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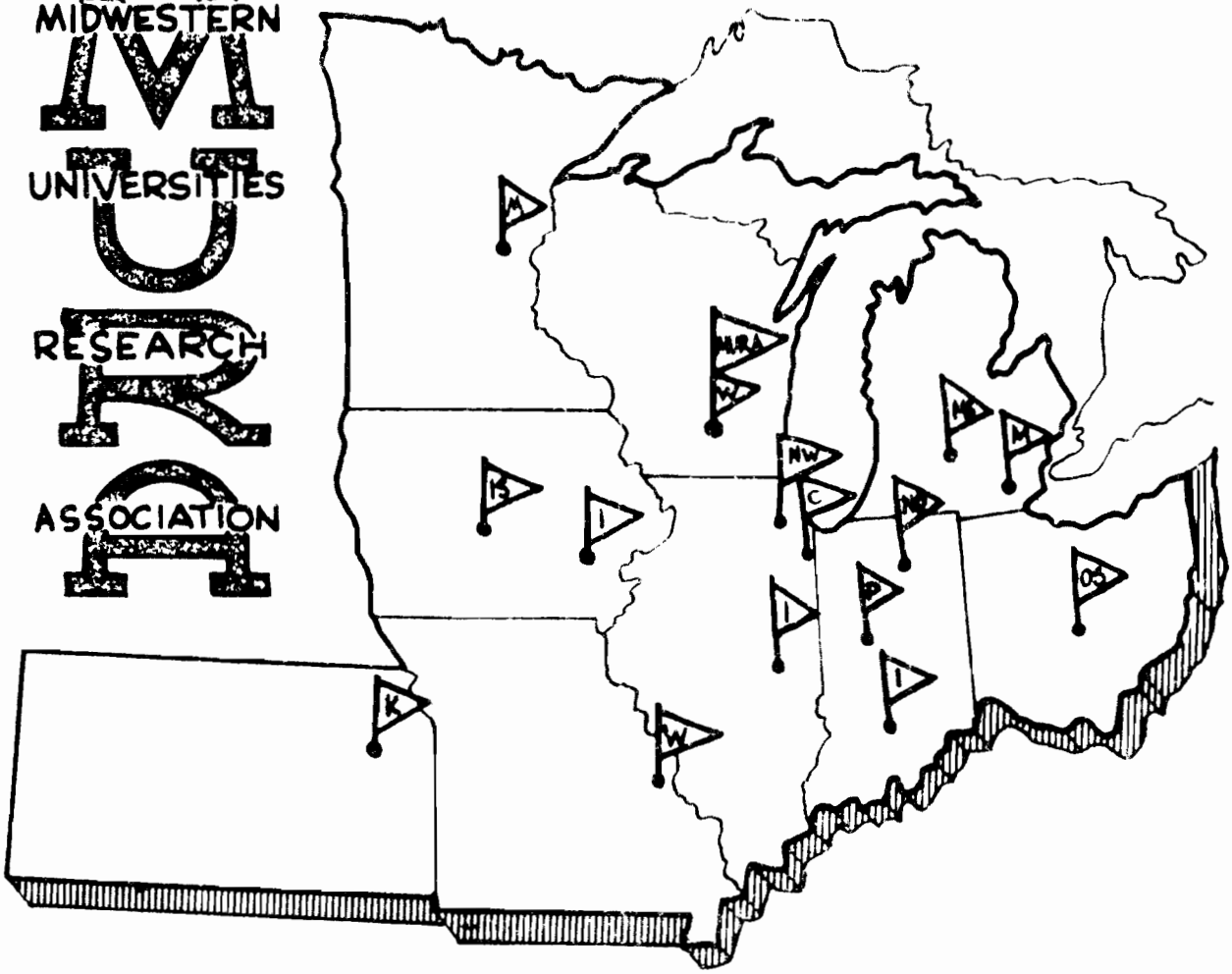

THE NON-LINEAR COUPLING RESONANCES OF AN ACCELERATOR

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ABSTRACT

This paper is a continuation of the reports MURA-258, MURA-273, and MURA-362. This paper treats the problem of finding the stability limits due to the $\nu_x + 2\nu_y = N$ coupling resonances and the threshold for y-growth due to the $\nu_x - 2\nu_y = 0$ resonance. The general spiral sector machine is treated in detail. The results of the theory are compared with numerically computed results.

