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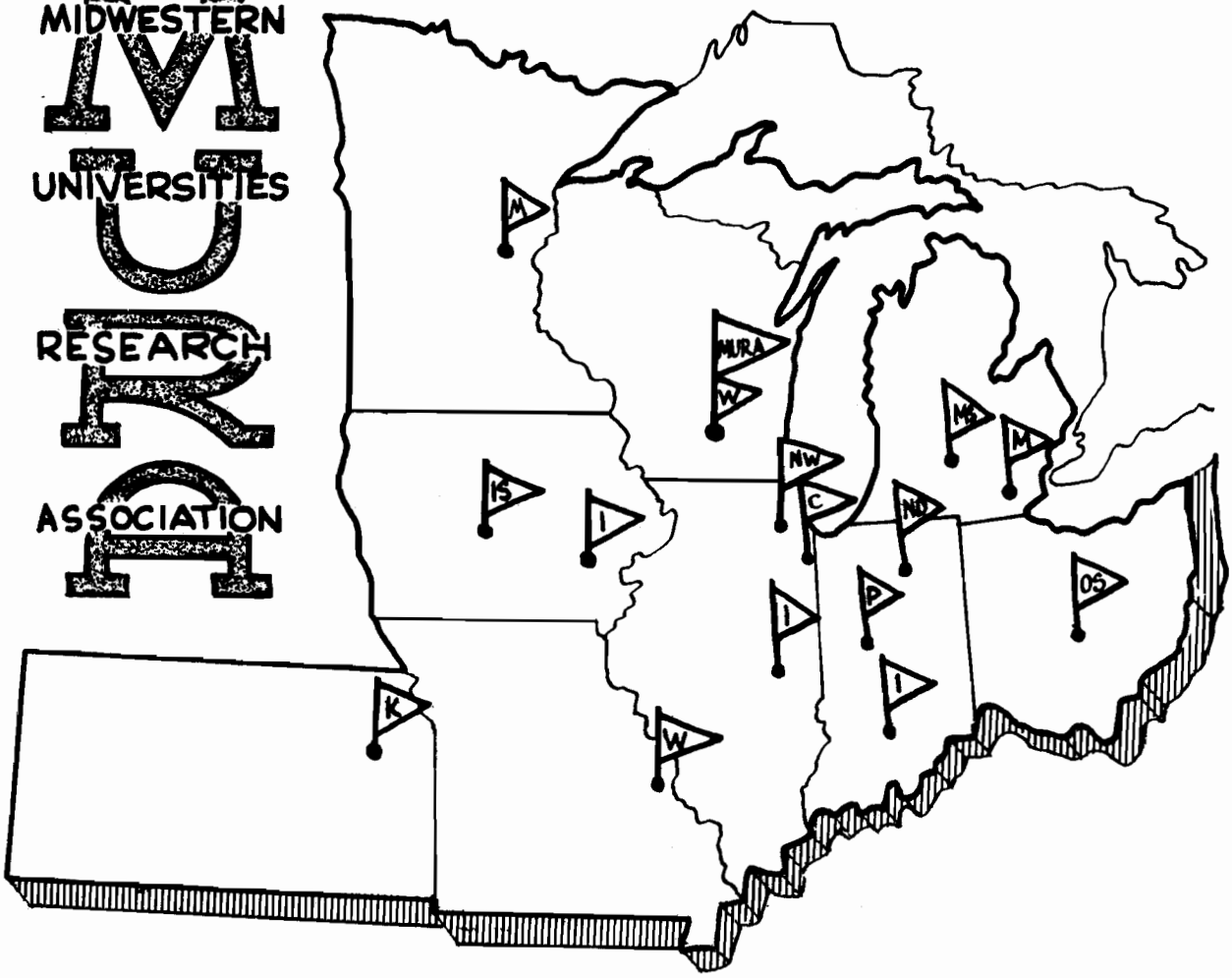

FIELD CALCULATIONS FOR DIGITAL COMPUTATION WITH NON-SCALING DYNAMICS

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FIELD CALCULATIONS FOR DIGITAL COMPUTATION WITH NON-SCALING DYNAMICS

F. T. Cole**

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ABSTRACT: Recursion relations are derived for computation of fields off the median plane when the median plane field is given in the case where k, $K = 1/w$ and the Fourier coefficients are functions of radius. Application to digital computation is discussed.

