned}
$$

produces variables which are complex conjugates through third degree terms, and

$$
\begin{aligned}
G = {} & \zeta_0{}'' \bar{\zeta}_3 + \eta_0{}'' \bar{\eta}_3 \\
& + \frac{1}{4} \sum_{123} l_1 k_2 [(l_1-1)k_3 - (k_2-1) l_3] S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_0{}''^{\,k_1+k_2+k_3-2} \bar{\zeta}_3^{\,l_1+l_2+l_3-2} \\
& \qquad \cdot \eta_0{}''^{\,k_1'+k_2'+k_3'} \bar{\eta}_3^{\,l_1'+l_2'+l_3'} \\
& + \frac{1}{2} \sum_{123} l_1 k_2' (k_3 l_1' - k_2 l_3') S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_0{}''^{\,k_1+k_2+k_3-1} \bar{\zeta}_3^{\,l_1+l_2+l_3-1} \\
& \qquad \cdot \eta_0{}''^{\,k_1'+k_2'+k_3'-1} \bar{\eta}_3^{\,l_1'+l_2'+l_3'-1} \\
& + \frac{1}{4} \sum_{123} l_1' k_2' [(l_1'-1)k_3' - (k_2'-1) l_3'] S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_0{}''^{\,k_1+k_2+k_3} \bar{\zeta}_3^{\,l_1+l_2+l_3} \\
& \qquad \cdot \eta_0{}''^{\,k_1'+k_2'+k_3'-2} \bar{\eta}_3^{\,l_1'+l_2'+l_3'-2}
\end{aligned}
$$

finishes this transformation by producing variables that are complex conjugates through fourth degree terms. These last two transformations do not change the cubic terms of the Hamiltonian.

A series of three new Hamiltonians are produced

$$\Omega_0' = \Omega_0 + \frac{\partial S_0^{(3)}}{\partial \theta}$$

$$\Omega_0'' = \Omega_0' + \frac{\partial F}{\partial \theta}$$

and

$$\Omega_3 = \Omega_0'' + \frac{\partial G}{\partial \theta}$$

The next two pages contain ζ_0 , $\bar{\zeta}_0$, η_0 , and $\bar{\eta}_0$ to fourth degree terms in ζ_3 , $\bar{\zeta}_3$, η_3 , $\bar{\eta}_3$. The following page gives Ω_3 in terms of the new variables and $S_0^{(3)}$. This is followed by the expanded expressions of the new additions to the fourth degree term of the Hamiltonian.