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I. INTRODUCTION

This paper is a continuation of reports MURA-258, MURA-273, and MURA-362. In this paper we will treat the problem of finding the stability limits due to the coupled sum resonance $\nu_x + 2\nu_y = N$ and the coupled difference resonance $\nu_x - 2\nu_y = 0$ in an accelerator having an arbitrary magnetic field.

We will limit ourselves to treating the general spiral sector machine. (Mark V and Mark I machines.) However, we believe that the procedure given can be applied with little change to any other magnetic field. Some of the results for the limiting case of $\frac{1}{w} \gg K$ and for small flutter have been previously found by Laslett and Sessler[1] using the expanded equations of motion found by F. T. Cole.[2]

II. SUMMARY OF RESULTS

In this section we will simply list some of the results for the stability limits due to the $\nu_x + 2\nu_y = N$ and $\nu_x - 2\nu_y = 0$ coupled resonances for the general spiral sector machine (Mark I and Mark V). The derivation is given in Section III, where more detailed results are also given. For the sake of simplicity, the most accurate results are not listed here.

The magnetic field for the general spiral sector machine is written as

$$H_z = -H\left(\frac{r}{r_1}\right)^K \sum_{-\infty}^{\infty} h_n e^{i\omega_n \varphi} , \tag{2.1a}$$

where $\omega_n = nN$ and

$$\varphi = \theta - \frac{1}{wN} \ln\left(\frac{r}{r_1}\right) . \tag{2.1b}$$

The $\nu_x + 2\nu_y = N$ Resonance

The x-stability limit amplitude A is given by

$$A_x = \frac{1}{N^2} \frac{\left| \left(\frac{\nu_{y0}}{N}\right)^2 - \left(\frac{\nu_{ys}}{N}\right)^2 \right|}{M}, \tag{2.2}$$

where

$$\nu_{ys} = \frac{1}{2}(N - \nu_{x0}), \tag{2.3}$$

(ν_{x0}, ν_{y0}) is the linear small amplutude tune, and

$$M = \left| \frac{b h_1}{2} \frac{K^2 + 1/w^2}{N^4} \right|. \tag{2.4}$$

A is measured in units of R, the average radius of the equilibrium orbit (See MURA-258 for a more accurate definition of R). $b = \{(h_0^2 + h_1^2 K/N^2)^{\frac{1}{2}} - h_0\}/(4 h_1^2 K/N^2)$.

The y-stability limit amplitude is given by

$$A_y = \frac{1}{N^2} \frac{\left\{ 2\left|\left(\frac{\nu_{x0}}{N}\right)^2 - \left(\frac{\nu_{xs}}{N}\right)^2\right| \left| \left(\frac{\nu_{y0}}{N}\right)^2 - \left(\frac{\nu_{ys}}{N}\right)^2 \right| \right\}^{\frac{1}{2}}}{M} \tag{2.5a}$$

where $\nu_{xs} + 2\nu_{ys} = N$, $\nu_{x0} \le \nu_{xs} \le \frac{1}{2}N$, $\frac{1}{4}N \le \nu_{ys} \le \frac{1}{2}(N - \nu_{x0})$ and (ν_{xs}, ν_{ys}) are so chosen as to give the largest value of A_y. One usually finds that A_y and A_x are roughly equal.

If (ν_{x0}, ν_{y0}) is very close to the resonance line, then one finds that

$$A_y = \sqrt{\frac{\nu_{x0}}{\nu_{y0}}}\, A_x . \tag{2.5b}$$

The $\nu_x - 2\nu_y$ Resonance

The threshold x-amplitude for y-growth which we will call A is given by

$$A_x = \frac{1}{N^2} \frac{\left| \left(\frac{\nu_{y0}}{N}\right)^2 - \left(\frac{\nu_{ys}}{N}\right)^2 \right|}{M} , \tag{2.6}$$

where

$$\nu_{ys} = \frac{1}{2} \nu_{x0} \tag{2.7}$$

and we will give M for the two cases, $\frac{1}{w} = 0$ (Mark I) and $\frac{1}{w} \gg K$ (Mark V).

$\frac{1}{w} = 0$, (Mark I, $\nu_x - 2\nu_y = 0$ Resonance)

$$M = \frac{b h_0}{2} \frac{K^2}{N^4} \tag{2.8}$$

Eq. (2.8) gives an infinite threshold amplitude A for the two-way machine which has $h_0 = 0$. This is not truly so, as terms neglected as being small then become important. One may expect that the two-way machine for which

$h_0 = 0$ will have a threshold amplitude which is considerably larger than the threshold amplitude of a similar one-way machine.

