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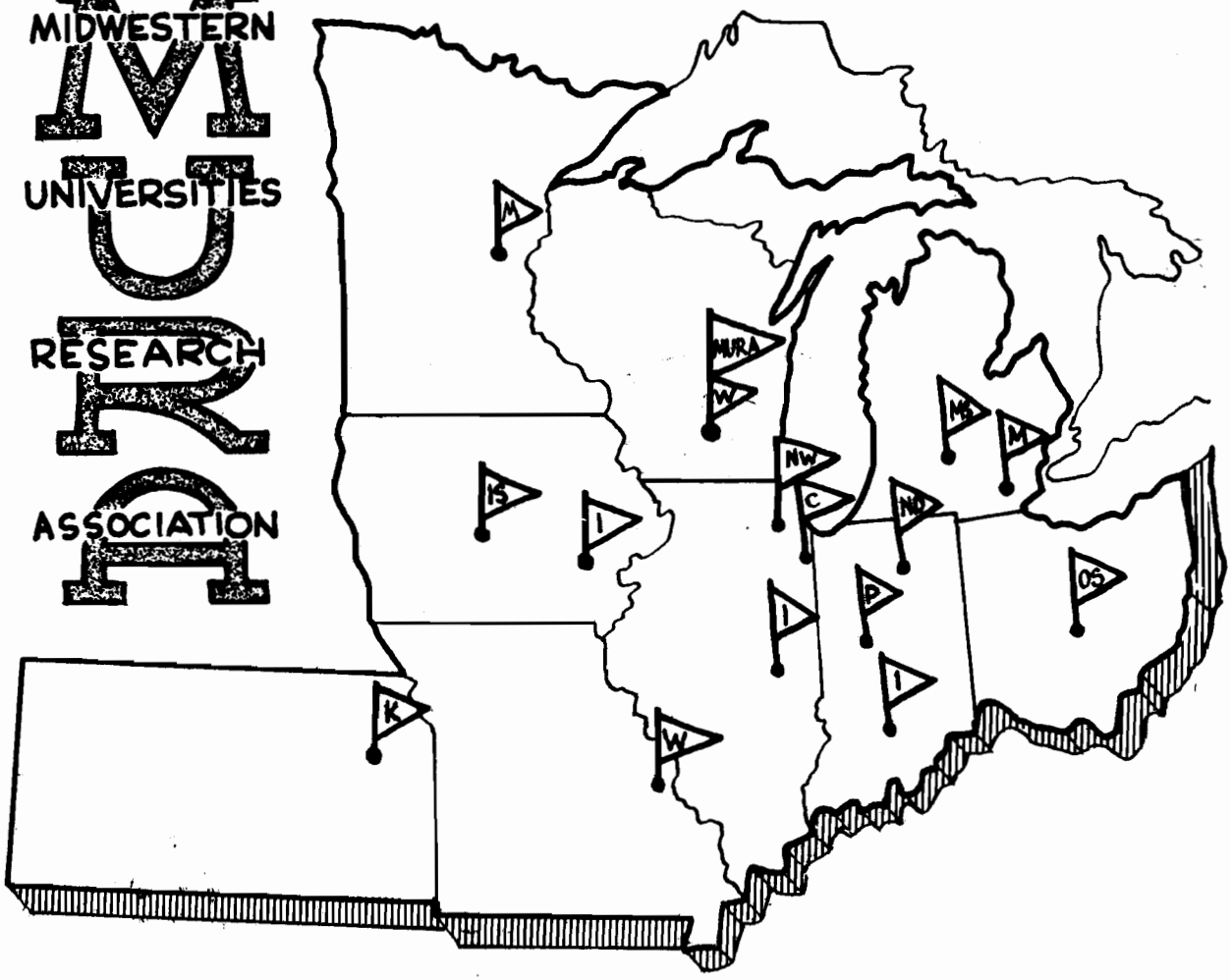

THEORY OF THE $2\nu_y - \nu_x = 0$ NON-LINEAR RESONANCE IN ALTERNATING GRADIENT ACCELERATORS

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THEORY OF THE $2\nu_y - \nu_x = 0$ NON-LINEAR RESONANCE IN ALTERNATING GRADIENT ACCELERATORS

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ABSTRACT

The method presented in reports MURA-200 and MURA-217 for treating the non-linear resonances in alternating gradient accelerators is applied to the $2\nu_y - \nu_x = 0$ coupled resonance. The results of the theory for the threshold amplitudes for the appearance of growth are compared with those of numerical calculations.

