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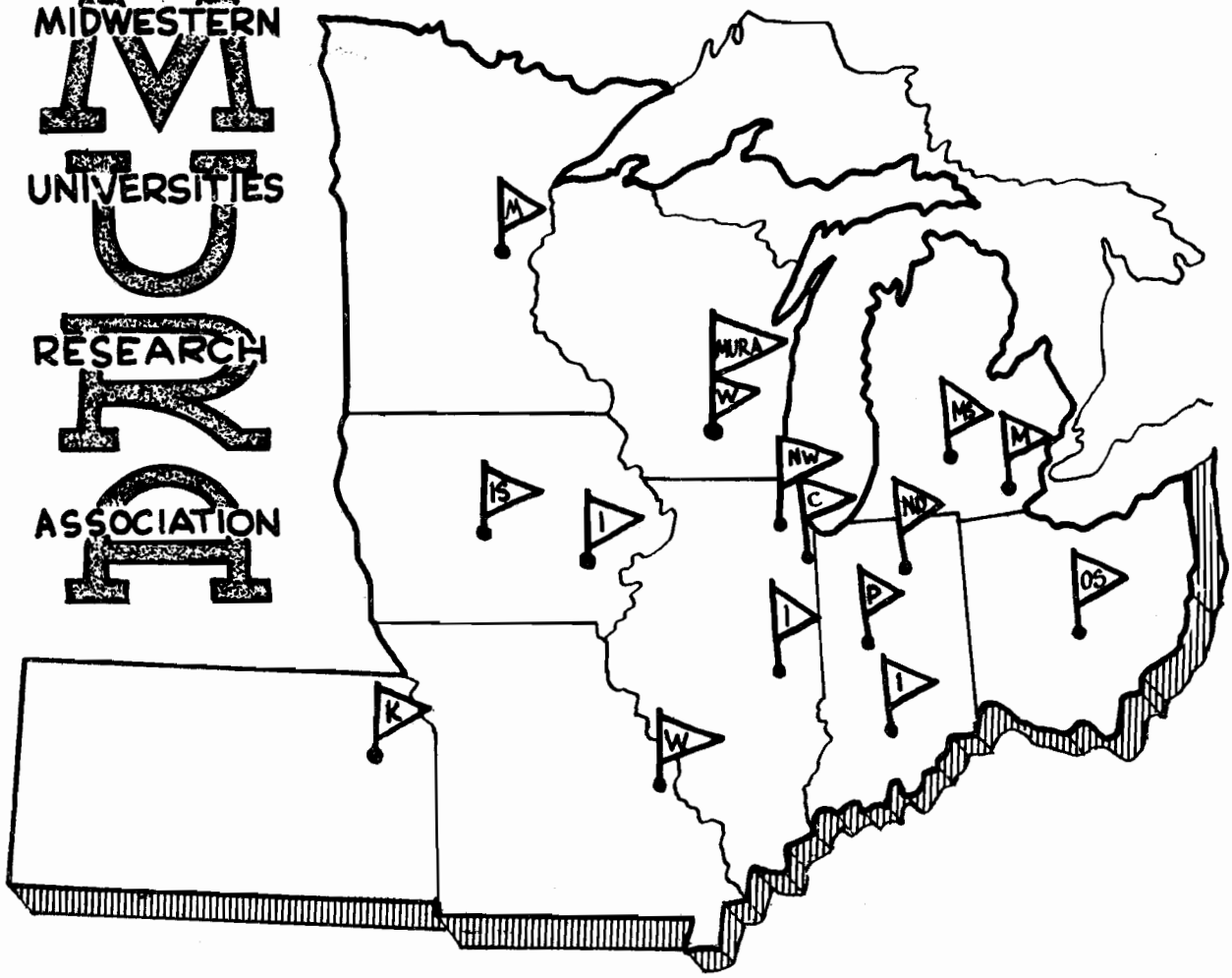

NON-HAMILTONIAN DYNAMICAL SYSTEMS

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ABSTRACT: Transformations of variables in dynamical systems which are not in general Hamiltonian are discussed. The time dependence of the Jacobian of the transformation is found, as well as transformed equations of motion. A generalized method of the variation of constants is developed.

