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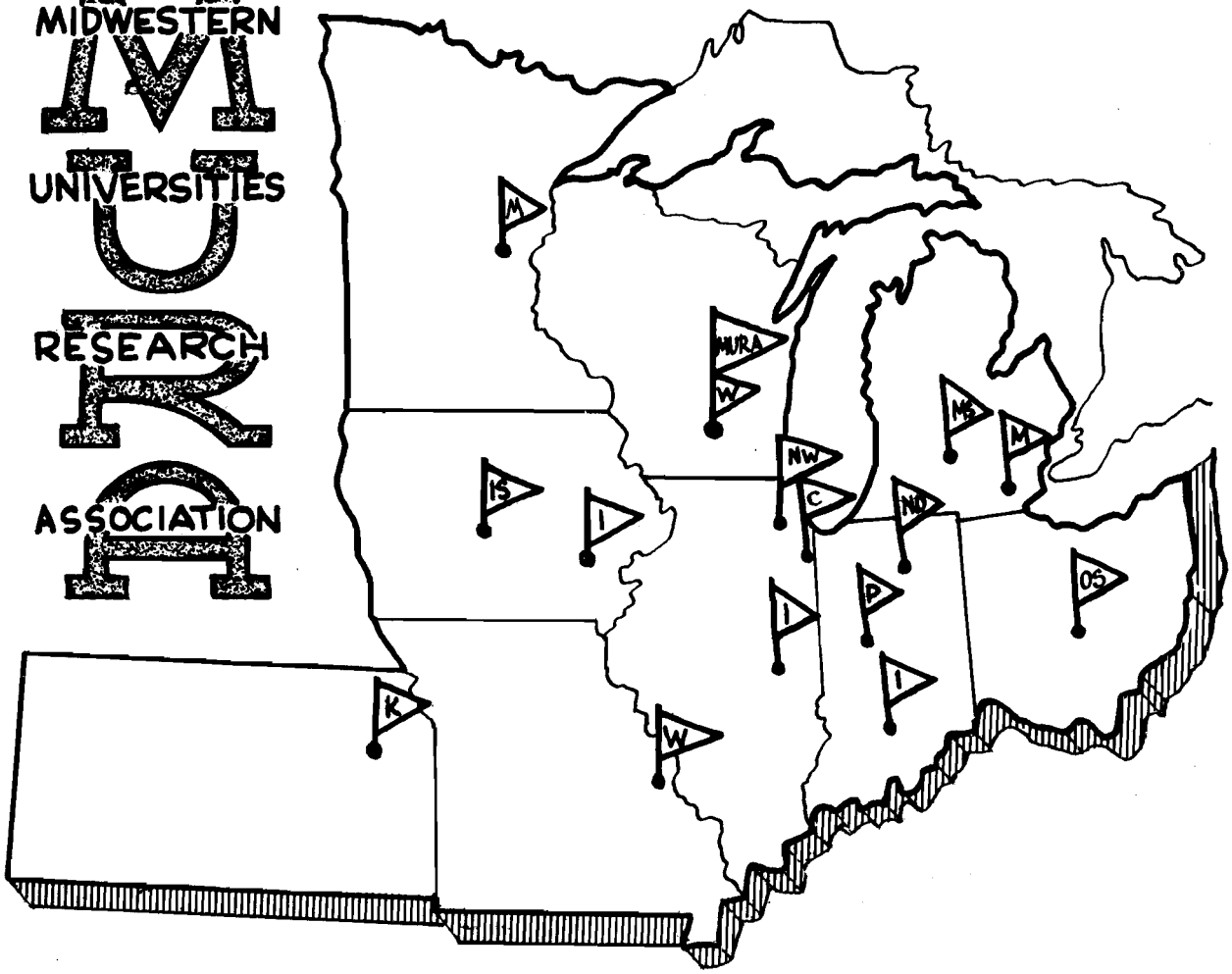

SPACE CHARGE LIMIT ON BEAM CURRENT IN ALTERNATING GRADIENT ACCELERATORS

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ABSTRACT

In alternating gradient accelerators, the beam envelope is not of constant cross-sectional dimensions, so that the usual methods of estimating space charge defocusing forces cannot be applied. However, it is shown that the space charge forces can be represented by Mathieu Equation terms, so that frequencies of betatron oscillation with and without space charge may be found using methods of Parzen in conjunction with the Tables of Laslett, Snyder and Hutchinson. The alternating gradient terms due to space charge are of such a sign as to reduce the defocusing slightly. The methods are applied to the New Model parameters as an example.

