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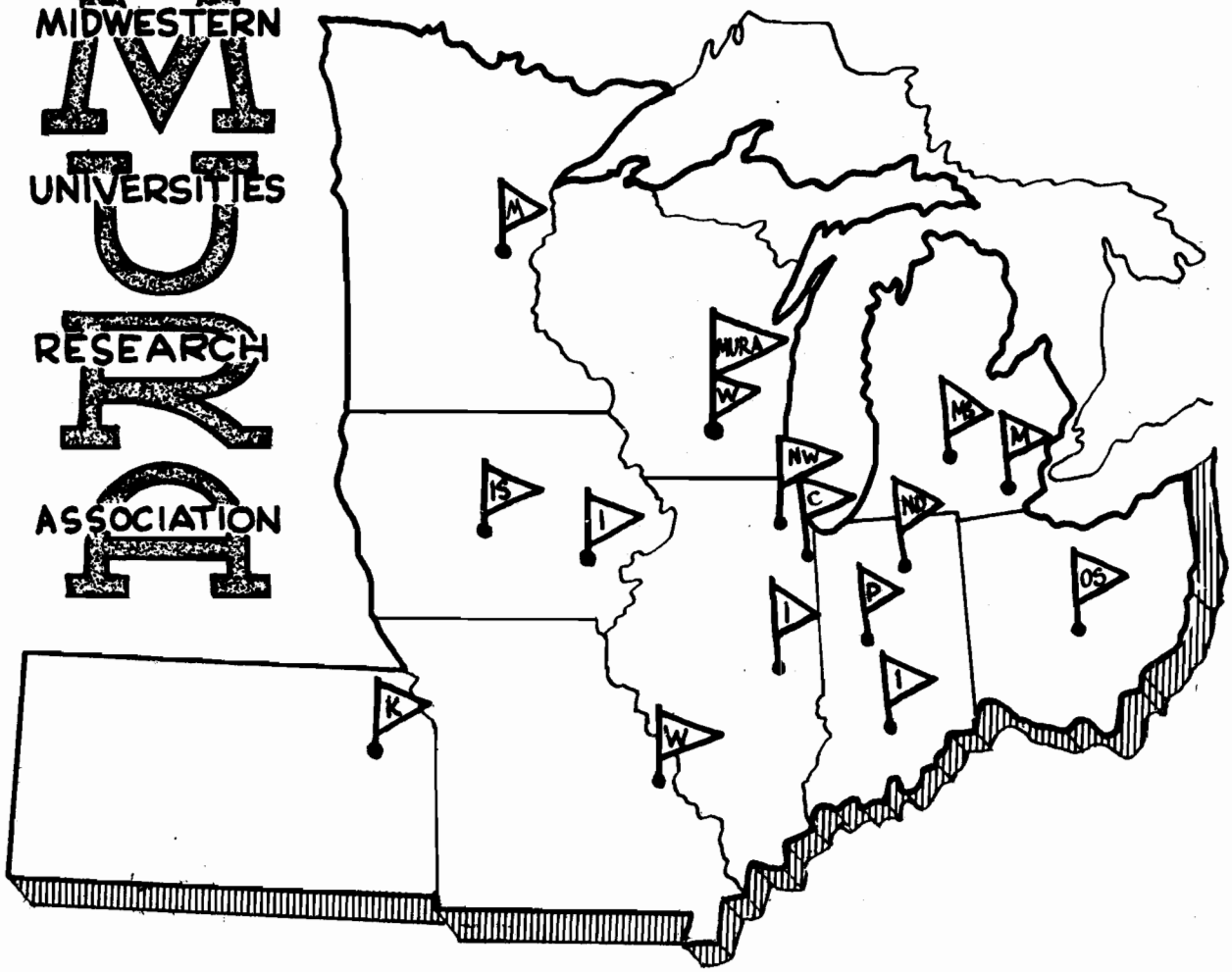

THE $\nu = \frac{1}{3} N$ NON-LINEAR RESONANCE OF A FIXED FIELD ACCELERATOR

G. Parzen

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MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

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G. Parzen**

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ABSTRACT

This paper is a continuation of the reports MURA-258, and MURA-273. This paper treats the problem of finding the stability limits, due to the $\nu = \frac{1}{3} N$ non-linear resonance in the radial motion, of an accelerator having an arbitrary magnetic field. The general spiral sector machine is treated in detail. The results of the theory are compared with numerically computed results.