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I. INTRODUCTION

Though most of the orbit problems connected with accelerators are Hamiltonian in nature, there are certain problems where more general systems come into consideration. Two prominent examples are the effects of radiation on motion and gas scattering. Consideration of these problems has led the writer to some generalizations of the transformation theory of Hamiltonian systems usually given in classical mechanics. No general theory comparable in scope to Hamiltonian transformation theory has been developed, but particular results which might be of some interest are given below. There are many questions (and corresponding answers) about a general theory which are not clear to the writer. One of the most interesting of these questions is the relation of this method to the Lie theory of differential equations.

II. NOTATION

Consider a dynamical system with f degrees of freedom. The equations of motion of this system can be written

$$\begin{cases} \dot{x}_i = f_i(x_j, y_j, t) \\ \dot{y}_i = g_i(x_j, y_j, t), \end{cases} \tag{1}$$

where dots denote total time derivatives and x_i and y_i are some set of variables, not in general canonical.

Let us first introduce a unified notation. Following Corben and Stehle,[1] we define $2f$ -dimensional vectors ξ , η and φ by

$$\begin{cases} \begin{cases} \xi^\alpha = x_j, \ (\alpha = j = 1, 2, \dots, f), \\ \quad\;\; = y_j, \ (\alpha = j+f = f+1, f+2, \dots, 2f) \end{cases} ; \\ \begin{cases} \eta_\alpha = y_j, \ (\alpha = j = 1, 2, \dots, f), \\ \quad\;\; = -x_j, \ (\alpha = j+f = f+1, f+2, \dots, 2f) \end{cases} ; \\ \begin{cases} \varphi^\alpha = f_j, \ (\alpha = j = 1, 2, \dots, f), \\ \quad\;\; = g_j, \ (\alpha = j+f = f+1, f+2, \dots, 2f) \end{cases} . \end{cases} \tag{2}$$

Greek indices run from 1 to $2f$ and the summation convention for repeated indices will be understood except where (N. S. on ___) appears after an equation. It will be assumed throughout that the ξ^α are independent coordinates, so that

$$\frac{\partial \xi^\alpha}{\partial \xi^\beta} = \delta^\alpha_\beta \ . \tag{3}$$

δ^α_β is the usual Kronecker delta symbol ($\delta^\alpha_\beta = 0, (\alpha \neq \beta)$, $= 1$, $(\alpha = \beta)$). The ξ^α and η_α are of course not independent; they are related by

$$\eta_\alpha = G_{\alpha\beta} \, \xi^\beta , \tag{4}$$

where $G_{\alpha\beta}$ is the appropriate element of the $2f$ -rank matrix

$$G = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \tag{5}$$

I being the unit matrix of rank f . We have also

$$\xi^\alpha = G^{\alpha\beta} \eta_\beta , \tag{6}$$

where $G^{\alpha\beta}$ is the element of the $2f$ -rank matrix

$$G' = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} , \tag{7}$$

the transpose of G. The relations

$$G_{\alpha\beta} G^{\beta\gamma} = G_{\beta\alpha} G^{\gamma\beta} = - G_{\alpha\beta} G^{\gamma\beta} = - G_{\beta\alpha} G^{\beta\gamma} = \delta_\alpha^\gamma \tag{8}$$

follow from (5) and (7).

In this notation Poisson Brackets have the form

$$(\rho, \pi) = \sum_i \left(\frac{\partial \rho}{\partial x_i} \frac{\partial \pi}{\partial y_i} - \frac{\partial \rho}{\partial y_i} \frac{\partial \pi}{\partial x_i} \right) = \frac{\partial \rho}{\partial \xi^\alpha} \frac{\partial \pi}{\partial \eta_\alpha} . \tag{9}$$