$\frac{1}{w} >> K$ (Mark V, $\nu_x - 2\nu_y = 0$ Resonance)

$$M = \left| \frac{b h_0}{2} \frac{K^2}{N^4} - 3 \frac{K}{N^2} \sum_{n \geq 1} \left| \frac{b h_n}{w N^2} \right|^2 + i\, 4 \frac{\nu_{x0}}{N} \frac{1}{|b h_1|} \left| \frac{b h_1}{w N^2} \right|^3 \right| . \tag{2.9}$$

III. DERIVATION OF THE STABILITY LIMITS

For the sake of simplicity we will treat the case of a general spiral sector machine whose median plane field is given by

$$H_z = -H \left(\frac{r}{r_1}\right)^K \sum_{n=-\infty}^{\infty} h_n e^{i \omega_n \varphi} , \tag{3.1}$$

where $\omega_n = n N$ and

$$\varphi = \theta - \frac{1}{w N} \ln\left(\frac{r}{r_1}\right). \tag{3.2}$$

We believe that any other magnetic field can be treated in the same way with little change.

The equations of motion can be written as (See MURA-273)

$$\frac{d}{d\theta}\frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = F_r , \tag{3.3a}$$

$$\frac{d}{d\theta}\frac{\partial L}{\partial z'} - \frac{\partial L}{\partial z} = F_z , \tag{3.3b}$$

where

$$F_r = \frac{e}{pc}\left(r H_z - z' H_\theta\right) , \tag{3.4}$$

$$F_z = \frac{e}{pc}\left(r' H_\theta - r H_r\right) , \tag{3.5}$$

$$L = \sqrt{r^2 + r'^2 + z'^2} . \tag{3.6}$$

In order to investigate the quadratic coupled non-linear resonances, we must expand Eqs. (3.3) about the equilibrium orbit $r = r_s(\theta)$ and keep up to quadratic terms.

The expansion of F_r up to quadratic terms is given in Appendix A of MURA-273.

$$F_r = \frac{e}{pc}\Big\{\cdots + \left(H_{z,r} + \tfrac{1}{2} H_{z,rr}\right)\rho^2 - \tfrac{1}{2} r\, \nabla_t^2 H_z\, z^2 - \frac{1}{r} H_{z,\theta}\, z z' + \cdots\Big\} . \tag{3.7}$$

In Eq. (3.6), ρ is defined by $r = r_s(\theta) + \rho$ and the linear terms have been omitted, and $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The coefficients of ρ^2, z^2 and $z z'$ are to be evaluated on the equilibrium orbit.

The expansion of F_z is given by

$$F_z = \frac{e}{pc}\left\{\ldots + \left[\frac{r'}{r}\left(H_{z,\theta r} - \frac{1}{r}H_{z,\theta}\right) - r H_{z,rr} - H_{z,r}\right]\rho z + \frac{1}{r}H_{z,\theta}\,\rho' z + \ldots\right\}. \tag{3.8}$$

The expansion of L is[2]

$$L = \ldots + \frac{1}{2} b_8 \rho' z'^2 + \frac{1}{2} b_7 \rho z'^2 + \ldots \tag{3.9}$$

where we have kept only the terms which contribute to the quadratic coupling terms.

$$b_7 = -r/L^3, \tag{3.9a}$$

$$b_8 = -r'/L^3. \tag{3.9b}$$

The coefficients b_7 and b_8 are to be evaluated on the equilibrium orbit which means we put $r = r_s(\theta)$, $r' = r_s'(\theta)$, $z = 0$.

Using the results of MURA-273 and MURA-362 for the linear and radial quadratic parts, Eq. (3.3a) becomes

$$
\begin{aligned}
a_3 \rho'' + a_3' \rho' + (a_2' - a_1)\rho = {} & \frac{e}{pc}\left(H_z + r H_{z,r}\right)\rho \\
& + \frac{1}{R^2}\Big(B\rho^2 + B^{(1)} \rho\rho' + B^{(2)} \rho\rho'' + B^{(3)} \rho'^2 \\
& \qquad + B^{(4)} \rho'\rho'' + B^{(5)} \rho^2 \Big) \\
& + \frac{1}{R^2}\Big(\left[C + C^{(1)}\right] z^2 + C^{(2)} z z' + C^{(3)} z' z'' \\
& \qquad + C^{(4)} z'^2 \Big),
\end{aligned}
\tag{3.10}
$$