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I. Introduction

In two previous reports, MURA-200 and MURA-217, a method was presented for treating non-linear resonances. In MURA-217, coupled sum resonances were treated. In this paper, we will apply the methods used in MURA-217 to coupled difference resonances. In particular we will treat the difference resonance $2\nu_y - \nu_x = 0$ in detail.

It has been pointed out by Hagedorn,[1] Moser[2] and Sturrock[3] that the $2\nu_y - \nu_x = 0$ resonance does not cause instability. However, it has been found by Walkinshaw[4], and Laslett and Sessler,[5] that when the amplitudes of motion exceeds certain limits, the motion is characterized by exceptional growth in the amplitude of the motion. We shall regard orbits which exhibit growth as "unstable," and in this sense there exist stability limit amplitudes or threshold amplitudes for growth which will be found in this paper.

Some properties of y-growth have been previously treated by Laslett and Sessler.[5] Work on this difference resonance has also been done by K. Symon using Moser's method.

II. Stability Limits of the $2\nu_y - \nu_x = 0$ Resonance

We will treat the $2\nu_y - \nu_x = 0$ resonance in detail. Similar methods should apply to the higher order difference resonances.

The procedure here is very similar to that used in MURA-217 for the sum resonances, and we will use the same notation.

The equations of motion, up to quadratic terms, are given by

$$\left\{\frac{d^2}{d\theta^2} + E_0 - g_1(\theta)\right\} u = B(\theta) u^2 + C(\theta) y^2, \tag{2.1a}$$

$$\left\{\frac{d^2}{d\theta^2} + E_0' - g_2(\theta)\right\} y = D(\theta) u y. \tag{2.1b}$$

Proceeding as in MURA-217, we replace Eqs. (2) by an eigenvalue problem. We consider the more general equations

$$\left\{\frac{d^2}{d\theta^2} + W - g_1(\theta)\right\} u = B(\theta) u^2 + C(\theta) y^2, \tag{2.2a}$$

$$\left\{\frac{d^2}{d\theta^2} + W' - g_2(\theta)\right\} y = D(\theta) u y, \tag{2.2b}$$

and we will ask for what real values of W and W' do Eqs. (2.2) have stable solutions.

As in MURA-217, we solve Eqs. (2.2) by expanding u and y in the flocquet solutions of the linear equations corresponding to Eqs. (2.2). We expand u and y as

$$u = \sum_i a_i u_i(\theta), \tag{2.3a}$$

$$y = \sum_i b_i y_i(\theta), \tag{2.3b}$$

and we find the equations for the a_i and b_i

$$(W-E_i)\,a_i = \sum_{j\ell} \left\{ B_{i,j\ell}\, a_j\, a_\ell + C_{i,j\ell}\, b_j\, b_\ell \right\}, \tag{2.4a}$$

$$(W'-E_i')\,b_i = \sum_{j\ell} D_{i,j\ell}\, a_j\, b_\ell , \tag{2.4b}$$

The $B_{i,j\ell}$, $D_{i,j\ell}$ and $C_{i,j\ell}$ are defined in MURA-217. We proceed as in MURA-217 to solve our eigenvalue problem by a perturbation procedure.

Consider the E -point (E_s, E_s') of the linear equations which has the tune (ν_s, K_s). If the tune of this E -point lies on the line $\nu_s - 2K_s = 0$ then the shift in this E -point due to the non-linear perturbation is given by Eqs. (2.18) in MURA-217, which gives

$$(W-E_s)\,a_s = C_{s,ss}\, b_s^2 , \tag{2.5a}$$

$$(W'-E_s')\,b_s = D_{s,s\bar{s}}\, a_s\, b_{\bar{s}} . \tag{2.5b}$$

The shift in the E -point (E_s, E_s') is given by $W-E_s$, $W'-E_s'$. The magnitude of the shift is given by

$$|W-E_s| = |C_{s,ss}| \; \frac{|b_s^2|}{|a_s|} , \tag{2.6a}$$

$$|W'-E_s'| = |D_{s,s\bar{s}}| \; |a_s| . \tag{2.6b}$$

We can show that ($W - E_s$) and ($W' - E_s'$) have the same sign. From Eqs. (2.5) we see that

$$ph(W - E_s) = ph\, C_{s,ss} + 2\beta - \alpha, \tag{2.7a}$$

$$ph(W' - E_s') = -ph\, C_{s,ss} - 2\beta - \alpha, \tag{2.7b}$$

where ph means phase and $ph\, a_s = \alpha$, $ph\, b_s = \beta$ and $ph\, D_{s,s\bar{s}} = -ph\, C_{s,ss}$.