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I. Introduction

In the past, MURA work has been confined to the case of scaling dynamics, where the parameters k, $\kappa = 1/w$ and the Fourier coefficients μ_{om} and ν_{om} of the median plane field are constants. All three accelerators thus far designed by this group have been scaling dynamically as closely as possible. There are slight non-scaling effects with k-tuning which have not been investigated in detail. It is interesting to consider theoretically relaxation of this requirement in order to treat problems of k-tuning, FFAG cyclotrons and because it is possible that the circumference factor can be improved with non-scaling fields. Analytical work on this problem has been done by Teng,[1] who has developed a theory from the "unified FFAG field" point of view and applied it mainly to FFAG cyclotrons, and by Roberts,[2] who has shown that improvement in spiral sector design can be gained in this way. Walkinshaw and his collaborators have discussed[3] the problem of radial straight sections in spiral sector accelerators, which is another example of a non-scaling field.

The present report is a suggestion for a digital computer program to treat dynamics with non-scaling fields in the manner of the "Well-Tempered Five" program.[4] It is being given a certain species of (pronounced specious) immortality as an internal report because interest in this topic appears to be widespread.

II. Development of Field Relations

It is desired to develop recursion relations which are generalizations of Powell's expansion,[5] where the fields off the median plane are developed from the field on that plane. We write the fields in the same form used by Powell,

$$\begin{cases} B_x = -(1+x)^k \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} (\alpha_{im} c_m + \beta_{im} s_m) \frac{1}{(2i-1)!} \left(\frac{y}{1+x}\right)^{2i-1} \\ B_y = (1+x)^k \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} (\mu_{im} c_m + \nu_{im} s_m) \frac{1}{(2i)!} \left(\frac{y}{1+x}\right)^{2i} \\ B_\theta = -(1+x)^k \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} (\gamma_{im} c_m + \delta_{im} s_m) \frac{1}{(2i-1)!} \left(\frac{y}{1+x}\right)^{2i-1}, \end{cases} \quad (1)$$

where the coefficient ($-B_0$) has been suppressed and where

$$\begin{cases} r = r_0(1+x) \\ z = r_0 y \\ c_m = \cos m\Phi \\ s_m = \sin m\Phi \\ \Phi = Kq - N\theta \\ q = \ln(1+x). \end{cases} \quad (2)$$

K is also known for historical reasons as 1/w. The Well-Tempered Five Program calculates initially and stores the coefficients μ_{im}, ν_{im}, α_{im} and β_{im} (γ_{im} and δ_{im} are given simply in terms of ν_{im} and μ_{im}). At each integration step the fields are calculated (four times) from (1), using the local values of x and y.

Recursion relations of the proper form follow if we consider k, K, μ_{om} and ν_{om} as functions of q. Derivatives with respect to q will be denoted by primes. We define

$$\begin{cases} \varphi_1 = k + k'q - 2i \\ \varphi_2 = K + K'q \quad . \end{cases} \quad (3)$$

Maxwell's equations can be written in these dimensionless cylindrical coordinates as

$$\nabla\cdot\underline{B} = \frac{1}{1+x}\left\{\frac{\partial}{\partial x}\left[(1+x)B_x\right] + \frac{\partial B_\theta}{\partial\theta} + (1+x)\frac{\partial B_y}{\partial y}\right\} = 0 \,.$$

$$\begin{aligned}\nabla\times\underline{B} = {} & \underline{i}_x\left\{\frac{1}{1+x}\frac{\partial B_y}{\partial\theta} - \frac{\partial B_\theta}{\partial y}\right\} + \underline{i}_\theta\left\{\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right\} \\ & + \underline{i}_y\left\{\frac{1}{1+x}\left[\frac{\partial}{\partial x}\left((1+x)B_\theta\right) - \frac{\partial B_x}{\partial\theta}\right]\right\} = 0 \,.\end{aligned} \tag{4}$$

The fields (1) can be derived from a scalar potential Ψ by

$$\underline{B} = -\nabla\Psi = -\left\{\underline{i}_x\frac{\partial\Psi}{\partial x} + \underline{i}_\theta\frac{1}{1+x}\frac{\partial\Psi}{\partial\theta} + \underline{i}_y\frac{\partial\Psi}{\partial y}\right\}. \tag{5}$$

Substitution of the fields (1) into (4) gives the recursion relations for B_θ

$$\begin{cases}\gamma_{i,m} = mN\nu_{i-1,m} \\ \delta_{i,m} = -mN\mu_{i-1,m} \,.\end{cases} \tag{6}$$

Thus it is not necessary to store γ_{im} and δ_{im}, since they can be computed easily from $\nu_{i-1,m}$ and $\mu_{i-1,m}$ at each step.