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I. Introduction

This paper is a continuation of reports MURA-258 and MURA-273, which will be referred to hereafter as I and II respectively. In this paper we will treat the problem of finding the stability limits, due to the $\nu = \frac{1}{3} N$ non-linear resonance in the radial motion, of an accelerator having an arbitrary magnetic field.

We will limit ourselves to treating the general spiral sector machine. (Mark V and Mark I machines) However, we believe that the procedures given can be applied with little change to any other magnetic field. The case when $1/w \gg K$ and the flutter is small was treated previously by Laslett and Sessler[1] and F. W. Cole.[2]

II. Summary of Results

In this section we will simply list some of the results for the stability limits due to the $\nu = \frac{1}{3} N$ resonance for the general spiral sector machine (Mark I and Mark V). The derivation is given in section III, where more detailed results for the stability limit orbit are also given. For the sake of simplicity, the most accurate results are not listed here.

The stability limit amplitude A due to the $\nu = \frac{1}{3} N$ resonance of a spiral sector machine is given by

$$A = 2 \frac{\left| \left(\frac{\nu}{N}\right)^2 - \left(\frac{1}{3}\right)^2 \right|}{M} \cdot \frac{1}{N^2} , \tag{2.1}$$

where the amplitude is measured in units of R. (See MURA-258 for definition of R ; usually, $R \simeq r_1$ if r_1 is chosen so that $pc = eHr_1$. The magnetic field, H , and r_1 are defined by Eq. (3.1).)

$$M = \frac{1}{N^4} \left| B_{\bar{5},ss} \right| ,$$

$$B_{\bar{5},ss} = \left(1 + \frac{54}{5} |\gamma_1|^2\right) B_{-1} + 27 \gamma_{-1}^2 B_1 + 9\gamma_{-1} B_0 + \frac{9}{5} \gamma_1 B_{-2} , \tag{2.2}$$

$$\gamma_1 = b h_1 \frac{(K+1 - i/w)}{N^2} \tag{2.3}$$

$$\frac{B_0}{N^4} = -\frac{b h_0}{2} \frac{K(K+1)}{N^4} - b^2 \sum_{n \geq 1} \frac{|h_n|^2}{n^2} \frac{K(K^2 - 1 - 3n^2/w^2)}{N^6} , \tag{2.4}$$

$$\frac{B_n}{N^4} = -\frac{b h_n}{2} \frac{(K - i\,n/w)(K+1 - i\,n/w)}{N^4} \tag{2.5}$$

For the case where $1/w = 0$, (Mark I machine), the above has an error of about 30% for large machines, and an error of about 100% for small machines. 30% accuracy can be had for the small machine case by including the effect of M_s (see Appendix A).

For the case when $1/w \gg K$, (Mark V machine), the above result has an error of about 20%, for either small or large machines provided $1/w \gg N$. If $1/w \simeq N$, then the error may get as large as 100%.

A rough estimate of A, good to a factor of two, can be found by putting

$$M \simeq \frac{1}{2} b h_1 \frac{(K^2 + 1/w^2)}{N^4} . \tag{2.6}$$

Eq. (2.6) is particularly good when $1/w \gg K$, and $1/w \gg N$, the error being of order 20% in this case.