The equations of motion (1) become

$$\dot{\xi}^\alpha = \varphi^\alpha = \varphi^\alpha(\xi^\beta, t). \tag{10}$$

III. TRANSFORMATIONS

We wish to discuss transformations from the variables ξ^α to some other set ρ^α. A set π_α "conjugate" to ρ^α in the same sense that η_α is "conjugate" to ξ^α can be defined by

$$\pi_\alpha = G_{\alpha\beta} \rho^\beta . \tag{11}$$

The transformation equations must have the general form

$$\rho^\alpha = \rho^\alpha (\xi^\beta, t), \tag{12}$$

giving the ρ^α as functions of the ξ^β and t. If the Jacobian of this transformation is different from zero, (12) can be inverted to give

$$\xi^\alpha = \xi^\alpha(\rho^\beta, t) . \tag{13}$$

This Jacobian

$$J(\rho/\xi) = \frac{\partial(\rho^1, \rho^2, \ldots, \rho^{2f})}{\partial(\xi^1, \xi^2, \ldots, \xi^{2f})} \tag{14}$$

is a determinant of rank $2f$ whose (α,β) element is

$$\frac{\partial \rho^\alpha}{\partial \xi^\beta} .$$

J satisfies

$$\begin{cases} J(\xi|\xi) = 1 \\ J(\rho|\xi) = (J(\xi|\rho))^{-1} \\ J(\rho|\xi) = J(\rho|\omega)\, J(\omega|\xi) , \end{cases} \tag{15}$$

where ω is some third set of coordinates. This last relation enables us to consider only the simpler case where one set of coordinates is constant, since the Jacobian of any transformation can be found from this case by (15).

Call ξ_0^α the initial value of ξ^α and consider the Jacobian $J(\xi|\xi_0)$. Its time derivative $\dot{J}(\xi|\xi_0)$ is a sum of $2f$ determinants, each of which has one row differentiated. Now

$$\frac{d}{dt}\left(\frac{\partial \xi^\alpha}{\partial \xi_0^\beta}\right) = \frac{\partial \dot{\xi}^\alpha}{\partial \xi_0^\beta} = \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\gamma}\frac{\partial \xi^\gamma}{\partial \xi_0^\beta} ,$$

so that each of the $2f$ determinants of $\dot{J}$ is itself a sum of $2f$ determinants. If the αth row is differentiated, the γth determinant has the coefficient

$$\frac{\partial \dot{\xi}^\alpha}{\partial \xi^\gamma}$$

and the (α,β) element

$$\frac{\partial \xi^\gamma}{\partial \xi_0^\beta} .$$

Unless $\gamma = \alpha$, the determinant vanishes because two rows are identical. If $\gamma = \alpha$, the determinant is just J. Thus

$$\dot{J}(\xi|\xi_0) = \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha} J(\xi|\xi_0) , \tag{16}$$

which may be immediately integrated to give

$$J(\xi|\xi_0) = \exp\left(\int_0^t \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha} dt\right). \tag{17}$$

Eq. (16) is stated without proof in reference 1 (p. 294). It is also clearly related to a theorem of Symon, Lichtenberg and Stehle,[2] according to which the time rate of change of a volume V of phase space occupied by a system obeying (1) is

$$\dot{V} = \int \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha} dV . \tag{18}$$

(18) can be derived from (16), since (choosing the ξ^α as canonical variables)

$$V(t) = \int dV(t) = \int J(\xi|\xi_0) dV(0)$$

and therefore

$$\dot{V} = \int \dot{J} \cdot dV(0) = \int \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha} J dV(0) = \int \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha} dV(t) .$$

The time dependence of any Jacobian can now be found from (15) and (17). Thus

$$J(\rho|\xi) = J(\rho_0|\xi_0) \exp\left\{\int_0^t \left(\frac{\partial \dot{\rho}^\alpha}{\partial \rho^\alpha} - \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha}\right) dt\right\} . \tag{19}$$

$J(\rho|\xi)$ can be expanded in minors of any row or column, i.e.,

$$J(\rho|\xi) = (-1)^{\alpha+\beta} \frac{\partial \rho^\alpha}{\partial \xi^\beta} D_\alpha^\beta \qquad \text{(N.S. on } \alpha \text{ or N.S. on } \beta\text{)}, \tag{20}$$

where D_α^β is the minor of $\frac{\partial \rho^\alpha}{\partial \xi^\beta}$. Define ψ_α^β by

$$D_\alpha^\beta = (-1)^{\alpha+\beta} \psi_\alpha^\beta J(\rho|\xi) \qquad \text{(N.S. on } \alpha, \beta\text{)}. \tag{21}$$