$$
\begin{aligned}
\dot{\zeta}_0 = \zeta_3 &- \sum_{1} l_1 S_{0,1}^{(3)} \zeta_3^{k_1} \bar{\zeta}_3^{l_1-1} \eta_3^{k_1'} \bar{\eta}_3^{l_1'} \\
&+ \frac{1}{2} \sum_{12} k_1 l_2 (l_1 - l_2 + 1) S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2-1} \bar{\zeta}_3^{l_1+l_2-2} \eta_3^{k_1'+k_2'} \bar{\eta}_3^{l_1'+l_2'} \\
&+ \frac{1}{2} \sum_{12} k_1' l_2' (l_1 - l_2) S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2} \bar{\zeta}_3^{l_1+l_2-1} \eta_3^{k_1'+k_2'-1} \bar{\eta}_3^{l_1'+l_2'-1} \\
&- \frac{1}{4} \sum_{123} k_1 l_2 \left[k_3 (l_2 - 1)(l_1 + l_2 + l_3 - 2) + l_1 l_3 (k_1 + k_2 + k_3 - 1) \right] \\
&\qquad \cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-2} \bar{\zeta}_3^{l_1+l_2+l_3-3} \eta_3^{k_1'+k_2'+k_3'} \bar{\eta}_3^{l_1'+l_2'+l_3'} \\
&- \frac{1}{2} \sum_{123} k_1' l_2 \left[l_1 l_3' (k_1 + k_2) + k_3 (l_1 + l_2 - 1)(l_2' + l_3') \right] \\
&\qquad \cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-1} \bar{\zeta}_3^{l_1+l_2+l_3-2} \eta_3^{k_1'+k_2'+k_3'-1} \bar{\eta}_3^{l_1'+l_2'+l_3'-1} \\
&- \frac{1}{4} \sum_{123} k_1' l_2' \left[l_1 l_3' (k_1' + k_2' + k_3' - 1) + k_3' l_2 (l_1' + l_2' + l_3' - 1) \right] \\
&\qquad \cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3} \bar{\zeta}_3^{l_1+l_2+l_3-1} \eta_3^{k_1'+k_2'+k_3'-2} \bar{\eta}_3^{l_1'+l_2'+l_3'-2}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\zeta}_0 = \bar{\zeta}_3 &+ \sum_{1} k_1 S_{0,1}^{(3)} \zeta_3^{k_1-1} \bar{\zeta}_3^{l_1} \eta_3^{k_1'} \bar{\eta}_3^{l_1'} \\
&- \frac{1}{2} \sum_{12} k_1 l_2 (k_1 - k_2 - 1) S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2-2} \bar{\zeta}_3^{l_1+l_2-1} \eta_3^{k_1'+k_2'} \bar{\eta}_3^{l_1'+l_2'} \\
&- \frac{1}{2} \sum_{12} k_1' l_2' (k_1 - k_2) S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2-1} \bar{\zeta}_3^{l_1+l_2} \eta_3^{k_1'+k_2'-1} \bar{\eta}_3^{l_1'+l_2'-1} \\
&+ \frac{1}{4} \sum_{123} k_1 l_2 \left[k_2 k_3 (l_1 + l_2 + l_3 - 1) + (k_1 - 1) l_3 (k_1 + k_2 + k_3 - 2) \right] \\
&\qquad \cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-3} \bar{\zeta}_3^{l_1+l_2+l_3-2} \eta_3^{k_1'+k_2'+k_3'} \bar{\eta}_3^{l_1'+l_2'+l_3'} \\
&+ \frac{1}{2} \sum_{123} k_1 l_2' \left[k_2 k_3' (l_1 + l_2) + l_3 (k_1 + k_2 - 1)(k_1' + k_3') \right] \\
&\qquad \cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-2} \bar{\zeta}_3^{l_1+l_2+l_3-1} \eta_3^{k_1'+k_2'+k_3'-1} \bar{\eta}_3^{l_1'+l_2'+l_3'-1} \\
&+ \frac{1}{4} \sum_{123} k_1' l_2' \left[k_2 k_3' (l_1' + l_2' + l_3' - 1) + k_1 l_3' (k_1' + k_2' + k_3' - 1) \right] \\
&\qquad \cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-1} \bar{\zeta}_3^{l_1+l_2+l_3} \eta_3^{k_1'+k_2'+k_3'-2} \bar{\eta}_3^{l_1'+l_2'+l_3'-2}
\end{aligned}
$$

$$\eta_0 = \eta_3 - \sum_{1} l_1' S_{0,1}^{(3)} \zeta_3^{k_1} \bar{\zeta}_3^{l_1} \eta_3^{k_1'} \bar{\eta}_3^{l_1'-1}$$

$$+ \frac{1}{2} \sum_{12} k_1 l_2 (l_1' - l_2') S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2-1} \bar{\zeta}_3^{l_1+l_2-1} \eta_3^{k_1'+k_2'} \bar{\eta}_3^{l_1'+l_2'-1}$$

$$+ \frac{1}{2} \sum_{12} k_1' l_2' (l_1' - l_2' + 1) S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2} \bar{\zeta}_3^{l_1+l_2} \eta_3^{k_1'+k_2'-1} \bar{\eta}_3^{l_1'+l_2'-2}$$

$$- \frac{1}{4} \sum_{123} k_1' l_2' \left[k_3' (l_2'-1)(l_1'+l_2'+l_3'-2) + l_1' l_3' (k_1'+k_2'+k_3'-1) \right]$$
$$\cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3} \bar{\zeta}_3^{l_1+l_2+l_3} \eta_3^{k_1'+k_2'+k_3'-2} \bar{\eta}_3^{l_1'+l_2'+l_3'-3}$$