III. Derivation of the Stability Limits

For the sake of simplicity, we will treat the case of a general spiral sector machine, (Mark V), whose median plane magnetic field is given by

$$H_z = -H \left(\frac{r}{r_1}\right)^K \sum_{n=-\infty}^{\infty} h_n e^{i \omega_n \varphi} , \tag{3.1}$$

where

$$\varphi = \theta - \frac{1}{wN} \ln \frac{r}{r_1} . \tag{3.2}$$

The general radial sector machine (Mark I) is a special case of the above machine corresponding to $1/w = 0$. We believe that any other magnetic field can be treated in the same way with little change.

We can write the r -equation of motion as (see report MURA-273)

$$\frac{d}{d\theta}\frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = F_r , \tag{3.3}$$

where

$$F_r = \frac{e}{pc}\left(r H_z - z' H_\theta\right) , \tag{3.4}$$

$$L = \sqrt{r^2 + r'^2 + z'^2} \tag{3.5}$$

In order to investigate the quadratic non-linear resonances, we must expand Eq. (3.3) about the equilibrium orbit $r = r_s(\theta)$ and keep up to the quadratic terms.

The expansion of F_r up to quadratic terms is given in Appendix A of MURA-273.

$$F_r = \ldots + \frac{e}{pc}\left(H_{z,r} + \frac{1}{2} r H_{z,rr}\right)\rho^2 + \ldots\ldots \tag{3.6}$$

In Eq. (3.6) , ρ is defined by $r = r_s(\theta) + \rho$, the linear terms in ρ have been omitted, since the quadratic terms interest us

here and the quadratic terms involving z have been omitted as we are concerned with the radial oscillations.

The expansion of L up to quadratic terms is[2]

$$L = \ldots\ldots + \frac{1}{6} b_1 \rho^3 + \frac{1}{6} b_2 \rho'^3 + \frac{1}{2} b_3 \rho^2 \rho' + \frac{1}{2} b_4 \rho'^2 \rho + \ldots\ldots \tag{3.7}$$

where

$$b_1 = -3 r'^2 r / L^5, \tag{3.8a}$$

$$b_2 = -3 r' r^2 / L^5, \tag{3.8b}$$

$$b_3 = r' (2r^2 - r'^2) / L^5, \tag{3.8c}$$

$$b_4 = r (2r'^2 - r^2) / L^5. \tag{3.8d}$$

The coefficients b_1, b_2, b_3, b_4 are all to be evaluated on the equilibrium orbit which means we put $r = r_s(\theta)$, $r' = r_s'(\theta)$, $z = 0$. The linear and quadratic terms have been omitted in Eq. (2.7) for L.

Using the results of II for the linear part, we find that Eq. (3.3) becomes

$$a_3 \rho'' + a_3' \rho' + (a_2' - a_1)\rho = \frac{e}{pc}(H_z + r H_{z,r})\rho$$
$$+ B \rho^2/R^2$$
$$+ B^{(1)} \rho\rho'/R^2 + B^{(2)} \rho\rho''/R^2 + B^{(3)} \rho'^2/R^2$$
$$+ B^{4} \rho'\rho''/R^2 + B^{(5)} \rho^2/R^2, \tag{3.9}$$

where

$$B = \frac{e}{pc} R^2 (H_{z,r} + \tfrac{1}{2} r H_{z,rr}),$$
$$B^{(1)} = -R^2 b_4',$$
$$B^{(2)} = -R^2 b_4,$$
$$B^{(3)} = -\tfrac{1}{2} R^2 (b_2' + b_4), \tag{3.10}$$
$$B^{(4)} = -R^2 b_2,$$
$$B^{(5)} = -\tfrac{1}{2} R^2 (b_3' - \tfrac{1}{2} b_1),$$

All the B - coefficients are to be evaluated on the equilibrium orbit. $B^{(1)}, B^{(2)} \ldots, B^{(5)}$ represent the contributions of L to the quadratic terms. The linear part of Eq. (3.9) was derived in MURA-273.