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**MURA and University of Michigan

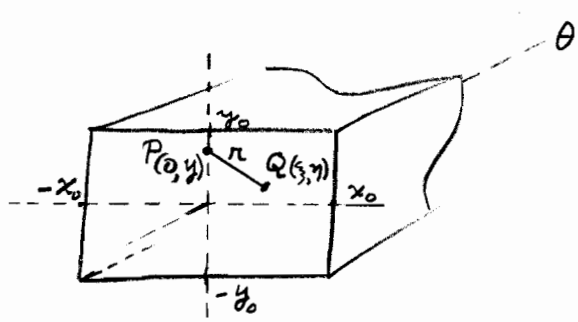
I. METHOD

Space charge forces are first found for a beam of rectangular cross-section and uniform charge distribution integration. The force is almost linear along the axes of the distribution, so that the force can be expressed as a constant, α, times the force due to a uniformly charged cylinder of the same total charge and a diameter the same as the dimension of the rectangle under consideration. Knowing the elements, β, of the linear transfer matrices, the range over which the force constant varies may be found. In first approximation, the force constant may be assumed to vary sinusoidally through a sector.

The coefficients representing this force are then added to those found by Parzen's expressions[1] and resulting σ_s and ν_s found from the Illiac Tables of Mathieu Equation coefficients and solutions[2].

II. ELECTRIC FIELD IN A RECTANGULAR DISTRIBUTION

Assume a beam of uniform charge density moving in the θ - direction (around the machine) and of reactangular dimensions $2x_0$ by $2y_0$.



1. G. Parzen, MURA-273

2. Laslett, Snyder and Hutchinson, "Tables for the Determination of Stability Boundaries and Characteristic Exponents for a Hill's Equation Characterizing the Mark V FFAG Synchrotron", April, 1955.

If the charge density is ρ e.s.u./cm^3, the contribution to the electric field at $P(0,y)$ due to a lineal charge element at $Q(\xi,\eta)$ in dynes/e.s.u. is given by

$$d\vec{E} = 2\rho\, d\xi\, d\eta \frac{\vec{r}}{r^2}$$

or

$$dE_y = \frac{2\rho\, d\xi\, d\eta\, (\eta - y)}{\xi^2 + (\eta - y)^2}.$$

Therefore

$$E_y = 2\rho \int_{-y_0}^{y_0} \int_{-x_0}^{x_0} \frac{(\eta - y)\, d\xi\, d\eta}{\xi^2 + (\eta - y)^2}$$

$$E_y = 2\rho \int_{-x_0}^{x_0} d\xi\, \frac{1}{2} \ln\left[\xi^2 + (\eta - y)^2\right]_{-y_0}^{y}$$

$$E_y = 2\rho\left[x_0 \ln\left(\frac{x_0^2 + (y_0 - y)^2}{x_0^2 + (y_0 + y)^2}\right) + 2(y_0 - y)\tan^{-1}\frac{x_0}{(y_0 - y)} - 2(y_0 + y)\tan^{-1}\left(\frac{x_0}{y_0 + y}\right)\right].$$

Since the charge per unit length, q/l, is given by

$$q/l = 4\rho x_0 y_0,$$

$$\frac{E_y}{2q/l} = \frac{E_y}{8\rho x_0 y_0} = \frac{1}{4y_0}\left[\ln\left(\frac{1 + \left(\frac{y_0 - y}{x_0}\right)^2}{1 + \left(\frac{y_0 + y}{x_0}\right)^2}\right) + 2\left(\frac{y_0 - y}{x_0}\right)\tan^{-1}\left(\frac{x_0}{y_0 - y}\right) - 2\left(\frac{y_0 + y}{x_0}\right)\tan^{-1}\left(\frac{x_0}{y_0 + y}\right)\right].$$

This field is rather linear in y for $x_0 \cong y_0$ (Figure 1). In Table I, $E/8\rho x_0$ is given as a function of y/y_0 for three values of x_0/y_0.