Thus

$$ph(W - E_s) = -ph(W' - E_s'). \tag{2.8}$$

Since $W - E_s$, and $W' - E_s'$ must be real, by the definition of W and E_s, W' and E_s', $ph(W - E_s)$ and $ph(W' - E_s')$ can only be zero or π. Thus if $ph(W - E_s) = 0$, then $ph(W' - E_s') = 0$; if $ph(W - E_s) = \pi$, $ph(W' - E_s') = -\pi$ and ($W - E_s$), ($W' - E_s'$) have the same signs.

Each E -point on the $2\nu - K = 0$ line is split into two points as shown in Fig. 1.

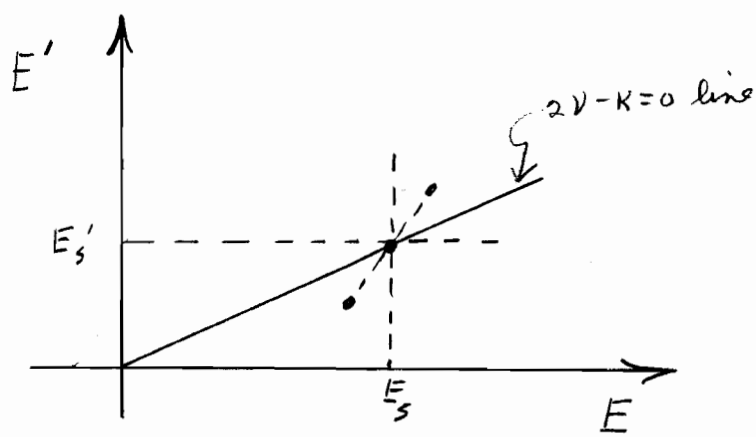

Fig. 1

A gap appears around the $2\nu - K = 0$ line, as shown in Fig. 2. The size of the gap depends on the amplitudes of motion a_s and b_s.

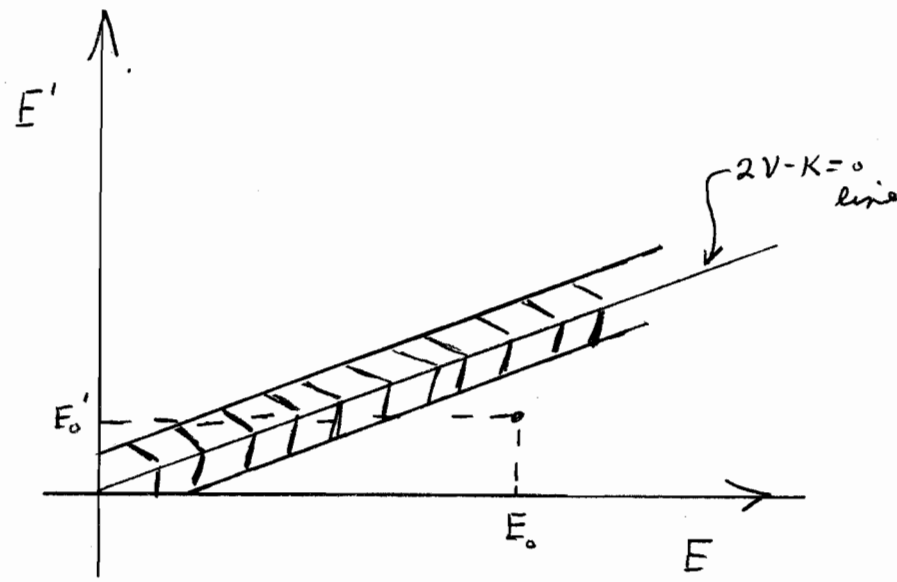

Fig. 2

The stability limit amplitudes a_s, b_s to drive the tune of the motion to (ν_s, K_s) which lies on the resonance line $2\nu - K = 0$ is given by

$$|a_s| = \frac{|E_0' - E_s'|}{2\,|C_{s,ss}|}, \tag{2.9a}$$

$$|b_s| = \left\{ \frac{|E_0 - E_s|\,|E_0' - E_s'|}{2\,|C_{s,ss}|^2} \right\}^{\frac{1}{2}}. \tag{2.9b}$$

One should note that the [illegible] tune can be driven only to the part of the resonance line shown in Fig. 3.