For B_x the recursion relations are

$$\begin{cases}\alpha_{i+1,m} = -\varphi_1\mu_{i,m} - \mu'_{i,m} - m\varphi_2\nu_{i,m} \\ \beta_{i+1,m} = -\varphi_1\nu_{i,m} - \nu'_{i,m} + m\varphi_2\mu_{i,m} \,.\end{cases} \tag{7}$$

The divergence equation gives

$$\begin{cases} \mu_{i+1,m} = m^2 N^2 \mu_{i,m} + \varphi_1 \alpha_{i+1,m} + \alpha'_{i+1,m} + m\varphi_2 \beta_{i+1,m} \\ \nu_{i+1,m} = m^2 N^2 \nu_{i,m} + \varphi_1 \beta_{i+1,m} + \beta'_{i+1,m} - m\varphi_2 \alpha_{i+1,m} . \end{cases} \tag{8}$$

(7) and (8) can be combined to eliminate $\alpha_{i+1,m}$ and $\beta_{i+1,m}$.

$$\begin{cases} \mu_{i+1,m} = \left\{ m^2[N^2+\varphi_2^2] - \varphi_1^2 - \varphi_1' \right\} \mu_{i,m} - 2\varphi_1 \mu'_{i,m} - \mu''_{i,m} \\ \qquad - m\left\{ 2\varphi_1\varphi_2 + \varphi_2' \right\} \nu_{i,m} - 2m\varphi_2 \nu'_{i,m} \\ \nu_{i+1,m} = \left\{ m^2[N^2+\varphi_2^2] - \varphi_1^2 - \varphi_1' \right\} \nu_{i,m} - 2\varphi_1 \nu'_{i,m} - \nu''_{i,m} \\ \qquad + m\left\{ 2\varphi_1\varphi_2 + \varphi_2' \right\} \mu_{i,m} + 2m\varphi_2 \mu'_{i,m} . \end{cases} \tag{9}$$

The relations (7) and (9) are the desired generalizations of Powell's recursion relations and with $\alpha_{0m} = \beta_{0m} = 0$, enable the fields to be calculated.

It is of interest to note that in the same way, the scalar potential

$$\Psi = (1+x)^{k+1} \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \left(\mu_{i-1,m} c_m + \nu_{i-1,m} s_m \right) \frac{1}{(2i-1)!} \left(\frac{y}{1+x} \right)^{2i-1} . \tag{10}$$

III. Application to Digital Computation

Eqs. (7) and (8) or (7) and (9) constitute a solution of the field problem (convergence difficulties aside). However, they are not in a form suitable for digital computation because the parameters which appear implicitly functions of x. Let us expand these parameters, viz.,

$$\begin{cases} k = \sum_{j=0}^{j_{max}} k_j x^j \\ \kappa = \sum_{j=0}^{j_{max}} \kappa_j x^j \\ \mu_{i,m} = \sum_{j=0}^{j_{max}} \mu_{i,m,j} x^j \\ \nu_{i,m} = \sum_{j=0}^{j_{max}} \nu_{i,m,j} x^j \\ \alpha_{i,m} = \sum_{j=0}^{j_{max}} \alpha_{i,m,j} x^j \\ \beta_{i,m} = \sum_{j=0}^{j_{max}} \beta_{i,m,j} x^j . \end{cases} \tag{11}$$

The φ_n can be expanded in powers of x, i.e.,

$$\varphi_n = \sum_{j=0}^{j_{max}} \varphi_{nj} x^j , \tag{12}$$

and from (11) and (3),

$$\begin{cases} \varphi_{10} = k_0 - 2i , \\ \varphi_{11} = 2k_1 , \\ \varphi_{12} = 3k_2 + \frac{1}{2} k_1 , \\ \varphi_{13} = 4k_3 + k_2 - \frac{1}{6} k_1 , \\ \varphi_{14} = 5k_4 + \frac{3}{2} k_3 - \frac{1}{3} k_2 + \frac{1}{12} k_1 , \\ \text{etc.,} \end{cases} \tag{13}$$

and

$$\begin{cases} \varphi_{20} = K_0 , \\ \varphi_{21} = 2K_1 , \\ \varphi_{22} = 3K_2 + \frac{1}{2} K_1 \\ \varphi_{23} = 4K_3 + K_2 - \frac{1}{6} K_1 \\ \varphi_{24} = 5K_4 + \frac{3}{2} K_3 - \frac{1}{3} K_2 + \frac{1}{12} K_1 , \\ \text{etc..} \end{cases} \tag{14}$$

Only φ_{10} must be recalculated for each i.