Then the ψ_α^β satisfy

$$\frac{\partial \rho^\beta}{\partial \xi^\alpha} \psi_\gamma^\alpha = \frac{\partial \rho^\alpha}{\partial \xi^\gamma} \psi_\alpha^\beta = \delta_\gamma^\beta , \tag{22}$$

by substituting (21) in (20). These relations are identically satisfied by

$$\psi_\gamma^\alpha = \frac{\partial \xi^\alpha}{\partial \rho^\gamma} \tag{23}$$

because of the assumption of (3).

IV. TRANSFORMED EQUATIONS OF MOTION

The transformed equations of motion in $\dot{\rho}^\beta$ can now be found from the original equations of motion (10). Now

$$\dot{\xi}^\alpha = \frac{\partial \xi^\alpha}{\partial \rho^\beta} \dot{\rho}^\beta + \frac{\partial \xi^\alpha}{\partial t} = \varphi^\alpha$$

or

$$\frac{\partial \xi^\alpha}{\partial \rho^\beta} \dot{\rho}^\beta = \varphi^\alpha - \frac{\partial \xi^\alpha}{\partial t} . \tag{24}$$

This set of linear equations in the $\dot{\rho}^\beta$ can be solved for the $\dot{\rho}^\beta$, since the coefficients are just the elements of $J(\xi/\rho)$. Thus

$$\begin{aligned} \dot{\rho}^\beta &= \frac{(-1)^{\alpha+\beta} D_\alpha^\beta}{J(\xi/\rho)} \left(\varphi^\alpha - \frac{\partial \xi^\alpha}{\partial t}\right) \quad \text{(N. S. on } \beta \text{)} \\ &= \psi_\alpha^\beta \left(\varphi^\alpha - \frac{\partial \xi^\alpha}{\partial t}\right) \\ &= \frac{\partial \rho^\beta}{\partial \xi^\alpha} \left(\varphi^\alpha - \frac{\partial \xi^\alpha}{\partial t}\right), \end{aligned} \tag{25}$$

where (21) and (23) have been used. Note that the right hand side of (25) can be expressed in terms of the ρ^α by use of the transformation equation (13).

These same equations can also be derived by differentiation and manipulation of (12) and (13). We note that

$$\dot{\rho}^{\beta} = \frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}} \dot{\xi}^{\alpha} + \frac{\partial \rho^{\beta}}{\partial t}$$

from (12). Then

$$\begin{aligned} \frac{\partial \rho^{\beta}}{\partial t} &= \dot{\rho}^{\beta} - \frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}} \dot{\xi}^{\alpha} \\ &= \dot{\rho}^{\beta} - \frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}} \left(\frac{\partial \xi^{\alpha}}{\partial \rho^{\gamma}} \dot{\rho}^{\gamma} + \frac{\partial \xi^{\alpha}}{\partial t} \right) \\ &= - \frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial t} . \end{aligned} \tag{26}$$

If (26) is substituted in the equation for $\dot{\rho}^{\beta}$, (25) results.

Eqs. (25) are a set of equations of motion in the new variables ρ^{β}. Thus the problem originally defined by (10) can be discussed in any coordinate system, provided only that the Jacobian of the transformation of coordinates is different from zero.

In particular, a system can always be found in which the Jacobian is unity for all t. Let us calculate

$$\frac{d}{dt}\left(\frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}} \right) .$$

From (22) and (23),

$$\frac{\partial \xi^{\gamma}}{\partial \xi_0^{\mu}} \frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}} = \delta_{\alpha}^{\gamma} .$$