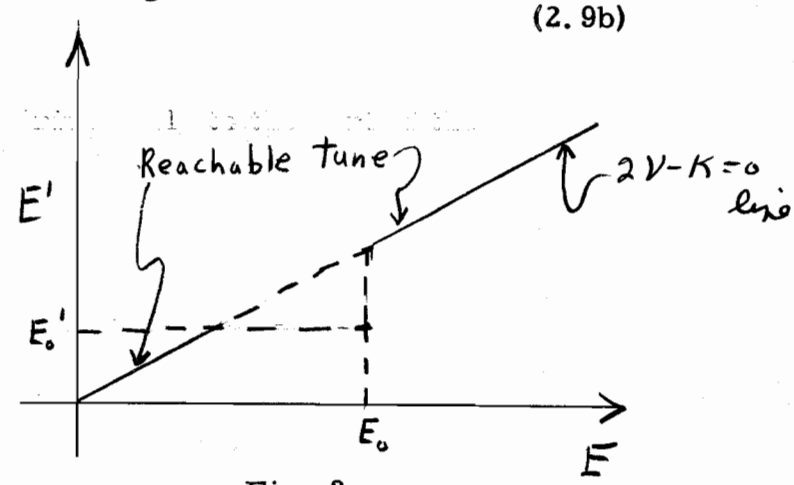

Fig. 3

In formulas (2.9) it is often adequate to put $E_s = \nu_s^2$, $E_s' = K_s^2$, and $E_0 = \nu_0^2$, $E_0' = K_0^2$. In calculating $C_{s,ss}$, one must usually be more careful than in the sum resonance case where one can use plane waves for the flocquet functions. We can then write Eqs. (2.9) as

$$|a_s| = \frac{|K_s^2 - K_0^2|}{2|C|}, \tag{2.10a}$$

$$|b_s| = \frac{1}{|C|}\left\{\frac{1}{2}|\nu_s^2 - \nu_0^2|\,|K_s^2 - K_0^2|\right\}^{\frac{1}{2}}, \tag{2.10b}$$

where $|C| = |C_{s,ss}|$ and the dependence of $C_{s,ss}$ on the tune (ν_s, K_s) can be neglected if one is near the resonance line.

We can use Eqs. (2.10) to plot $|b_s|$ against $|a_s|$. Such a plot is a good indication of the shape of the stability surface.

$|b|$ is plotted against $|a|$ in Fig. 4. There are two branches to this curve. One branch corresponds to driving the tune to points on the upper part of the resonance line. This is branch 1 in Fig. 4. The other branch corresponds to tunes on the lower part of the resonance line. This is branch 2.

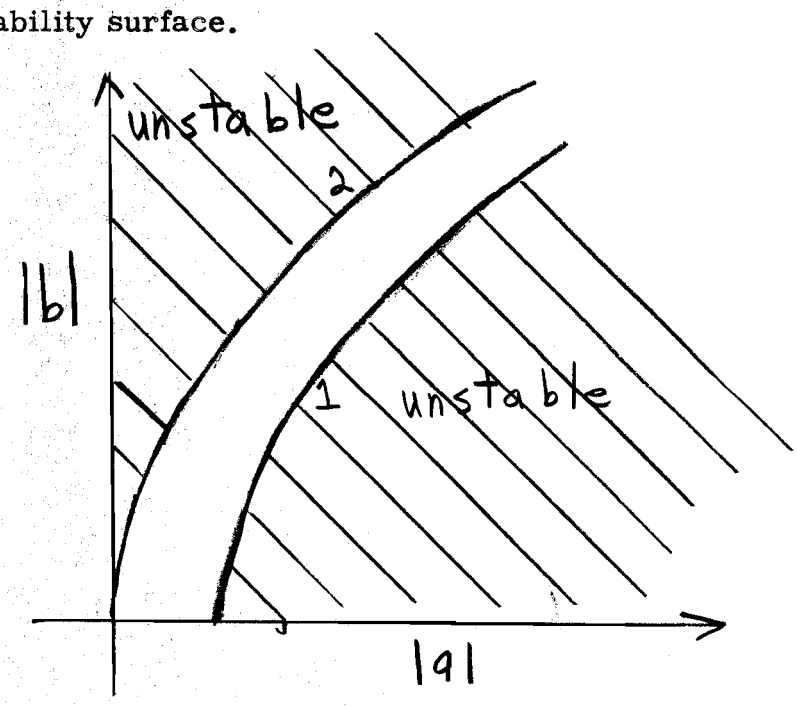

Fig. 4

Branch 1 is given by the equation

$$b = 2.8 \sqrt{a(a - a_{th})}, \tag{2.11}$$

where

$$a_{th} = \frac{\left| K_0^2 - \left(\frac{1}{2}\nu_0\right)^2 \right|}{2|c|}, \tag{2.12}$$