$$- \frac{1}{2} \sum_{123} k_1 l_2' \left[l_1' l_3 (k_1'+k_2') + k_3' (l_2+l_3)(l_1'+l_2'-1) \right]$$
$$\cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-1} \bar{\zeta}_3^{l_1+l_2+l_3-1} \eta_3^{k_1'+k_2'+k_3'-1} \bar{\eta}_3^{l_1'+l_2'+l_3'-2}$$

$$- \frac{1}{4} \sum_{123} k_1 l_2 \left[l_1' l_3 (k_1+k_2+k_3-1) + k_3 l_2' (l_1+l_2+l_3-1) \right]$$
$$\cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-2} \bar{\zeta}_3^{l_1+l_2+l_3-2} \eta_3^{k_1'+k_2'+k_3'} \bar{\eta}_3^{l_1'+l_2'+l_3'-1}$$

$$\bar{\eta}_0 = \bar{\eta}_3 + \sum_{1} k_1' S_{0,1}^{(3)} \zeta_2^{k_1} \bar{\zeta}_3^{l_1} \eta_3^{k_1'-1} \bar{\eta}_3^{l_1'}$$

$$- \frac{1}{2} \sum_{12} k_1 l_2 (k_1' - k_2') S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2-1} \bar{\zeta}_3^{l_1+l_2-1} \eta_3^{k_1'+k_2'-1} \bar{\eta}_3^{l_1'+l_2'}$$

$$- \frac{1}{2} \sum_{12} k_1' l_2' (k_1' - k_2' - 1) S_{0,1}^{(3)} S_{0,2}^{(3)} \zeta_3^{k_1+k_2} \bar{\zeta}_3^{l_1+l_2} \eta_3^{k_1'+k_2'-2} \bar{\eta}_3^{l_1'+l_2'-1}$$

$$+ \frac{1}{4} \sum_{123} k_1' l_2' \left[(k_1'-1) l_3' (k_1'+k_2'+k_3'-2) + k_2' k_3' (l_1'+l_2'+l_3'-1) \right]$$
$$\cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_2^{k_1+k_2+k_3} \bar{\zeta}_3^{l_1+l_2+l_3} \eta_3^{k_1'+k_2'+k_3'-3} \bar{\eta}_3^{l_1'+l_2'+l_3'-2}$$

$$+ \frac{1}{2} \sum_{123} k_1' l_2 \left[k_2' k_3 (l_1'+l_2') + l_3' (k_1+k_3)(k_1'+k_2'-1) \right]$$
$$\cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-1} \bar{\zeta}_3^{l_1+l_2+l_3-1} \eta_3^{k_1'+k_2'+k_3'-2} \bar{\eta}_3^{l_1'+l_2'+l_3'-1}$$

$$+ \frac{1}{4} \sum_{123} k_1 l_2 \left[k_2' k_3 (l_1+l_2+l_3-1) + k_1' l_3 (k_1+k_2+k_3-1) \right]$$
$$\cdot S_{0,1}^{(3)} S_{0,2}^{(3)} S_{0,3}^{(3)} \zeta_3^{k_1+k_2+k_3-2} \bar{\zeta}_3^{l_1+l_2+l_3-2} \eta_3^{k_1'+k_2'+k_3'-1} \bar{\eta}_3^{l_1'+l_2'+l_3'}$$

$$\delta_1 = 1 \quad \text{if} \quad \Omega_{0,1m_1} \text{ is a resonant term}$$
$$= 0 \quad \text{otherwise}$$

$$\Omega_3 = -i\nu_x \zeta_3 \bar{\zeta}_3 - i\nu_y \eta_3 \bar{\eta}_3 + \Omega_3^{(3)} + \Omega_3^{(4)} + \Omega_3^{(5)} + \cdots$$

$$\Omega_3^{(3)} = \sum_{1,m_1} \delta_1 \Omega^{(3)}_{0,1m_1} e^{im_1N\theta} \zeta_3^{k_1} \bar{\zeta}_3^{l_1} \eta_3^{k'_1} \bar{\eta}_3^{l'_1}$$