Eq. (3.9) will hold for any magnetic field. We will now specialize to the general spiral sector field given by Eq. (3.1). The dominant quadratic term is the $B(0)\rho^2$ term which depends directly on H_z. The $B^{(1)}, B^{(2)} \ldots B^{(5)}$ terms, which stem from expanding L, give contributions

to the stability limits which vanish for large N and go like $1/N^2$. These terms are significant however for small machines ($N \lesssim 12$) having a Mark I field. These small machine terms will be dropped here and their effect is discussed in Appendix A.

Let us now find the coefficient $B(\theta)$ for the spiral sector field. It is convenient to write the magnetic field as

$$H_z = -H\left(\frac{R}{r_1}\right)^K \sum_n (1+x)^{K-in/w}\, \bar{h}_n\, e^{i w_n \theta}, \tag{3.11}$$

where

$$\bar{h}_n = h_n \exp\left[-i(n/w)\ln(R/r_1)\right]. \tag{3.12}$$

Then as $B(\theta) = (eR^2/pc)\left(H_{z,r} + \frac{1}{2} r H_{z,rr}\right)$, we get

$$B(\theta) = -\frac{1}{2}\frac{eR}{pc} H \left(\frac{R}{r_1}\right)^K \sum_n \Big\{(1+x)^{K-1-in/w} \cdot (K+1-in/w)(K-in/w)\, \bar{h}_n e^{i w_n \theta}\Big\}. \tag{3.13}$$

$$= -\frac{b}{2}\sum_n \Big\{\left[1+(K-1-in/w)x\right] (K+1-in/w)(K-in/w)\, \bar{h}_n e^{i w_n \theta}\Big\}.$$

where $b = (eHR/pc)(R/r_1)^K$ is determined as in report MURA-258.

We can then write for the r - equation, up to quadratic terms,

$$\left\{ \frac{d^2}{d\theta^2} + E_0 - g(\theta) \right\} u = B(\theta)\, u^2 , \tag{3.14}$$

where $B(\theta) = \sum_n B_n \exp(i \omega_n \theta)$, and

$$\frac{B_0}{N^4} = -\frac{b}{2} \frac{K(K+1)}{N^4} h_0 - b^2 \sum_{n \geq 1} \frac{|h_n|^2}{n^2} \frac{K(K^2-1-3n^2/\nu^2)}{N^6} , \tag{3.15a}$$

$$\frac{B_n}{N^4} = -\frac{b}{2} \frac{(K - in/\nu)(K+1-in/\nu)}{N^4} \bar{h}_n . \tag{3.15b}$$

E_0 and $g(\theta)$ are given in MURA-273 and $u = \rho/R$.

Now that we have the equation for the r - motion, we can apply the results of a previous report[2] to calculate the stability limits due to the $\nu = \frac{1}{3} N$ resonance.

According to the results of the report MURA-200, the stability limit orbit is given by

$$u = a\, u_{1/3}(\theta) + a^* u^*_{1/3}(\theta) , \tag{3.16}$$

where $u_{1/3}(\theta)$ is the flocquet solution having the tune $\nu = \frac{1}{3}N$ and which is found by solving the linear equation of motion when E_0 is replaced by $E_{1/3}$, $E_{1/3}$ being so chosen to give the tune $\nu = \frac{1}{3}N$.

The amplitude a is given by

$$|a| = \frac{|E_{1/3} - E_0|}{|B_{s,\bar{s}\bar{s}}|} \tag{3.17}$$

where

$$B_{\bar{s},ss} = \frac{N}{2\pi}\int_0^{2\pi/N} d\theta \; u^*_{-\frac{1}{3}} \; B(\theta)\, u_{1/3}\, u_{1/3}\,, \tag{3.18}$$

and

$$\text{phase } a = \frac{1}{3}\left\{ -\text{ phase } B_{\bar{s},ss} \quad - \text{ phase}\left(E_0 - E_{1/3}\right)\right\} \tag{3.19}$$