Table I

y/y_0	$x_0/y_0 = 0.5$ $E/8\rho x_0$	$x_0/y_0 = 1$ $E/8\rho x_0$	$x_0/y_0 = 2$ $E/8\rho x_0$
0.1	.095	.078	.055
0.5	.495	.403	.280
1.0	1.19	.872	.565
$\alpha(x_0/y_0)$	.95	.80	.55

From a uniform cylindrical charge distribution of radius y_0

$$\frac{E}{2q/l} = \frac{y}{y_0^2}$$

Therefore in the linear approximation, it is convenient to represent the field for the rectangular dimension in terms of the field for a cylinder of the same charge and the same radius as the coordinate of the rectangle under consideration (in this case y_0). Thus, for the rectangle,

$$\frac{E}{2q/l} = \frac{\alpha(x_0/y_0)\, y}{y_0^2}$$

where the function $\alpha(x_0/y_0)$ is given also in Table I for the three values of x_0/y_0 .

III. EQUATIONS OF MOTION

The y equation of motion due to space charge forces alone for a particle in a cylindrical charge distribution is found in the usual way[3]

3. LJL(MAC)-2, MURA-LWJ-8 (Internal), MURA-208, and others.

$$\frac{d^2y}{dt^2} - \frac{eE}{m} = 0$$

$$\frac{d^2y}{dt^2} - \frac{2me^2y}{m y_0^2} = 0$$

$$\frac{d^2y}{d\theta^2} - \left(\frac{2R^2me^2}{m v^2 y_0^2}\right) y = 0;$$

or $$\frac{d^2y}{d\theta^2} - a_y^0 \frac{y}{y_0^2} = 0 \quad , \text{ where } a_y^0 = \frac{2R^2 e c i}{10 m_0 \gamma^3 v^3}$$

Where i is the beam current in amperes, R is the machine radius and y_0 the beam minor radius in centimeters, $\gamma = E/E_0$, e is in esu , m in grams, and v and c in centimeters per second. The expression for a_y^0 includes relativistic effects.

In the alternating gradient case, the radial dimensions vary by a ratio of $\sqrt{\beta_{x\,max}/\beta_{x\,min}}$ between the center of a radial focusing magnet and a radial defocusing magnet. The ratio of vertical dimensions y_0 between the same two points is given by $\sqrt{\beta_{y\,min}/\beta_{y\,max}}$. Thus a beam will change in x_0/y_0 by

$$\left(\frac{x_0}{y_0}\right)_{max} \Big/ \left(\frac{x_0}{y_0}\right)_{min} = \sqrt{\frac{\beta_{x\,max}\ \beta_{y\,max}}{\beta_{x\,min}\ \beta_{y\,min}}} .$$

Now in the absence of detailed knowledge about couplings, the simplest assumption is that $x_0 = y_0$ for some median, so that

$$\left(\frac{x_0}{y_0}\right)_{max} \times \left(\frac{x_0}{y_0}\right)_{min} = 1$$

The y equation of motion due to space charge forces alone, assuming a beam whose rectangular dimensions are changing according to the known magnetic focusing, is given by

$$\frac{d^2y}{d\theta^2} + \left[-a_y + b_y \cos N\theta \right] y = 0,$$

where a_y and b_y are given by

$$b_y = \frac{a_y^0}{2} \left[\left\{ \frac{\alpha(x_0/y_0)}{y_0^2} \right\}_{y\,min} - \left\{ \frac{\alpha(x_0/y_0)}{y_0^2} \right\}_{max} \right],$$

$$a_y = \frac{a_y^0\, \alpha(1)}{y_0^2}.$$

It may be noticed that the repulsive force in y is greatest when y_0 is small (e.g., in a vertical defocusing sector), therefore the alternating gradient term due to space charge is in phase with the magnetic a. g. term. It is evident that in the radial equation the AG space charge force is also in phase with the magnet AG force.