$$\begin{aligned}
\Omega_3^{(4)} = & \sum_{4,m_4} \Omega^{(4)}_{0,4,m_4} e^{im_4N\theta} \zeta_3^{k_4} \bar{\zeta}_3^{l_4} \eta_3^{k'_4} \bar{\eta}_3^{l'_4} \\
& + \frac{1}{2} \sum_{12m_1m_4} (1+\delta_1)(k_2 l_1 - k_1 l_2) \Omega^{(3)}_{0,1m_1} S^{(3)}_{0,2,m_4-m_1} e^{im_4N\theta} \zeta_3^{k_1+k_2-1} \bar{\zeta}_3^{l_1+l_2-1} \eta_3^{k'_1+k'_2} \bar{\eta}_3^{l'_1+l'_2} \\
& + \frac{1}{2} \sum_{12m_1m_4} (1+\delta_1)(k'_2 l'_1 - k'_1 l'_2) \Omega^{(3)}_{0,1m_1} S^{(3)}_{0,2,m_4-m_1} e^{im_4N\theta} \zeta_3^{k_1+k_2} \bar{\zeta}_3^{l_1+l_2} \eta_3^{k'_1+k'_2-1} \bar{\eta}_3^{l'_1+l'_2-1}
\end{aligned}$$

$$\begin{aligned}
\Omega_3^{(5)} = & \sum_{5,m_5} \Omega^{(5)}_{0.5,m_5} e^{im_5N\theta} \zeta_3^{k_5} \bar{\zeta}_3^{l_5} \eta_3^{k'_5} \bar{\eta}_3^{l'_5} \\
& + \sum_{14m_4m_5} (k_1 l_4 - k_4 l_1) \Omega^{(4)}_{0,4m_4} S^{(3)}_{0,1,m_5-m_4} e^{im_5N\theta} \zeta_3^{k_1+k_4-1} \bar{\zeta}_3^{l_1+l_4-1} \eta_3^{k'_1+k'_4} \bar{\eta}_3^{l'_1+l'_4} \\
& + \sum_{14m_4m_5} (k'_1 l'_4 - k'_4 l'_1) \Omega^{(4)}_{0,4m_4} S^{(3)}_{0,1,m_5-m_4} e^{im_5N\theta} \zeta_3^{k_1+k_4} \bar{\zeta}_3^{l_1+l_4} \eta_3^{k'_1+k'_4-1} \bar{\eta}_3^{l'_1+l'_4-1} \\
& + \frac{1}{4} \sum_{123m_1m_2m_5} \Big\{ \big[(1+\delta_1)k_1 l_2 - 2k_2 l_1\big] l_3(k_1+k_2+k_3-1) + \big[(1+\delta_1)k_2 l_1 - 2k_1 l_2\big] k_3(l_1+l_2+l_3-1) \Big\} \\
& \qquad \cdot \Omega^{(3)}_{0,1m_1} S^{(3)}_{0,2m_2} S^{(3)}_{0,3,m_5-m_1-m_2} e^{im_5N\theta} \zeta_3^{k_1+k_2+k_3-2} \bar{\zeta}_3^{l_1+l_2+l_3-2} \eta_3^{k'_1+k'_2+k'_3} \bar{\eta}_3^{l'_1+l'_2+l'_3} \\
& + \frac{1}{2} \sum_{123m_1m_2m_5} \Big\{ k_2 l_1 \big[k'_3(l'_1+\delta_1 l'_2) - l'_3(k'_1+k'_2)\big] - k_1 l_2 \big[k'_3(l'_1+l'_2) - \delta_1 l'_3(k'_1+k'_2)\big] \\
& \qquad + k'_2 l'_1 \big[\delta_1 k_3(l_1+l_2) - l_3(k_1+k_2)\big] - k'_1 l'_2 \big[k_3(l_1+l_2) - l_3(k_1+\delta_1 k_2)\big] \Big\} \\
& \qquad \cdot \Omega^{(3)}_{0,1m_1} S^{(3)}_{0,2m_2} S^{(3)}_{0,3,m_5-m_1-m_2} e^{im_5N\theta} \zeta_3^{k_1+k_2+k_3-1} \bar{\zeta}_3^{l_1+l_2+l_3-1} \eta_3^{k'_1+k'_2+k'_3-1} \bar{\eta}_3^{l'_1+l'_2+l'_3-1} \\
& + \frac{1}{4} \sum_{123m_1m_2m_5} \Big\{ \big[(1+\delta_1)k'_1 l'_2 - k'_2 l'_1\big] l'_3(k'_1+k'_2+k'_3-1) + \big[(1+\delta_1)k'_2 l'_1 - k'_1 l'_2\big] k'_3(l'_1+l'_2+l'_3-1) \Big\} \\
& \qquad \cdot \Omega^{(3)}_{0,1m_1} S^{(3)}_{0,2m_2} S^{(3)}_{0,3m_5-m_1-m_2} e^{im_5N\theta} \zeta_3^{k_1+k_2+k_3} \bar{\zeta}_3^{l_1+l_2+l_3} \eta_3^{k'_1+k'_2+k'_3-2} \bar{\eta}_3^{l'_1+l'_2+l'_3-2}
\end{aligned}$$