$E_{1/3}$ may be calculated from the relation that the E_ν to give the tune ν is given by

$$E_\nu = \alpha\nu^2 - 2\sum_{n\geq 1}\frac{|g_n|^2}{\omega_n^2}\,, \tag{3.20a}$$

$$\alpha = 1 - 8\sum_{n\geq 1}\left|\frac{g_n}{\omega_n^2}\right|^2. \tag{3.20b}$$

The solution $\mu_{1/3}(\theta)$ is given by

$$\mu_\nu = e^{i\nu\theta}\left\{1 - \sum_{n\neq 0} \frac{g_n}{\omega_n^2}\,\frac{1}{1+\frac{2\nu}{\omega_n}}\, e^{i\omega_n\theta}\right\} \tag{3.21}$$

where we put $\nu = \frac{1}{3}N$. One should note however, that in Eq. (3.16), $\mu_{1/3}(\theta)$ should be normalized so that the average value of $|\mu_{1/3}|^2$ over a period is one. This has not been done yet in Eq. (3.21).

In order to make use of formulas Eqs. (3.17) to (3.19) for the stability limit orbit, we must first calculate the matrix element $B_{\bar{s},ss}$. According to Eq. (3.21) we can write $\mu_{1/3}(\theta)$ as

$$\mu_{1/3} = e^{i\frac{N\theta}{3}}\cdot\left\{1 + \frac{3}{5}\gamma_1 e^{iN\theta} + 3\gamma_{-1} e^{-iN\theta} + \ldots\right\}, \tag{3.22}$$

where

$$\gamma_n = -g_n/\omega_n^2,$$

$$\frac{g_n}{N^2} = -b\,\bar{h}_n\,\frac{(K+1-in/w)}{N^2}. \tag{3.23}$$

We have not normalized $\mu_{1/3}$ as this is usually a 10% correction. If the normalization factors become important, this correction can be put in. Using the above result for $\mu_{1/3}$ and Eq. (3.15) for $B(\theta)$, we find for $B_{\bar{s},ss}$

$$B_{\bar{5},ss} = \left(1 + \frac{54}{5}|\gamma_1|^2\right) B_1 + 27\,\gamma_{-1}^2 B_1 + 9\,\gamma_{-1} B_0 + \frac{9}{5}\gamma_1 B_{-2} \tag{3.24}$$

In calculating $B_{\bar{5},ss}$, we have kept in mind that $\gamma_1 \simeq .1$ and we have omitted some terms which seemed small.

In most cases, one gets a good measure of the stability limits by simply considering the quantity $A = 2|a|$, which we will call the stability limit amplitude. According to Eq. (3.17) and Eq. (3.20), we find that A is given by

$$A = \frac{1}{N^2} \cdot \frac{2\left|\left(\frac{\nu_0}{N}\right)^2 - \left(\frac{1}{3}\right)^2\right|}{M}, \tag{3.25}$$

where

$$M = \frac{1}{N^4}\left|B_{\bar{5},ss}\right| . \tag{3.26}$$

ν_0 is the r - tune for small oscillations and we have put $\alpha = 1$ in Eq. (3.20). We have factored out the $1/N^2$ in Eq. (3.25) as it turns out that $1/N^2$ is the natural unit in which to measure A.

We will now apply the above formulae for the stability limit orbit to some special cases.

Case I. Radial Sector Machine, $1/w = 0$.

If we chose the h_n to be real then as $1/w = 0$, the B_n are real and negative, and phase $B_{\bar{s},ss} = \pi$. Thus phase $a = \frac{1}{3}\{\pi - ph(E_0 - E_{1/3})\}$ and phase a may be taken as π if $\nu_0 > \frac{1}{3}N$, or zero if $\nu_0 < \frac{1}{3}N$.