The factor 2.8 comes from $2\sqrt{\nu_0/K_0}$, and $\nu_0/K_0 \simeq 2$ near the resonance line.

Branch 2 is given by

$$b = 2.8 \sqrt{a(a + a_{th})}. \tag{2.13}$$

One can get an idea of the shape of stability surface from Fig. 4. The surface is not closed. The stable region inside the surface is the unshaded region.

We have not said yet what is meant by instability here. From the point of view of this theory, instability means that the motion has a character which is very different from the linear motion; or to put it mathematically, the motion cannot be described by a linear combination of flocquet functions whose tunes do not differ much from the small amplitude tune. Instability here does not necessarily imply that the amplitude will grow very large.

We find, experimentally, that the unstable region below branch 1 in Fig. 4 is characterized by η -growth. The unstable region above branch 2 is characterized by χ-growth. These two kinds of motion, η -growth and χ -growth, will be discussed in the next sections.

Orbits where growth takes place may not be unstable for an accelerator, since the growth may not be so large as to take the particle outside the vacuum chamber and for small initial amplitude, the growth is also small. The results of Moser, Hagedorn and Sturrock which say that the $\nu_x - 2\nu_y = 0$ resonance causes no instability are correct since motions for sufficiently small initial amplitude will continue to have small amplitudes.

However, there is some evidence due to Laslett and Sessler that an orbit showing growth may be particularly susceptible to bumps.

III. η -growth

It has been found experimentally that the stability limit amplitudes given by Eqs. (2.9) and Eqs. (2.10), which correspond to driving the tune to the upper part of the resonance line and which correspond to branch 1 in Fig. 4, are really threshold amplitudes for η -growth. Motions having amplitudes larger than the stability limit amplitudes will not, in general, grow exceedingly large; but if the motion is started with a small η -amplitude, the η -amplitude will grow until it is about 2 or 3 times the χ -amplitude. For motions having amplitudes smaller than the stability limit amplitudes, the η -amplitude will remain small if it is started at a small value.

Eqs. (2.9) for the stability limit amplitudes can be converted into threshold values of μ and η for η -growth, by using the result for the stability

limit orbits,

$$u = a_s u_s + a_s^* u_s^*, \tag{3.1a}$$

$$y = b_s y_s + b_s^* y_s^*, \tag{3.1b}$$

where $\nu_s - 2K_s = 0$ and (ν_s, K_s) lie on the upper part of the resonance line as shown in Fig. 3.

To illustrate the threshold values of u and y, let us consider the case where $g_1(\theta) = g_2(\theta) = 0$, and $C(\theta) = C_0 + 2C_1 \cos N\theta$. Then the fiocquet functions are just plane waves, and $C_{s,ss} = C_0$. We assume that C_0 is positive.

Eqs. (3.1) then become

$$u = 2|a| \cos(\nu\theta + \alpha), \tag{3.2a}$$

$$y = 2|b| \cos(K\theta + \beta). \tag{3.2b}$$

α and β are not independent.

According to Eq. (2.6), α and β are related by

$$\alpha - 2\beta = \pi \tag{3.3a}$$

on the upper part of the resonance line.

$$\alpha - 2\beta = 0 \tag{3.3b}$$

on the lower part of the resonance line.

Thus all points on the stability surface which have $\mu' = 0$, $y' = 0$ at $\theta = 0$, and which correspond to tunes on the upper resonance line are given by

$$\mu = -2|a| \quad , \quad y = \pm 2|b| \quad ,$$

and these points are shown in Fig. 5.