Expanded Fourth Degree Terms

This section gives explicitly the terms in $\Omega^{(4)}_{3,\,k_4 l_4 k_4' l_4'}$ as a sum of terms of the type $c_{k_4 l_4 k_4' l_4'}\, \Omega_{k_1 l_1 k_1' l_1'}\, S_{k_2 l_2 k_2' l_2'}$. Ω and S are the terms which have been called $\Omega^{(3)}_{0,1}$ and $S^{(3)}_{0,1}$ up to this time. If $\Omega_{k_1 l_1 k_1' l_1'}$ contains a resonant term, the given $c_{k_4 l_4 k_4' l_4'}$ must be doubled for this resonant term. Note, however, that if all cubic resonant lines which go through ν_{x_0}, ν_{y_0} have been retained in the cubic part of the Hamiltonian, and if $\Omega^{(4)}_{3,\,k_4 l_4 k_4' l_4' m_4}$ also passes through ν_{x_0}, ν_{y_0} then each $\Omega^{(3)}_{0,1 m_1}$, which is a cubic resonant term is multiplied by an $S^{(3)}_{0,2,m_4-m_1}$ which is zero. So if one is interested only in finding the fourth degree resonant terms the doubling of $c_{k_4 l_4 k_4' l_4'}$ can be ignored. Only half of the fourth degree terms are given, the others can be found as follows. As an example $\Omega^{(4)}_{0004}$ is not given, it is the same as $-\Omega^{(4)}_{0040}$ with $k_1, l_1;\ k_1', l_1';\ k_2, l_2;$ and k_2', l_2' reversed.

$$\Omega^{(4)}_{0040} = \frac{1}{2}\Omega_{0120} S_{1020} - \frac{1}{2}\Omega_{1020} S_{0120} + \frac{3}{2}\Omega_{0021} S_{0030} - \frac{3}{2}\Omega_{0030} S_{0021}$$

and

$$\Omega^{(4)}_{0004} = -\frac{1}{2}\Omega_{1002} S_{0102} + \frac{1}{2}\Omega_{0102} S_{1002} - \frac{3}{2}\Omega_{0012} S_{0003} + \frac{3}{2}\Omega_{0003} S_{0012}$$

$$\Omega^{(4)}_{4000} = \tfrac{3}{2}\Omega_{2100}S_{3000} - \tfrac{3}{2}\Omega_{3000}S_{2100} + \tfrac{1}{2}\Omega_{2001}S_{2010} - \tfrac{1}{2}\Omega_{2010}S_{2001}$$

$$\Omega^{(4)}_{3100} = 3\Omega_{1200}S_{3000} - 3\Omega_{3000}S_{1200} + \tfrac{1}{2}\Omega_{2001}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{2001} + \tfrac{1}{2}\Omega_{1101}S_{2010} - \tfrac{1}{2}\Omega_{2010}S_{1101}$$

$$\Omega^{(4)}_{2200} = \tfrac{9}{2}\Omega_{0300}S_{3000} - \tfrac{9}{2}\Omega_{3000}S_{0300} + \tfrac{3}{2}\Omega_{1200}S_{2100} - \tfrac{3}{2}\Omega_{2100}S_{1200} + \tfrac{1}{2}\Omega_{0201}S_{2010} - \tfrac{1}{2}\Omega_{2010}S_{0201} + \tfrac{1}{2}\Omega_{2001}S_{0210} - \tfrac{1}{2}\Omega_{0210}S_{2001} + \tfrac{1}{2}\Omega_{1101}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{1101}$$