The three unstable fixed points at $\theta = 0$ are found from Eq. (3.16) by putting $N\theta = 0, \pm 2\pi$. For the radial sector machine, they will lie as shown in Fig. 1 if $\nu_0 < \frac{1}{3}N$. The triangle points to the left, if $\nu_0 > \frac{1}{3}N$.

If the coordinates of the fixed points are given by (u_i, p_i), $i = 1, 2, 3$, then Eq. (3.16) gives the following relations:

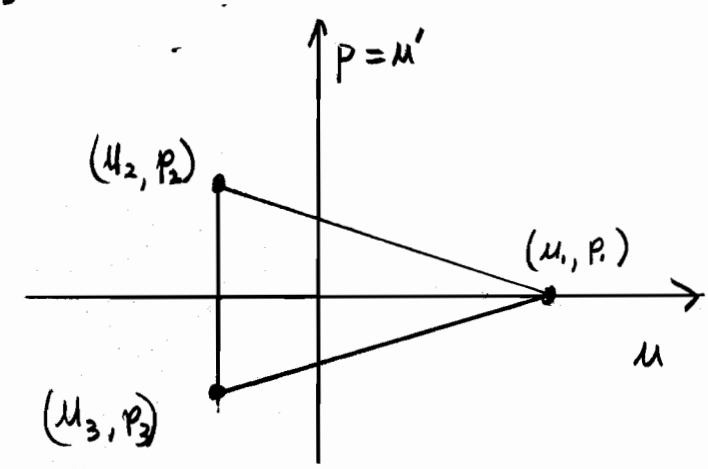

Fig. 1

$$p_1 = 0 \qquad p_2 = -p_3 , \tag{3.27}$$

$$|u_2| = |u_3| = \frac{1}{2}|u_1| , \tag{3.28}$$

$$|u_1| = (1 + 3.6\gamma_1)A , \tag{3.29}$$

$$|p_2| = |p_3| = .289\,(1 - 3.6\gamma_1)NA \tag{3.30}$$

The area of the triangle which is a measure of the available stable phase space at $\theta = 0$ is given by $|p_2|(|u_1| + |u_2|)$ or

$$\text{Area} = .433\,(1 - 12.95\,\gamma_1^2)\,NA^2 \tag{3.31}$$

The stability limit orbit is given by

$$u = \pm A\left\{\cos N\theta + 3\gamma_1 \cos\left(-\tfrac{2}{3}N\theta\right) + .6\gamma_1 \cos\left(\tfrac{4}{3}N\theta\right)\right\}, \tag{3.32}$$

the $+$ sign is for $\nu_0 < \frac{1}{3}N$, and the $-$ sign for $\nu_0 > \frac{1}{3}N$.

Case II.

For the spiral sector machine (Mark V), we usually have $1/w \gg K$. If $1/w \neq 0$, we can simplify our results for the stability limit orbit by shifting the $\theta = 0$ point to $\theta = (1/wN)\ln(r/r_1)$; that is, we introduce the new angle variable θ' where

$$\theta' = \theta - (1/wN)\ln(r/r_1). \tag{3.33}$$

All our results for the stability limit orbit will hold when we replace θ by θ' provided we also replace $\bar{h}_n$ by h_n.

We will assume that h_1 is pure imaginary, $h_1 = -i\,|h_1|$. This choice gives a symmetrical set of unstable fixed points, for γ_1 is then real and positive if $1/w \gg K$. When $1/w \gg K$, we may neglect B_0 compared with B_1, and B_1 is pure imaginary

with the phase $-\pi/2$. Then phase $\beta_{5,SS} = \pi/2$, and phase $a = \frac{1}{3}\left\{-\frac{\pi}{2} - \text{phase}\,(E_0 - E_{1/3})\right\}$. Phase a may be taken as -90° if $\nu_0 > \frac{1}{3}N$, and $+90^\circ$ if $\nu_0 < \frac{1}{3}N$.

The three unstable fixed points at $\theta' = 0$ will lie as shown in Fig. 2 when $\nu_0 > \frac{1}{3}N$.