Differentiate to get

$$\frac{\partial \dot{\xi}^{\gamma}}{\partial \xi_0^{\mu}} \frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}} + \frac{\partial \xi^{\gamma}}{\partial \xi_0^{\mu}} \frac{d}{dt}\left(\frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}} \right) = 0$$

or

$$\frac{\partial \xi^{\gamma}}{\partial \xi_0^{\mu}} \frac{d}{dt}\left(\frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}}\right) = -\frac{\partial \dot{\xi}^{\gamma}}{\partial \xi^{\beta}}\left(\frac{\partial \xi^{\beta}}{\partial \xi_0^{\mu}} \frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}}\right) = -\frac{\partial \dot{\xi}^{\gamma}}{\partial \xi^{\alpha}}.$$

Now multiply by $\frac{\partial \xi_0^{\epsilon}}{\partial \xi^{\gamma}}$. Use of (22) and (23) gives

$$\frac{d}{dt}\left(\frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}}\right) = -\frac{\partial \dot{\xi}^{\gamma}}{\partial \xi^{\alpha}} \frac{\partial \xi_0^{\mu}}{\partial \xi^{\gamma}}. \tag{27}$$

Now we calculate

$$\frac{\partial \xi^{\alpha}}{\partial \rho^{\beta}} \frac{d}{dt}\left(\frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}}\right) = \frac{\partial \xi^{\alpha}}{\partial \rho^{\beta}}\left\{\frac{d}{dt}\left(\frac{\partial \rho^{\beta}}{\partial \rho_0^{\gamma}} \frac{\partial \rho_0^{\gamma}}{\partial \xi_0^{\mu}} \frac{\partial \xi_0^{\mu}}{\partial \xi^{\alpha}}\right)\right\}$$

$$= \frac{\partial \xi^{\alpha}}{\partial \rho^{\beta}}\left\{\frac{\partial \dot{\rho}^{\beta}}{\partial \xi^{\alpha}} - \frac{\partial \rho^{\beta}}{\partial \xi^{\mu}} \frac{\partial \dot{\xi}^{\mu}}{\partial \xi^{\alpha}}\right\}$$

by using (27). The sums on the right hand side collapse to give

$$\frac{\partial \xi^{\alpha}}{\partial \rho^{\beta}} \frac{d}{dt}\left(\frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}}\right) = \frac{\partial \dot{\rho}^{\beta}}{\partial \rho^{\beta}} - \frac{\partial \dot{\xi}^{\beta}}{\partial \xi^{\beta}}. \tag{28}$$

We may interchange ρ^{β} and α^{α} and add to find

$$\frac{\partial \dot{\rho}^{\beta}}{\partial \rho^{\beta}} - \frac{\partial \dot{\xi}^{\alpha}}{\partial \xi^{\alpha}} = \frac{1}{2}\left\{\frac{\partial \xi^{\alpha}}{\partial \rho^{\beta}} \frac{d}{dt}\left(\frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}}\right) - \frac{\partial \rho^{\beta}}{\partial \xi^{\alpha}} \frac{d}{dt}\left(\frac{\partial \xi^{\alpha}}{\partial \rho^{\beta}}\right)\right\}. \tag{29}$$

Suppose that

$$\int_0^t \frac{\partial \dot{\xi}^{\beta}}{\partial \xi^{\beta}} dt = 2f\, T(t),$$

where f is the number of degrees of freedom. Choose the transformation

$$\xi^{\alpha} = e^{T(t)} \rho^{\alpha}$$

Then a short calculation shows that

$$\frac{\partial \dot{\rho}^{\beta}}{\partial \rho^{\beta}} = 0$$

and $J(\rho|\rho_0) = 1$ for all t.

V. VARIATION OF CONSTANTS

A special case of great interest is the theory of the variation of constants. This is a generalization of the theory of the variation of canonical constants for Hamiltonian systems and of the elementary method of treating inhomogeneous linear differential equations.