where

$$C = -\frac{e}{pc} R^2 \frac{r}{2}\left(H_{z,rr} + \frac{1}{r} H_{z,r}\right), \tag{3.11a}$$

$$C^{(1)} = -\frac{e}{pc} R^2 \frac{1}{r} H_{z,\theta\theta}, \tag{3.11b}$$

$$C^{(2)} = -\frac{e}{pc} R^2 \frac{1}{r} H_{z,\theta}, \tag{3.11c}$$

$$C^{(3)} = R^2 b_8, \tag{3.11d}$$

$$C^{(4)} = R^2 \left(\frac{1}{2} b_8' - \frac{1}{2} b_7\right). \tag{3.11e}$$

Eq. (3.3b) becomes

$$a_6 z' + a_6' z' = \frac{e}{pc}\left(\frac{r'}{r} H_{z,\theta} - r H_{z,r}\right) z$$

$$+ \frac{1}{R^2}\Big([D + D^{(1)}] \rho z + D^{(2)} \rho' z$$

$$+ D^{(3)} u' y' + D^{(4)} u'' y' + D^{(5)} u' y''$$

$$+ D^{(6)} u y' + D^{(7)} u y'' \Big) , \tag{3.12}$$

where

$$D = \frac{e}{pc} R^2 \left\{ -r H_{z,rr} - H_{z,r} + \frac{r'}{r}\left(H_{z,\theta r} - \frac{1}{r} H_{z,\theta}\right) \right\} , \tag{3.13a}$$

$$D^{(1)} = \frac{e}{pc} R^2 \frac{r'}{r}\left(H_{z,\theta r} - \frac{1}{r} H_{z,\theta}\right) , \tag{3.13b}$$

$$D^{(2)} = \frac{e}{pc} R^2 \frac{1}{r} H_{z,\theta} , \tag{3.13c}$$

$$D^{(3)} = R^2 (b_8' + b_7) , \tag{3.13d}$$

$$D^{(4)} = D^{(5)} = R^2 b_8 , \tag{3.13e}$$

$$D^{(6)} = R^2 b_7' , \tag{3.13f}$$

$$D^{(7)} = R^2 b_7 . \tag{3.13g}$$

Eqs. (3.10) and (3.13) will hold for any magnetic field. We will now specialize to the general spiral sector field given by Eq. (3.1). The dominant quadratic coupling terms are the $C(\theta)\, z^2$ term in Eq. (3.10) and the $D(\theta)\, p z$ term in Eq. (3.12). All the other terms give contributions to the stability limits which get small for large N like $1/N^2$. These $1/N^2$ terms may be significant, however, for small machines ($N \lesssim 12$). We will drop the $1/N^2$ terms here. Their effect may be calculated as was done in MURA-362 for the $\nu = \frac{1}{3} N$ resonance. Dropping the $1/N^2$ terms means that our results may be off by a factor of two for small machines ($N \lesssim 12$).

Let us now find the coefficients $C(\theta)$ and $D(\theta)$ for the spiral sector magnetic field. It is convenient to write the magnetic field as

$$H_z = -H\left(\frac{K}{r_1}\right)^K \sum_n (1+x)^{K - in/w}\, \bar{h}_n\, e^{i\omega_n \theta}, \tag{3.14}$$

where

$$\bar{h}_n = h_n \exp\left[-i \frac{n}{w} \ln\left(\frac{r}{r_1}\right)\right]. \tag{3.15}$$

We will find $C(\theta)$ and note that $D(\theta) = 2\, C(\theta)$. Then as

$$C(\theta) = -(eR^2/pc)\,(r/a)\left[H_{z,rr} + \frac{1}{r} H_{z,r}\right],$$

we see from MURA-362 that

$$C(\theta) = -B(\theta) \tag{3.16}$$

We can then write for the equations of motion up to quadratic terms

$$\left\{ \frac{d^2}{d\theta^2} + E_0 - g(\theta) \right\} u = B(\theta)\, u^2 + C(\theta)\, y^2 \tag{3.17a}$$

$$\left\{ \frac{d^2}{d\theta^2} + E_0' - f(\theta) \right\} y = D(\theta)\, u y , \tag{3.17b}$$

where $B(\theta) = -C(\theta)$, $D(\theta) = 2C(\theta)$ and

$B(\theta) = \sum_n B_n \exp(i\omega_n \theta)$, where the B_n are given by (See MURA-362)

$$\frac{B_0}{N^4} = -\frac{b}{2}\,\frac{K(K+1)}{N^4}\, h_0 - b^2 \sum_{n \geq 1} \frac{|h_n|^2}{n^2}\,\frac{K\left[(K^2-1) - 3n^2/w^2\right]}{N^6} , \tag{3.18a}$$

$$\frac{B_n}{N^4} = -\frac{b}{2}\,\frac{\left(K - i\frac{n}{w}\right)\left(K+1 - i\frac{n}{w}\right)}{N^4}\,\bar{h}_n \tag{3.18b}$$