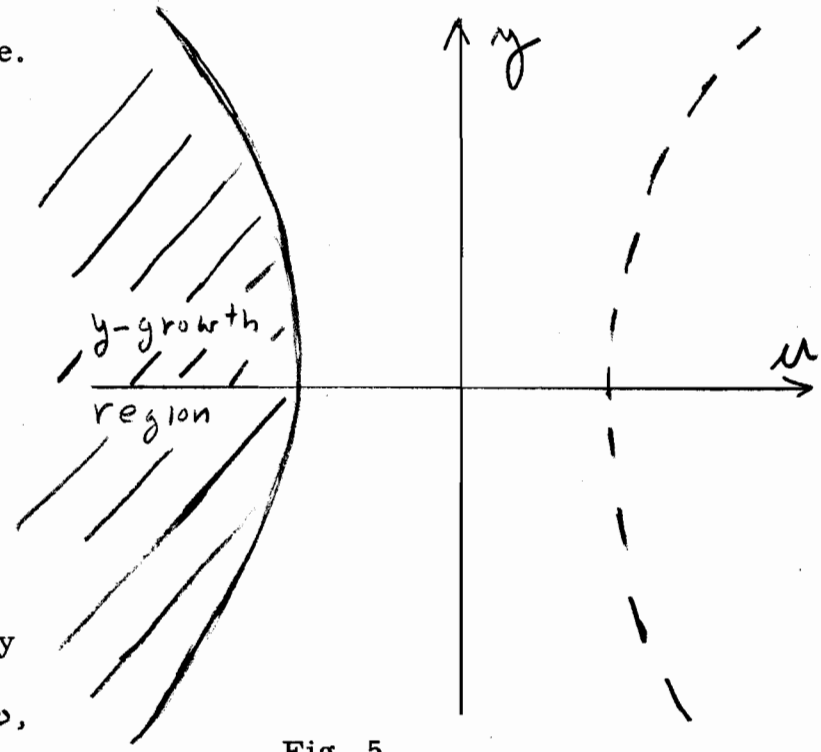

Fig. 5

Although the stability surface only has points for $-\mu$ when $\mu' = y' = 0$, as shown in Fig. 5, the stability region for y -growth does have a boundary on the right side shown by the dashed curve. The points on the dashed curve lie inside the stability surface.

If we start with some initial value of y, we can then vary μ coming in from $-\infty$ to find the threshold for y -growth. The threshold values of μ and y are related by Eq. (2.11). Thus they are related by

$$y = 2.8 \sqrt{|\mu|\,(|\mu| - \mu_{th})} \quad , \tag{3.4}$$

where

$$\mu_{th} = \frac{\left| K_0^2 - \left(\frac{1}{2}\nu_0\right)^2 \right|}{C_0} \quad . \tag{3.5}$$

Eq. (3.4) has been tested by numerical calculation. In Table I, the threshold μ for growth is given as a function of the initial y as determined numerically and theoretically for a case where $\nu_0 = .23$, $K_0 = .105$, $C_0/N^4 = .04$, $C_1/N^4 = .075$. μ and y are given in units of $1/N^2$.

The threshold for y-growth is somewhat ambiguous to determine numerically. We have arbitrarily assumed that a run has no growth if the y-motion increases by less than .02 in units of $1/N^2$ for this problem.

We would like to point out that both the formulae in Eq. (2.9) and in Eq. (3.4) start to break down when the tune change caused by the increasing amplitude becomes too large. Too large in this case means a tune change much larger than $|2K_0 - \nu_0|$. When the tune change becomes too large, other resonances, like the $2K + \nu = N$ resonance, affect the motion.

TABLE I

$y_{initial}$	$\mu_{threshold}$	
	Theory	Numerical
.0001	.0525	.05
.1347	.0788	.074
.2200	.105	.095
.3812	.158	.09

IV. x-growth

It has been found experimentally that the stability limit amplitudes given by Eq. (2.9) and Eq. (2.10) which correspond to driving the tune to the lower part of the resonance line, and which correspond to branch 2 in Fig. 4, are

really threshold amplitudes for x -growth. The x -growth is not as marked as the x -growth, but it is in many ways a similar phenomenom and the threshold value of μ and y for x -growth can be found as was done for y-growth in section III.

For the case when $g_1(\theta) = g_2(\theta) = 0$, the threshold values of μ and y for x -growth are related

$$y = 2.8 \sqrt{|\mu|(|\mu| + \mu_{th})} \, . \tag{3.5}$$

In Table II, the threshold y for x -growth is given as a function of the initial μ as determined numerically and theoretically for the same numerical example treated in section III.

In determining the threshold values for x -growth, we have arbitrarily assumed that a run has no growth if the x -motion increases by less than .005 in units of $1/N^2$ for this problem.