Suppose that

$$\varphi^{\alpha} = \varphi_0^{\alpha} + \varphi_1^{\alpha} \tag{30}$$

and that the solution of the problem

$$\dot{\xi}^{\alpha} = \varphi_0^{\alpha}$$

is known. This solution has the form

$$\xi^{\alpha} = \xi^{\alpha}(\xi_0^{\beta}, t) . \tag{31}$$

We now view this ξ^{α} as a solution of the complete problem $\dot{\xi}^{\alpha} = \varphi_0^{\alpha} + \varphi_1^{\alpha}$ with the ξ_0^{β} now taken as functions of t. (31) is then the transformation equation of the form (13). Now

$$\frac{\partial \xi^{\alpha}}{\partial t} = \varphi_0^{\alpha}$$

for every set of values ξ_0^{β}, since ξ^{α} was a solution of that problem. Then we find in eqs. (25)

$$\begin{aligned} \dot{\xi}_0^{\beta} &= \frac{\partial \xi_0^{\beta}}{\partial \xi^{\alpha}} \left(\varphi_0^{\alpha} + \varphi_1^{\alpha} - \frac{\partial \xi^{\alpha}}{\partial t} \right) \\ &= \frac{\partial \xi_0^{\beta}}{\partial \xi^{\alpha}} \varphi_1^{\alpha} \end{aligned} \tag{32}$$

where the right hand side is to be expressed in terms of the ξ_0^{β} by use of (31).

VI. HAMILTONIAN SYSTEMS

The usual Hamiltonian theory is a special case of the above, with

$$\begin{cases} f_i = \dfrac{\partial H}{\partial y_i} \\ g_i = -\dfrac{\partial H}{\partial x_i} \end{cases} \tag{33}$$

in the notation of eq. (1) or

$$\dot{\xi}^\alpha = \frac{\partial H}{\partial \eta_\alpha} \tag{34}$$

in the notation of (10).

Let us begin with the following theorem: A necessary and sufficient condition that the equations of motion (10) be derivable from a Hamiltonian H according to (34) is that

$$\frac{\partial \dot{\xi}^\alpha}{\partial \eta_\gamma} = \frac{\partial \dot{\xi}^\gamma}{\partial \eta_\alpha} . \tag{35}$$

The necessity is obvious, for if a Hamiltonian exists

$$\frac{\partial \dot{\xi}^\alpha}{\partial \eta_\gamma} = \frac{\partial^2 H}{\partial \eta_\gamma \partial \eta_\alpha} = \frac{\partial \dot{\xi}^\gamma}{\partial \eta_\alpha} .$$

To prove the sufficiency, note from (35) and (10) that

$$\frac{\partial \varphi^\alpha}{\partial \eta_\gamma} - \frac{\partial \varphi^\gamma}{\partial \eta_\alpha} = 0 ,$$

which is the well known integrability condition.[3] Then the form $\varphi^\alpha d\eta_\alpha$ is an exact differential and the function $H = H(\eta_\alpha, t)$ exists and is given by

$$H = \int \varphi^\alpha d\eta_\alpha .$$

The conditions (35) imply that

$$\frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha} = 0$$

for a Hamiltonian system, for

$$G_{\mu\rho} \frac{\partial \dot{\xi}^\rho}{\partial \eta_\mu} = G_{\mu\rho} G^{\alpha\mu} \frac{\partial \dot{\xi}^\rho}{\partial \xi^\alpha} = \frac{\partial \dot{\xi}^\alpha}{\partial \xi^\alpha}$$

by using (8). But also, from (35)

$$G_{\mu\rho}\frac{\partial\dot{\xi}^{\rho}}{\partial\eta_{\mu}} = G_{\mu\rho}\frac{\partial\dot{\xi}^{\mu}}{\partial\eta_{\rho}} = G_{\mu\rho}\,G^{\alpha\rho}\frac{\partial\dot{\xi}^{\mu}}{\partial\xi^{\alpha}} = -\frac{\partial\dot{\xi}^{\alpha}}{\partial\xi^{\alpha}}.$$

Thus $J(\xi/\xi_0) = 1$ for all Hamiltonian system. Of course, the conditions (35) are more restrictive, so that not all systems with $J(\xi/\xi_0) = 1$ are Hamiltonian, except where $f = 1$, where J and the Poisson bracket are identical.