The triangle points down if

$\nu_0 < \frac{1}{3}N$.

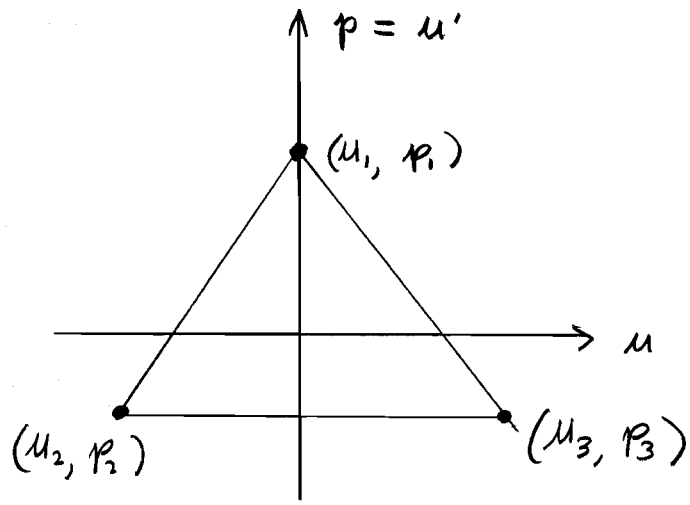

Fig. 2

The stability limit orbit is given by

$$u = \pm A\left\{\sin\frac{N\theta}{3} + 3\gamma_1 \sin\left(-\frac{2}{3}N\theta\right) + .6\gamma_1 \sin\left(\frac{4}{3}N\theta\right)\right\}, \tag{3.34}$$

where the + sign is for $\nu_0 > \frac{1}{3}N$ and the − sign for $\nu_0 < \frac{1}{3}N$.

The fixed points are given by the following relations

$$u_1 = 0 \,, \quad u_2 = -u_3 \,, \tag{3.35}$$

$$|p_2| = |p_3| = \frac{1}{2}|p_1| \,, \tag{3.36}$$

$$|u_2| = .865A\,(1 + 3.6\gamma_1) \,, \tag{3.37}$$

$$|p_1| = \frac{1}{3}NA\,(1 - 3.6\gamma_1) \,. \tag{3.38}$$

The area of the triangle determined by the fixed points at $\theta' = 0$ is given by $|u_2| \left(|p_1| + |p_2| \right)$ or

$$\text{Area} = .433\, N A^2 \left(1 - 12.95\, \gamma_1^2 \right) \tag{3.39}$$

IV. Comparison with Numerical Results

We shall compare the theory with numerical results for the two machines treated in detail in section III.

Case I. $1/w = 0$, Mark I machine

As a measure of the stability amplitude, we will use the unstable fixed point (u_1, p_1) shown in Fig. 1. $p_1 = 0$ in this case, and we will calculate u_1 theoretically and numerically for a number of different machine parameters which we will indicate by Machines A, B, C, D. Machines A, B, C, D, are defined in Table I.

The theoretical and numerical results are compared in Table II. Note that u_1 is multiplied by N^2.

Case II. $1/w \gg K$, Mark V machine

In this case we will compute $u_2 = -u_3$ shown in Fig. 2. For the machine parameters K = 128, N = 40 $1/w$ = 2112, h_0 = 1, $h_1 = -i/8$, $\nu_0/N = \frac{1}{3} + .01$, we find

u_2 = .195 , Theory

u_2 = .19 , Exp.

The numerical results here are due to N. Vogt-Nilsen.

TABLE I

Machine	K	N	h_0	$h_1 = h_{-1}$	ν/N
A	6.85	12	1	3.1	$\frac{1}{3} - .011$
B	8	12	1	3.1	$\frac{1}{3} + .024$
C	190	54	1	3.1	$\frac{1}{3} + .0092$
D	168	54	1	3.1	$\frac{1}{3} - .022$

Machine	$N^2 \mu_1$	
	Theo.	Exp.
A	2.7	2.19
B	4.58	3.37
C	2.16	1.60
D	7.1	10.4

APPENDIX A.