Now that we have the equations of motion expanded out to quadratic terms we can apply the results of previous report, MURA-217 and MURA-365, to calculate the stability limits of the coupling resonances $\nu_x + 2\nu_y = N$ and $\nu_x - 2\nu_y = 0$. We will treat the sum resonance $\nu_x + 2\nu_y = N$

first and then the difference resonance $\nu_x - 2\nu_y = 0$.

The $2\nu_y + \nu_x = N$ Resonance

According to the results of the report MURA-217, there is not just one stability limit orbit but many stability limit orbits depending on which tune on the resonance line $2\nu_y + \nu_x = N$ the motion is driven to. Every stability limit orbit has a tune (ν_x, ν_y) which lies on the resonance line $2\nu_y + \nu_x = N$ and this tune depends on the initial conditions.

The stability limit orbits having the tune (ν_{xs}, ν_{ys}), lying on the resonance line $2\nu_{ys} + \nu_{xs} = N$, are given by

$$u = a_s u_s(\theta) + a_s^* u_s^*(\theta) , \tag{3.19a}$$

$$y = b_s y_s(\theta) + b_s^* y_s^*(\theta) . \tag{3.19b}$$

where $u_s(\theta)$ and $y_s(\theta)$ are the flocquet solutions having the tune ν_{xs} and ν_{ys} , where $2\nu_{ys} + \nu_{xs} = N$, and which are found by solving the linear equations of motion when (E_0, F_0') are replaced by (E_s, E_s'), (E_s, E_s') being so chosen as to give the tune (ν_{xs}, ν_{ys}).

The amplitudes a_s and b_s are given by

$$|a_s| = \frac{|E_0' - E_s'|}{|D_{s,\bar{s}\bar{s}}|} , \tag{3.20a}$$

$$|b_s| = \left\{ \frac{|E_0 - E_s|\,|E_0' - E_s'|}{|C_{s,\bar{s}\bar{s}}|\,|D_{s,\bar{s}\bar{s}}|} \right\}^{\frac{1}{2}} , \tag{3.20b}$$

where

$$C_{s,\bar{s}\bar{s}} = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} d\theta \; u_s^*(\theta)\; C(\theta)\; y_{-s}(\theta)\; y_{-s}(\theta), \tag{3.21}$$

and phase $a = \alpha$ and phase $b = \beta$ are related by

$$\alpha + 2\beta = \text{phase } C_{s,\bar{s}\bar{s}} - \text{phase}\,(E_0 - E_s). \tag{3.22}$$

Note that $|D_{s,\bar{s}\bar{s}}| = 2\,|C_{s,\bar{s}\bar{s}}|$.

The E_s and E_s', and $u_s(\theta)$ and $y_s(\theta)$, may be calculated by Eqs. (3.20) and (3.21) of MURA-362. u_s and y_s, which we will need to calculate $C_{s,\bar{s}\bar{s}}$ are then given by

$$u_s = e^{i\nu_{xs}\theta}\left\{1 + \gamma_1 \frac{1}{1+\frac{2\nu_{xs}}{N}} e^{iN\theta} + \gamma_{-1}\frac{1}{1-\frac{2\nu_{xs}}{N}} e^{-iN\theta} + \dots\right\}, \tag{3.23}$$

where

$$\gamma_n = -\frac{g_n}{\omega_n^2}, \tag{3.24a}$$

$$\frac{g_n}{N^2} = -b\bar{h}_n \frac{(K+1-in/w)}{N^2}. \tag{3.24b}$$

$$y_s = e^{i\nu_{ys}\theta}\left\{1 + \gamma_1' \frac{1}{1+2\nu_{ys}/N} e^{iN\theta} + \gamma_{-1}' \frac{1}{1-2\nu_{ys}/N} e^{-iN\theta} + \dots\right\} \tag{3.25}$$

where

$$\gamma_n' = - f_n / \omega_n^2 , \tag{3.26a}$$

$$\frac{f_n}{N^2} = b \bar{h}_n \frac{K - i\, n/\omega}{N^2} . \tag{3.26b}$$

One may note that $\gamma_n' \simeq -\gamma_n$, $f_n \simeq - g_n$, if we assume $K >> 1$.