Then

$$\begin{cases} \alpha_{i+1,m,j} = -\sum_{\ell=0}^{j} \Big\{ \varphi_{1\ell} \mu_{i,m,j-\ell} + \ell \mu_{i,m,\ell} + (\ell+1) \mu_{i,m,\ell+1} \\ \qquad\qquad + m \varphi_{2\ell} \nu_{i,m,j-\ell} \Big\} \\ \beta_{i+1,m,j} = -\sum_{\ell=0}^{j} \Big\{ \varphi_{1\ell} \nu_{i,m,j-\ell} + \ell \nu_{i,m,\ell} + (\ell+1) \nu_{i,m,\ell+1} \\ \qquad\qquad - m \varphi_{2\ell} \mu_{i,m,j-\ell} \Big\} \end{cases} \tag{15}$$

and

$$\begin{cases} \mu_{i+1,m,j} = m^2 N^2 \mu_{i,m,j} + \sum_{\ell=0}^{j} \Big\{ \varphi_{1\ell} \alpha_{i+1,m,j-\ell} + \ell \alpha_{i+1,m,\ell} \\ \qquad + (\ell+1) \alpha_{i+1,m,\ell+1} + m \varphi_{2\ell} \beta_{i+1,m,j-\ell} \Big\} \\ \nu_{i+1,m,j} = m^2 N^2 \nu_{i,m,j} + \sum_{\ell=0}^{j} \Big\{ \varphi_{1\ell} \beta_{i+1,m,j-\ell} + \ell \beta_{i+1,m,\ell} \\ \qquad + (\ell+1) \beta_{i+1,m,\ell+1} - m \varphi_{2\ell} \alpha_{i+1,m,j-\ell} \Big\} . \end{cases} \tag{16}$$

The suggested field computation would proceed in the following way. The user would supply the k_j, K_j, $\mu_{0,m,j}$ and $\nu_{0,m,j}$. The program would then calculate

(A) The φ_{nj} from (13) and (14),

(B) The $\alpha_{1,m,j}$ and $\beta_{1,m,j}$ from (15),

(C) The $\mu_{1,m,j}$ and $\nu_{1,m,j}$ from (16),

(D) A new value of φ_{10} for $i = 1$,

(E) Repeat B and C with $i = 1$,

and so on through $i = i_{max}$. The quantities $\mu_{i,m,j}$, $\nu_{i,m,j}$, $\alpha_{i,m,j}$ and $\beta_{i,m,j}$ would be stored. At this point the routines used to calculate them could be cleared to give additional memory space.

At each step of the integration, the program would calculate k, K, $\mu_{i,m}$, $\nu_{i,m}$, $\alpha_{i,m}$ and $\beta_{i,m}$ from (11) using the appropriate x and y and, using these, the fields which enter the equations of motion from (1).

The additional time per step over Well-Tempered Five can be estimated. That program appears to require approximately

$$10M + i_{max} + 1 + 3\, i_{max}\, M$$

multiplications and

$$3M + 3$$

additions in the field calculation at each step, where i_{max} is the largest i and M the largest m (both set by the user). The same calculation done with non-scaling fields as outlined above would require approximately

$$2\, j_{max} \left(i_{max} M + \tfrac{1}{2} \right)$$

additional multiplications and

$$i_{max} M j_{max}$$

additional additions. The time for floating point additions is approximately 84 microseconds, while the time for floating point multiplications is approximately 204 microseconds, so that we may neglect the addition time. The time for field calculation is long compared to the time for the Runge-Kutta process, so that a rough answer can be obtained by comparing only multiplication times for field calculation. Then the ratio of the time per step of the non-scaling program to that of Well-Tempered Five is about

$$\frac{10M + i_{max} + 1 + 3 i_{max} M + 2 j_{max} i M}{10M + i_{max} + 1 + 3 i_{max} M} .$$

This ratio appears to be approximately 2 over the range of interest. It appears that such a program could produce worthwhile results if used carefully. The specification of a j_{max}, similar to the i_{max}, of Well-Tempered Five is essential for speed. Most investigations will probably have $j_{max} = 1$.

Since ($j_{max} + 1$) times as many quantities are to be stored, it is clear that i_{max} and M must be reduced from the generous limits of Well-Tempered Five ($i_{max} = 24$ and $M = 30$). It is suggested that the following limits might be appropriate.

$$i_{max} = 12.$$
$$M = 15.$$
$$j_{max} = 4.$$

It is suggested finally that an appropriate name for such a program would be "Ill-Tempered Five," reflecting both the physical change of harmonics with radius and the possible emotional response of a programmer confronted with this task.

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