$$\Omega^{(4)}_{3010} = \tfrac{3}{2}\Omega_{1110}S_{3000} - \tfrac{3}{2}\Omega_{3000}S_{1110} + \Omega_{2100}S_{2010} - \Omega_{2010}S_{2100} + \Omega_{2001}S_{1020} - \Omega_{1020}S_{2001} + \tfrac{1}{2}\Omega_{1011}S_{2010} - \tfrac{1}{2}\Omega_{2010}S_{1011}$$

$$\Omega^{(4)}_{2110} = 3\Omega_{0210}S_{3000} - 3\Omega_{3000}S_{0210} + 2\Omega_{1200}S_{2010} - 2\Omega_{2010}S_{1200} + \tfrac{1}{2}\Omega_{1110}S_{2100} - \tfrac{1}{2}\Omega_{2100}S_{1110} + \Omega_{2001}S_{0120} - \Omega_{0120}S_{2001} + \Omega_{1101}S_{1020} - \Omega_{1020}S_{1101} + \tfrac{1}{2}\Omega_{1011}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{1011} + \tfrac{1}{2}\Omega_{0111}S_{2010} - \tfrac{1}{2}\Omega_{2010}S_{0111}$$

$$\Omega^{(4)}_{1210} = 3\Omega_{0300}S_{2010} - 3\Omega_{2010}S_{0300} + 2\Omega_{0210}S_{2100} - 2\Omega_{2100}S_{0210} + \tfrac{1}{2}\Omega_{1200}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{1200} + \Omega_{1101}S_{0120} - \Omega_{0120}S_{1101} + \Omega_{0201}S_{1020} - \Omega_{1020}S_{0201} + \tfrac{1}{2}\Omega_{0111}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{0111} + \tfrac{1}{2}\Omega_{1011}S_{0210} - \tfrac{1}{2}\Omega_{0210}S_{1011}$$

$$\Omega^{(4)}_{0310} = \tfrac{3}{2}\Omega_{0300}S_{1110} - \tfrac{3}{2}\Omega_{1110}S_{0300} + \Omega_{0210}S_{1200} - \Omega_{1200}S_{0210} + \Omega_{0201}S_{0120} - \Omega_{0120}S_{0201} + \tfrac{1}{2}\Omega_{0111}S_{0210} - \tfrac{1}{2}\Omega_{0210}S_{0111}$$

$$\Omega^{(4)}_{2020} = \tfrac{3}{2}\Omega_{0120}S_{3000} - \tfrac{3}{2}\Omega_{3000}S_{0120} + \Omega_{1110}S_{2010} - \Omega_{2010}S_{1110} + \tfrac{1}{2}\Omega_{2100}S_{1020} - \tfrac{1}{2}\Omega_{1020}S_{2100} + \tfrac{3}{2}\Omega_{2001}S_{0030} - \tfrac{3}{2}\Omega_{0030}S_{2001} + \Omega_{1011}S_{1020} - \Omega_{1020}S_{1011} + \tfrac{1}{2}\Omega_{0021}S_{2010} - \tfrac{1}{2}\Omega_{2010}S_{0021}$$

$$\Omega^{(4)}_{1120} = 2\Omega_{0210}S_{2010} - 2\Omega_{2010}S_{0210} + \Omega_{0120}S_{2100} - \Omega_{2100}S_{0120}$$
$$+ \Omega_{1200}S_{1020} - \Omega_{1020}S_{1200} + \Omega_{1011}S_{0120} - \Omega_{0120}S_{1011}$$
$$+ \Omega_{0111}S_{1020} - \Omega_{1020}S_{0111} + \tfrac{3}{2}\Omega_{1101}S_{0030} - \tfrac{3}{2}\Omega_{0030}S_{1101}$$
$$+\tfrac{1}{2}\Omega_{0021}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{0021}$$