Parzen[1] gives the coefficients for the vertical equation of motion(due to magnetic forces) as

$$\frac{d^2y}{d\theta^2} + \left[E_0' + 2f_1 \cos N\theta \right] y = 0.$$

In the presence of space charge forces, we have

$$\frac{d^2y}{d\theta^2} + \left[(E_0' - a_y) + (2f_1 + b_y) \cos N\theta \right] y = 0$$

The Illiac Tables[2] give solutions (cosine σ) for the equations

$$\frac{d^2y}{dt^2} + [A + B\cos 2t + C\cos 4t + D\cos 6t]y = 0$$

The σ's and hence ν's for the above equations can then be found from these Tables (setting C = D = 0) and the detuning due to space charge deduced. Note that

$$A = \frac{4E_0'}{N^2} \quad \text{or} \quad \frac{4}{N^2}(E_0' - a_y),$$

$$B = \frac{8g_1}{N^2} \quad \text{or} \quad \frac{4}{N^2}(2g_1 + b_y).$$

IV. NUMERICAL EXAMPLE: 50 Mev ELECTRON FFAG

The parameters for the New Model (one-way operation) are:

N = 16	$\nu_x = 4.3$	$\sigma_x/\pi = .535$
k = 9.3	$\nu_y = 2.3$	$\sigma_y/\pi = .288$
E = 100 kev) injection	$\beta_x\,\text{max} = .47$	$\beta_y\,\text{max} = .60$
R = 125 cm)	$\beta_x\,\text{min} = .14$	$\beta_y\,\text{min} = .30$

For a cylindrical beam of radius $y_0 = 1\,\text{cm}$ the space charge limit for detuning ν_y by $\Delta\nu_y = 0.3$ has been found to be 0.172 amperes. We may evaluate for this current and the above parameters using the methods outlined above for a rectangular beam of varying dimensions.

$$a_y^0 = 8.7\frac{i}{y_0^2} \quad : \text{for } y_0 = 1\,\text{cm},\ i = .172 \text{ amps},$$

$$a_y^0 = 1.50$$

$$4\frac{a_x^0}{N^2} = 0.234$$

$$a(1) = 0.8 \qquad \frac{4a_x}{N^2} = 0.0187$$

Assuming β s as above and $x_0 = y_0$ at the median,

$$\left(\frac{x_0}{y_0}\right)_{max} \Big/ \left(\frac{x_0}{y_0}\right)_{min} = \sqrt{\frac{.47 \times .60}{.14 \times .30}} = 2.60$$

$$\frac{y_0^2{}_{max}}{y_0^2{}_{min}} = \frac{.60}{.30} = 2.00$$

From a smooth interpolation of

at $y_{0\,max}$: $y_0^2 = 1.4$ cm

$x_0/y_0 = 0.62$

$\alpha = 0.90$

at $y_{0\,min}$ $y_0^2 = .7$ cm

$x_0/y_0 = 1.61$

$\alpha = 0.625$

Therefore,

$$b_y = a_y^0 \frac{\left(\frac{.625}{.7} - \frac{.9}{1.4}\right)}{2} = a_y^0 \left(\frac{.89 - .64}{2}\right)$$

$$b_y = .125\, a_y^0 = 0.187$$

$$\frac{4 b_y}{N^2} = 0.0829$$

From Parzen's expressions in the absence of space charge

$$A = \frac{4 E_0'}{N^2} = +.0212$$

$$B = \frac{8 f_1}{N^2} = 0.664$$

using

$b = 0.558$

$h_1 = 3.88$

$h_0 = 1.0$

Using either Parzen's approximate solution or these coefficients with the Illiac Tables gives σ_y/π = .28 and ν_y = 2.24; in agreement with Cole's exact result for the machine neglecting space charge.