TABLE II

$\mu_{initial}$	y threshold	
	Theory	Numerical
0	0	.07
.0262	.1347	.127
.0525	.22	.17
.105	.38	.24

V. Turnover point for y-growth

If the motions showing y-growth may be considered as stable, then an important question is how far will y grow for a given initial u.

We will treat the case where $y_1(0) = y_2(0) = 0$. Similar arguments can be used if they are not zero. In this case, the threshold values of u and y for y-growth are given by Eq. (3.4). It seems at least qualitatively plausible, that if one starts with an initial u, the y should not grow past the y for which the initial u is the threshold value. Thus we would guess that the y-turnover point for an initial u is given by

$$y = 2.8 \sqrt{|u|(|u| - u_{th})} \quad . \tag{5.1}$$

Eq. (5.1) is only a plausible guess; however, numerical computations show that it holds quite well.

In Table III the theoretical and numerical y-turnover points are compared for the numerical example given in Section III.

One can present a similar argument for motions having x-growth. The x-turnover point for an initial y is given through the relation

$$|y| = 2.8 \sqrt{u(u + u_{th})} \tag{5.2}$$

In Table IV, the theoretical and numerical x-turnover points are compared for the numerical example given in Section III.

TABLE III

Initial ν = .0001

Initial μ	ν -turnover	
	Numerical	Theory
.016	.30	.38
.08	.12	.135
.06	.058	.06
.052	.0007	.0001
.04	.0001	.0001
.02	.0001	.0001

Initial ν = .135

Initial μ	ν -turnover	
	Numerical	Theory
.11	.23	.235
.09	.19	.18
.0788	.167	.135
.07	.150	.135
.05	.135	.135

Initial ν = .22

Initial μ	ν -turnover	
	Numerical	Theory
.18	.6	.45
.14	.35	.32
.12	.32	.27
.105	.26	.22
.09	.23	.22
.07	.22	.22

TABLE IV

Initial μ = 0

Initial γ	μ -turnover	
	Numerical	Theory
.2	.06	.045
.08	.02	.015
.05	.003	.003

Initial μ = .026

Initial γ	μ -turnover	
	Numerical	Theory
.25	.09	.065
.17	.045	.038
.15	.038	.030
.135	.033	.026
.12	.029	.026
.10	.026	.026

Initial μ = .0525

Initial γ	μ -turnover	
	Numerical	Theory
.40	.23	.11
.26	.10	.065
.24	.09	.06
.22	.08	.0525
.20	.07	.0525
.18	.06	.0525
.16	.056	.0525
.14	.0525	.0525

VI. The Stability Limits

The difference resonance $2\nu_y - \nu_x = 0$ does not give rise to instability. Nevertheless, for sufficiently large amplitudes the motion will become unstable. These stability limits do not vary very much when the tune is varied across the resonance line $2\nu_y - \nu_x = 0$. It is reasonable to assume that the stability limits are due to the other resonance lines present like the $\nu_x = \frac{1}{3}N$ or $2\nu_y + \nu_x = N$ resonances.

It is difficult to calculate the stability limits with accuracy as the linear tune is usually far from the resonance line causing the instability. We can estimate the stability limits by the following procedure. We consider all the resonance lines which are near by, and calculate the x and y stability limits due to each resonance by assuming it is present all alone. We then take the smallest stability limits so found as the actual stability limits, and the resonance line which gives rise to the smallest stability limits is the "closest" and the dominant resonance line. It has been found in a number of numerical examples that this procedure gives the stability limits within a factor of 2 or 3. Usually the procedure gives too large a stability limit by a factor of 2 or 3.

For the numerical example defined in Section III, the closest resonance is found to be the $\nu_x = 0$ coupling resonance (The theory of this resonance will be presented in a future report. We will assume the results here.). For the $\nu_x = 0$ resonance, the x -stability limit is given for this example by

$$A = \frac{\nu_y^2}{C_0} \tag{6.1}$$

The η -stability amplitude is given by

$$B = 2\sqrt{\frac{\nu_x}{\nu_\eta}} \cdot A \tag{6.2}$$

We find the following comparison of theory and the numerical calculation

$$A = \begin{cases} .27 & \text{Theory} \\ .25 & \text{Numerical} \end{cases}$$

$$B = \begin{cases} .75 & \text{Theory} \\ .45 & \text{Numerical} \end{cases}$$

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