We now calculate the matrix element $C_{s,\bar{s}\bar{s}}$ and find that

$$\begin{aligned} C_{s,\bar{s}\bar{s}} = \; & C_{-1} \\ & + \gamma_{-1} \left(\frac{1}{1 - 2\nu_{xs}/N} - \frac{2}{1 - 2\nu_{ys}/N} \right) C_{-} \\ & + \gamma_{1} \left(\frac{1}{1 + 2\nu_{xs}/N} - \frac{2}{1 + 2\nu_{ys}/N} \right) C_{-2} . \end{aligned} \tag{3.27}$$

In calculating $C_{s,\bar{s}\bar{s}}$, we have assumed that $\gamma_1 \simeq - \gamma_1'$ and have kept only up to terms which are linear in γ_1 since we expect that $\gamma_1 \simeq .1.$ $C_0, C_1,$ are defined by $C(\theta) = \sum_n C_n e^{i\omega_n \theta}$.

According to Eqs. (3.20), we have many stability limit amplitudes instead of just one. We can get a good measure of the stability limits by finding the maximum x-amplitude and y-amplitude allowed by Eqs. (3.20), which we will call a_{max} and b_{max} . We will call $A_x = 2|a_{max}|$, the x-stability limit amplitude and $A_y = 2|b_{max}|$ the y-stability limit amplitude.

According to MURA-217, a_s takes on its maximum value when the tune $\nu_{xs} = \nu_{xo}$ and $\nu_{ys} = \frac{1}{2}(N - \nu_{xo})$, where

(ν_{x0}, ν_{y0}) is the linear small amplitude tune. Thus from Eq. (3.20), we find for A_x, the x-stability amplitude

$$A_x = \frac{1}{N^2} \frac{\left| \left(\frac{\nu_{y0}}{N}\right)^2 - \left(\frac{\nu_{ys}}{N}\right)^2 \right|}{M}, \qquad (3.28)$$

where

$$M = \frac{1}{N^4} \left| C_{s,\bar{s}\bar{s}} \right| = \frac{1}{N^4} \left| \frac{1}{2} D_{s,\bar{s}\bar{s}} \right|, \qquad (3.29)$$

and

$$\nu_{ys} = \frac{1}{2} (N - \nu_{x0}). \qquad (3.30)$$

According to MURA-217, A_y, the y-stability limit amplitude is given by

$$A_y = \sqrt{\frac{\nu_x}{\nu_y}} \; A_x \qquad (3.31)$$

Eq. (3.31) for A_y holds only if (ν_{x0}, ν_{y0}) is close to the resonance line $2\nu_y + \nu_x = N$. If one is rather far from the resonance line, one should go back and use Eq. (3.20b) for A_y and find the tune (ν_{xs}, ν_{ys}) that gives the largest value of A_y.

For the spiral sector magnetic field, we see from Eqs. (3.27) and (3.18) that

$$C_{s,\bar{s}\bar{s}} = C_{-1} \qquad (3.32)$$

to a good approximation. Thus M is given by

$$M = \frac{1}{2} b h_1 \frac{K^2 + 1/w^2}{N^4} \tag{3.33}$$

We assumed $K \simeq K+1$ in getting Eq. (3.33).

The $2\nu_y - \nu_x = 0$ Resonance

In this case, the instability is not of the sort met in the $2\nu_y + \nu_x = N$ or the $\nu_x = \frac{1}{3} N$ resonances. There is, however, a threshold x-amplitude for the onset of y-growth. This threshold x-amplitude will depend on the initial y-amplitude. Thus there will be many threshold orbits which give the limits on the x and y amplitudes for motion having no y-growth. Eash threshold orbit has a tune (ν_x , ν_y) which lies on the resonance line $2\nu_y - \nu_x = 0$ and this tune depends on the initial conditions.

The results of theory for the $2\nu_y - \nu_x = 0$ resonance, which will be used in the following, are given in MURA-365.