Combining the above, in the presence of space charge

$$A = .0212 - .0187 = .0025$$

$$B = .664 + .0029 = .667.$$

From the Illiac Tables,

$$\sigma_y/\pi \cong .253$$

$$\nu_y \cong 2.03$$

so

$$\Delta\nu_y \cong 0.21$$

While this is indeed less than the result obtained assuming a cylindrical, uniform charge distribution, in this particular case it seems clear that almost the entire difference is due to choosing a rectangular rather than a cylindrical distribution and that the alternating gradient effects are quite small.

V. CONCLUSIONS

This solution to the space charge limit problem, while it reveals no startling results, does give one some confidence in predicting current available in alternating gradient accelerators. Previous estimates of current limits were greeted by statements to the effect that one does not know how the motion is affected by the a. g. properties of the beam. It also seems not worthwhile to press calculations to higher terms in the Mathieu Equations, since even the dominant terms have so little effect.

The same analysis could be carried through for radial motion, however, in FFAG accelerators ν_x is usually somewhat higher than ν_y and therefore space charge effects will most probably be felt first in the vertical motion.

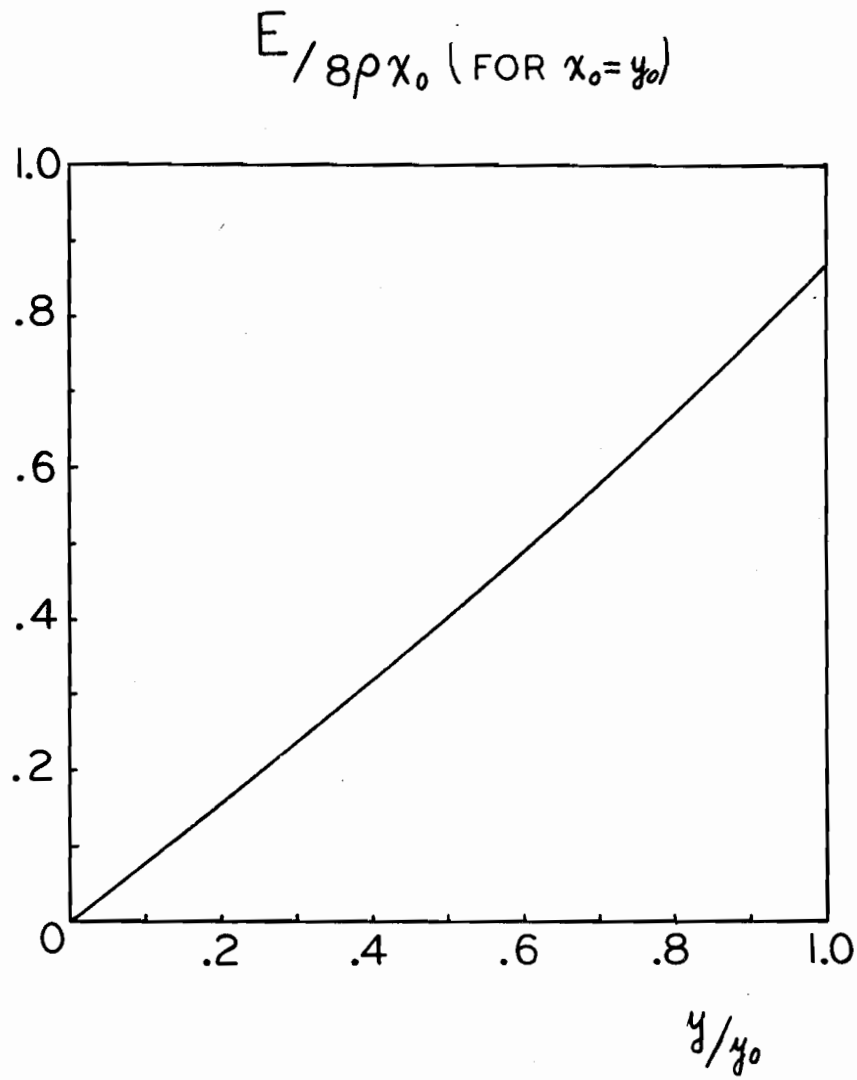


FIGURE 1