$$\Omega^{(4)}_{0220} = \tfrac{3}{2}\Omega_{0300}S_{1020} - \tfrac{3}{2}\Omega_{1020}S_{0300} + \Omega_{0210}S_{1110} - \Omega_{1110}S_{0210}$$
$$+\tfrac{1}{2}\Omega_{0120}S_{1200} - \tfrac{1}{2}\Omega_{1200}S_{0120} + \tfrac{3}{2}\Omega_{0201}S_{0030} - \tfrac{3}{2}\Omega_{0030}S_{0201}$$
$$+ \Omega_{0111}S_{0120} - \Omega_{0120}S_{0111} + \tfrac{1}{2}\Omega_{0021}S_{0210} - \tfrac{1}{2}\Omega_{0210}S_{0021}$$

$$\Omega^{(4)}_{2011} = \tfrac{3}{2}\Omega_{0011}S_{3000} - \tfrac{3}{2}\Omega_{3000}S_{0011} + \Omega_{1101}S_{2010} - \Omega_{2010}S_{1101}$$
$$+ \Omega_{1110}S_{2001} - \Omega_{2001}S_{1110} + \tfrac{1}{2}\Omega_{2100}S_{1011} - \tfrac{1}{2}\Omega_{1011}S_{2100}$$
$$+2\Omega_{1002}S_{1020} - 2\Omega_{1020}S_{1002} + \Omega_{2001}S_{0021} - \Omega_{0021}S_{2001}$$
$$+ \Omega_{0012}S_{2010} - \Omega_{2010}S_{0012}$$

$$\Omega^{(4)}_{1111} = 2\Omega_{0201}S_{2010} - 2\Omega_{2010}S_{0201} + 2\Omega_{0210}S_{2001} - 2\Omega_{2001}S_{0210}$$
$$+ \Omega_{1200}S_{1011} - \Omega_{1011}S_{1200} + \Omega_{0111}S_{2100} - \Omega_{2100}S_{0111}$$
$$+2\Omega_{1002}S_{0120} - 2\Omega_{0120}S_{1002} + 2\Omega_{0102}S_{1020} - 2\Omega_{1020}S_{0102}$$
$$+ \Omega_{1101}S_{0021} - \Omega_{0021}S_{1101} + \Omega_{0012}S_{1110} - \Omega_{1110}S_{0012}$$

$$\Omega^{(4)}_{0211} = \tfrac{3}{2}\Omega_{0300}S_{1011} - \tfrac{3}{2}\Omega_{1011}S_{0300} + \Omega_{0210}S_{1101} - \Omega_{1101}S_{0210}$$
$$+ \Omega_{0201}S_{1110} - \Omega_{1110}S_{0201} + \tfrac{1}{2}\Omega_{0111}S_{1200} - \tfrac{1}{2}\Omega_{1200}S_{0111}$$
$$+2\Omega_{0102}S_{0120} - 2\Omega_{0120}S_{0102} + \Omega_{0201}S_{0021} - \Omega_{0021}S_{0201}$$
$$+ \Omega_{0012}S_{0210} - \Omega_{0210}S_{0012}$$

$$\Omega^{(4)}_{1030} = \tfrac{1}{2}\Omega_{1110}S_{1020} - \tfrac{1}{2}\Omega_{1020}S_{1110} + \Omega_{0120}S_{2010} - \Omega_{2010}S_{0120}$$
$$+\tfrac{3}{2}\Omega_{1011}S_{0030} - \tfrac{3}{2}\Omega_{0030}S_{1011} + \Omega_{0021}S_{1020} - \Omega_{1020}S_{0021}$$

$$\Omega^{(4)}_{0130} = \Omega_{0210}S_{1020} - \Omega_{1020}S_{0210} + \tfrac{1}{2}\Omega_{0120}S_{1110} - \tfrac{1}{2}\Omega_{1110}S_{0120}$$
$$+\tfrac{3}{2}\Omega_{0111}S_{0030} - \tfrac{3}{2}\Omega_{0030}S_{0111} + \Omega_{0021}S_{0120} - \Omega_{0120}S_{0021}$$