The threshold orbits, having the tune (ν_{xs} , ν_{ys}) lying on the resonance line $2\nu_{ys} - \nu_{xs} = 0$, are given by results very similar to those for the $2\nu_y + \nu_x = N$ resonance.

These orbits are given by

$$u = a_s u_s(\theta) + a_s^* u_s^*(\theta) , \tag{3.32a}$$

$$y = b_s y_s(\theta) + b_s^* y_s^*(\theta) . \tag{3.32b}$$

The amplitudes a_s and b_s are given by

$$|a_s| = \frac{|E_0' - E_s'|}{|D_{s,s\bar{s}}|}, \tag{3.33a}$$

$$|b_s| = \left\{ \frac{|E_0' - E_s'|\,|E_0 - E_s|}{|D_{s,s\bar{s}}|\,|C_{s,ss}|} \right\}^{\frac{1}{2}}, \tag{3.33b}$$

where

$$C_{s,ss} = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} d\theta \; u_s^*(\theta)\, C(\theta)\, y_s(\theta)\, y_s(\theta), \tag{3.33c}$$

and phase $a = \alpha$ and phase $b = \beta$ are related by

$$\alpha - 2\beta = \text{phase } C_{s,ss} - \text{phase } (E_0 - E_s). \tag{3.34}$$

Note that $|D_{s,s\bar{s}}| = 2\,|C_{s,ss}|$.

The (E_s, E_s') and $u_s(\theta)$ and $y_s(\theta)$ may be calculated as in the case of the $2\nu_y + \nu_x = N$ resonance.

We can now calculate the matrix element $C_{s,ss}$ and find that

$$\begin{aligned} C_{s,ss} = \; & C_0 \\ & + \left(\frac{\gamma_{-1}^*}{1 - 2\nu_{xs}/N} - \frac{2\gamma_1}{1 + 2\nu_{ys}/N} \right) C_{-1} \\ & + \left(\frac{\gamma_1^*}{1 + 2\nu_{xs}/N} - \frac{2\gamma_{-1}}{1 - 2\nu_{ys}/N} \right) C_1 . \end{aligned} \tag{3.35}$$

In calculating $C_{s,ss}$, we have assumed that $\gamma_1 \simeq -\gamma_1'$ and have kept only up to terms linear in γ_1.

A good measure of the threshold x-amplitude for y-growth is the minimum x-amplitude allowed by Eqs. (3.33) which we will call a_{min}. We will call $A = 2|a_{min}|$ the threshold x-amplitude for y-growth.

a_s takes on its minimum value when $\nu_{xs} = \nu_{x0}$ and $\nu_{ys} = \frac{1}{2}\nu_{x0}$, where (ν_{x0}, ν_{y0}) is the linear small amplitude tune. Thus, from Eq. (333a) we find for A, the threshold x-amplitude,

$$A = \frac{1}{N^2} \frac{\left| \left(\frac{\nu_{y0}}{N}\right)^2 - \left(\frac{\nu_{ys}}{N}\right)^2 \right|}{M}, \tag{3.36}$$

where

$$M = \frac{1}{N^4} |C_{s,ss}| = \frac{1}{N^4} \left| \frac{1}{2} D_{s,ss} \right|, \tag{3.37}$$

and

$$\nu_{ys} = \frac{1}{2}\nu_{x0}. \tag{3.38}$$

We would like to find the explicit form of M for the spiral sector field. We will treat the two cases, $1/\Delta r = 0$, Mark I, and $\frac{1}{\Delta r} \gg K$, Mark V.

<u>$1/\Delta r = 0$, Mark I ($2\nu_y - \nu_x = 0$ Resonance)</u>

If we assume that the h_n are real and $h_1 = h_{-1}$, then γ_1 is real and $\gamma_1 = \gamma_{-1}$, and $C_1 = C_{-1}$. From Eqs. (3.35) and Eqs. (3.18), we find that

$$\frac{C_{s,ss}}{N^4} = \frac{1}{2} b h_0 \frac{k(k+1)}{N^4} \tag{3.39}$$

and

$$M = \frac{1}{2} b h_0 \frac{k(k+1)}{N^4} \tag{3.40}$$

One might note that for the two-way machine,[3] the matrix element vanishes. $M = 0$ since $h_0 = 0$. This does not mean the threshold amplitude A becomes infinite, because the terms which have been neglected in the calculation as small should now be considered. Nevertheless, one would expect the two-way machine to have an exceptionally large threshold amplitude as compared to a similar one-way machine. Some numerical results comparing the thresholds for these two machines are given in Section IV.