In the Hamiltonian case eqs. (22) are just the Poisson and LaGrange brackets and are satisfied by

$$\psi_{\nu}^{\alpha} = \frac{\partial\pi_{\nu}}{\partial\eta_{\alpha}}. \tag{36}$$

One can prove the theorem:

A necessary and sufficient condition that a system be Hamiltonian is that

$$\frac{\partial\xi_0^{\alpha}}{\partial\xi^{\gamma}} = \frac{\partial\eta_{\gamma}}{\partial\eta_{0\alpha}}. \tag{37}$$

The necessity is a trivial consequence of (36). For the proof of sufficiency, note that

$$\frac{d}{dt}\left(\frac{\partial\eta_{\gamma}}{\partial\eta_{0\alpha}}\right) = G_{\gamma\beta}\frac{\partial\dot{\xi}^{\beta}}{\partial\eta_{\mu}}\frac{\partial\eta_{\mu}}{\partial\eta_{0\alpha}}$$

and

$$\frac{d}{dt}\left(\frac{\partial\xi_0^{\alpha}}{\partial\xi^{\gamma}}\right) = -G_{\rho\gamma}\frac{\partial\dot{\xi}^{\beta}}{\partial\eta_{\rho}}\frac{\partial\eta_{\beta}}{\partial\eta_{0\alpha}}.$$

Multiply both of these by $G^{\nu\gamma}\dfrac{\partial\xi^{\delta}}{\partial\xi_0^{\alpha}}$. Then

$$G^{\nu\gamma} G_{\gamma\beta} \frac{\partial \dot{\xi}^{\beta}}{\partial \eta_{\mu}} \frac{\partial \xi^{\delta}}{\partial \xi_0^{\alpha}} \frac{\partial \eta_{\mu}}{\partial \eta_{0\alpha}} = \frac{\partial \dot{\xi}^{\nu}}{\partial \eta^{\delta}}$$

$$= - G^{\nu\gamma} G_{\rho\gamma} \frac{\partial \dot{\xi}^{\beta}}{\partial \eta_{\rho}} \frac{\partial \xi^{\delta}}{\partial \xi_0^{\alpha}} \frac{\partial \eta_{\beta}}{\partial \eta_{0\alpha}} = \frac{\partial \dot{\xi}^{\delta}}{\partial \eta_{\nu}} ,$$

which is just (35), implying the existence of a Hamiltonian. Since (37) is equivalent to the Poisson bracket expressions

$$\frac{\partial \xi^{\rho}}{\partial \xi_0^{\alpha}} \frac{\partial \eta_{\delta}}{\partial \eta_{0\alpha}} = \delta^{\rho}_{\delta} ,$$

these expressions are themselves equivalent to the existence of a Hamiltonian.

In the Hamiltonian case, with both ξ^{α} and ρ^{β} canonical coordinates, the transformation can be derived from a generating function $S = S(\xi^{\alpha}, \pi_{\beta}, t)$ by the relations

$$\begin{cases} \dfrac{\partial S}{\partial \xi^{\alpha}} = \eta_{\alpha} \\ \dfrac{\partial S}{\partial \pi_{\beta}} = \rho^{\beta} , \end{cases} \tag{38}$$

(where half the equations are redundant because of the notation). Now

$$\frac{\partial \rho^{\alpha}}{\partial t} = - \frac{\partial \rho^{\alpha}}{\partial \xi^{\beta}} \frac{\partial \xi^{\beta}}{\partial t} = \frac{\partial^2 S}{\partial \pi_{\alpha} \partial t}$$

from (26). The transformed equations of motion can now be written as

$$\dot{\rho}^{\beta} = \frac{\partial \eta_{\alpha}}{\partial \pi_{\beta}} \left(\varphi^{\alpha} - \frac{\partial \xi^{\alpha}}{\partial t} \right)$$

$$= \frac{\partial}{\partial \pi_{\beta}} \left(H + \frac{\partial S}{\partial t} \right),$$

which was only to be expected.

VII. INCONCLUSIVE REMARKS

It might be well to point out that amid all the verbiage, the important results of this report are eqs. (19), giving the time dependence of the Jacobian of the transformation, (25), the transformed equations of motion and (32), the variation of constants differential equations. (19) differs from well-known results only by trivial steps, but (25) and (32) certainly do not seem to be familiar in the literature. The variation of constants has been developed previously only for the Hamiltonian case[4] and for one-dimensional non-Hamiltonian equations of motion.[5]

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