$$\begin{aligned}\Omega_{1021} = {} & \Omega_{0111} S_{2010} - \Omega_{2010} S_{0111} + \Omega_{0120} S_{2001} - \Omega_{2001} S_{0120} \\ & + \tfrac{1}{2}\Omega_{1110} S_{1011} + \tfrac{1}{2}\Omega_{1101} S_{1020} - \tfrac{1}{2}\Omega_{1020} S_{1101} - \tfrac{1}{2}\Omega_{1011} S_{1110} \\ & + 3\Omega_{1002} S_{0030} - 3\Omega_{0030} S_{1002} + 2\Omega_{0012} S_{1020} - 2\Omega_{1020} S_{0012} \\ & + \tfrac{1}{2}\Omega_{1011} S_{0021} - \tfrac{1}{2}\Omega_{0021} S_{1011}\end{aligned}$$

$$\begin{aligned}\Omega_{0121} = {} & \Omega_{0210} S_{1011} - \Omega_{1011} S_{0210} + \Omega_{0201} S_{1020} - \Omega_{1020} S_{0201} \\ & + \tfrac{1}{2}\Omega_{0120} S_{1101} - \tfrac{1}{2}\Omega_{1101} S_{0120} + \tfrac{1}{2}\Omega_{0111} S_{1110} - \tfrac{1}{2}\Omega_{1110} S_{0111} \\ & + 3\Omega_{0102} S_{0030} - 3\Omega_{0030} S_{0102} + 2\Omega_{0012} S_{0120} - 2\Omega_{0120} S_{0012} \\ & + \tfrac{1}{2}\Omega_{0111} S_{0021} - \tfrac{1}{2}\Omega_{0021} S_{0111}\end{aligned}$$

$$\Omega_{0040} = \tfrac{1}{2}\Omega_{0120} S_{1020} - \tfrac{1}{2}\Omega_{1020} S_{0120} + \tfrac{3}{2}\Omega_{0021} S_{0030} - \tfrac{3}{2}\Omega_{0030} S_{0021}$$

$$\begin{aligned}\Omega_{0031} = {} & \tfrac{1}{2}\Omega_{0120} S_{1011} - \tfrac{1}{2}\Omega_{1011} S_{0120} + \tfrac{1}{2}\Omega_{0111} S_{1020} - \tfrac{1}{2}\Omega_{1020} S_{0111} \\ & + 3\Omega_{0012} S_{0030} - 3\Omega_{0030} S_{0012}\end{aligned}$$

$$\begin{aligned}\Omega_{0022} = {} & \tfrac{1}{2}\Omega_{0102} S_{1020} - \tfrac{1}{2}\Omega_{1020} S_{0102} + \tfrac{1}{2}\Omega_{0111} S_{1011} - \tfrac{1}{2}\Omega_{1011} S_{0111} \\ & + \tfrac{1}{2}\Omega_{0120} S_{1002} - \tfrac{1}{2}\Omega_{1002} S_{0120} + \tfrac{9}{2}\Omega_{0003} S_{0030} - \tfrac{9}{2}\Omega_{0030} S_{0003} \\ & + \tfrac{3}{2}\Omega_{0012} S_{0021} - \tfrac{3}{2}\Omega_{0021} S_{0012}\end{aligned}$$

PART II.

When the canonical transformation is applied to the fourth degree terms, there is no change in the lower order terms or in the fifth degree terms. As a result, the fourth and fifth degree terms can be transformed away together with the generating function

$$S = \zeta_3 \bar{\zeta} + \eta_3 \bar{\eta} + \sum_4 S_4 \zeta_3^{k_4} \bar{\zeta}^{l_4} \eta_3^{k_4'} \bar{\eta}^{l_4'} + \sum S_5 \zeta_3^{k_5} \bar{\zeta}_3^{l_5} \eta_3^{k_5'} \bar{\eta}^{l_5'}$$

where

$$S_4 = \sum_{m=-\infty}^{\infty} S_{k_4 l_4 k_4' l_4' m_4} e^{i m_4 \theta}$$

$$S_{4 m_4} = \frac{i \Omega_{4 m_4}}{m_4 - (k_4 - l_4)\nu_x - (k_4' - l_4')\nu_y}$$

if $\Omega_{4 m_4}$ is not a resonant term $S_{4, m_4} = 0$ if $\Omega_{4 m_4}$ is a resonant term. This leaves $\Omega^{(5)} = \Omega^{(3)}$ — all nonresonant terms of fourth and fifth degree.

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