$\frac{1}{w} >> K$, Mark V, ($2\nu_y - \nu_x = 0$ Resonance)

If we assume that h_1 is pure imaginary, $h_1 = -i|h_1|$, then γ_1 is real and positive, C_1 is pure imaginary and $C_1 = -C_{-1}$.
(Note we again make the angle transformation $\theta \longrightarrow \theta' = \theta - (1/wN) \ln \frac{r}{r_1}$ explained in MURA-362 which replaces $\overline{h_n}$ by h_n in our formuli.)

We find for the matrix element

$$\frac{1}{N^4} C_{s,ss} = \frac{C_0}{N^4} + 8 \frac{\nu_{xs}}{N} \frac{C_1}{N^4}$$

$$\frac{1}{N^4} C_{s,ss} = \frac{1}{2} b h_0 \frac{k(k+1)}{N^4}$$

$$- 3 \frac{k}{N^2} \sum_{n \geq 1} \left| \frac{b h_n}{\mu N^2} \right|^2 \tag{3.41}$$

$$+ i \frac{4}{|b h_1|} \frac{\nu_{xs}}{N} \left(\frac{b h_1}{\mu N^2} \right)^3$$

and

$$M = \frac{1}{N^4} \left| C_{s,ss} \right| \tag{3.42}$$

For small flutter machines $b h_1 << 1$, the third term in Eq. (3.41) becomes dominant, and our result for A then becomes similar to that previously found by Laslett and Sessler.[1]

IV. COMPARISON WITH NUMERICAL RESULTS

We will compare the theory with numerical results for the two resonances treated in Section III.

Case I, $\nu_x + 2\nu_y = N$ Resonance

The quantities which are found numerically using the IBM 704 are $\mu(0)$ and $\eta(0)$ which are the largest values of μ and η, at $\theta = 0$ which give rise to stable orbits; $\mu'(0)$ and $\eta'(0)$ are put equal to zero.

In Table I, a comparison is made between the theoretical and numerical results for two machines. The agreement is somewhat better for Machine B

which has a larger N.

TABLE I $2\nu_y + \nu_x = N$ Resonance

Machine	K	$\frac{1}{w}$	N	h_0	$\lvert h_0 \rvert$	ν_x/N	ν_y/N	$2\frac{\nu_y}{N} + \frac{\nu_x}{N}$
A	7.5	0	18	0	4.46	.2914	.3613	1.014
B	96	2524	40	1	.25	.3036	.3571	1.018

	$N^2 \mu$		$N^2 \eta$	
Machine	Theo.	Numerical	Theo.	Numerical
A	5.9	2.4	2.5	2.1
B	.07	.11	.013	.015

Case II. $\nu_x - 2\nu_y = 0$ **Resonance**

For this resonance the quantity that is found numerically using the IBM 704 is the threshold value of μ at $\theta = 0$ for y-growth. The motion is started with $\mu'(0) = \eta'(0) = 0$ and with a very small y.

In Table II, a comparison is made between the theoretical and numerical results for three machines. One may notice that agreement is much better for Machine B where N = 16 than for Machine A where N = 8. This is mostly due to the larger N for Machine B. The error due to neglected small machine terms should decrease like $1/N^2$. Machine C is a two-way machine, $h_0 = 0$, for which the theory gives an infinite threshold to first approximation. The threshold $-\mu$ for Machine C is much larger than that for

Machine B although they are both about the same distance from the resonance line. The threshold $-\mu$ found for Machine C is quite close to the stability limit and it is possible that the threshold measured is not due to the $2\nu_y - \nu_x = 0$ but to some other nearby resonance.

TABLE II. $2\nu_y - \nu_x = 0$ Resonance

Machine	K	$\frac{1}{w}$	N	h_0	h_1	$\frac{\nu_x}{N}$	$\frac{\nu_y}{N}$	$2\frac{\nu_y}{N} - \frac{\nu_x}{N}$
A	3.36	0	8	1	1.6	.3248	.1635	.0022
B	9.3	0	16	1	3.89	.268	.1438	.02
C	90	0	64	0	4.78	.2266	.1043	.018

Machine	$N^2\mu$	
	Theo.	Numerical
A	.36	.15
B	3.4	3.6
C	∞	18.5

REFERENCES

1. Laslett and Sessler, MURA-263
2. F. T. Cole, MURA-FTC/3
3. T. Ohkawa, MURA-318