In the treatment given in section III, the "small machine" terms $B^{(1)}, B^{(2)}, \ldots, B^{(5)}$ were dropped from Eq. (3.9). In this Appendix, we will keep these terms and calculate their effect.

If we keep the small machine terms, Eq. (3.14) for the r - motion becomes

$$\left\{ \frac{d^2}{d\theta^2} + E_0 - g(\theta) \right\} u = B(\theta)\, u^2 + B^{(1)} u u' + B^{(2)} u u'' + B^{(3)} u'^2 + B^{(4)} u' u'' + B^{(5)} u^2 \tag{A.1}$$

If one repeats the derivation given in MURA-200 for the stability limit orbit, one finds that the effect of these extra terms can be found by making the following substitution in the formulae for the stability limit orbit,

$$B_{\bar{s},ss} \longrightarrow B_{\bar{s},ss} + B^{(1)}_{\bar{s},ss} + B^{(2)}_{\bar{s},ss} + \ldots + B^{(5)}_{\bar{s},ss} \tag{A.2}$$

where

$$\begin{aligned} B^{(1)}_{\bar{s},ss} &= \frac{N}{2\pi} \int_0^{\frac{2\pi}{N}} d\theta\, u^*_{\bar{s}}\, B^{(1)} u_s u'_s , \\ B^{(2)}_{\bar{s},ss} &= \frac{N}{2\pi} \int_0^{\frac{2\pi}{N}} d\theta\, u^*_{\bar{s}}\, B^{(2)} u_s u''_s , \\ B^{(3)}_{\bar{s},ss} &= \frac{N}{2\pi} \int_0^{2\pi/N} d\theta\, u^*_{\bar{s}}\, B^{(3)} u'_s u'_s , \end{aligned} \tag{A.3}$$

$$B^{(4)}_{\bar{s},ss} = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} d\theta\, u_{\bar{s}}^{*}\, B^{(4)}\, u_s'\, u_s'',$$

$$B^{(5)}_{\bar{s},ss} = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} d\theta\, u_{\bar{s}}^{*}\, B^{(5)}\, u_s\, u_s .$$

where $u_s = u_{1/3}(\theta)$, $u_{\bar{s}} = u_{-\frac{1}{3}}(\theta)$.

Our result for the stability limit amplitude can then be written as

$$A = \frac{1}{N^2}\,\frac{2\left|\left(\frac{\nu_0}{N}\right)^2 - \left(\frac{1}{3}\right)^2\right|}{M + M_s}\,, \tag{A. 4}$$

and

$$M_s = \frac{1}{N^4}\left| B^{(1)}_{\bar{s},ss} + B^{(2)}_{\bar{s},ss} + \cdots B^{(5)}_{\bar{s},ss}\right|. \tag{A. 5}$$

M_s is appreciable only for small machines ($N \simeq 10$ for Mark I). We find that M_s is given by

$$\begin{aligned} M_s = &-\frac{1}{N^2}\left(2\gamma_1 - .5\gamma_{-1}\right) \\ &-\frac{1}{N^2}\, b\, h_{-1}\left(-.056 + 6.8\,|\gamma_1|^2\right) \\ &-\frac{1}{N^2}\, b\, h_1 \cdot 6\gamma_{-1}^2 . \end{aligned} \tag{A. 6}$$

It is difficult to calculate M_s accurately because it involves derivatives of the flocquet functions and derivatives of the equilibrium orbit.

For Mark V type machines for which $1/\mu \gg K$, M_s is usually small compared with M, and can be neglected.

References

1. Laslett and Sessler, MURA Notes 6/1/56

2. F. T. Cole, MURA-FTC/3

3. G. Parzen, MURA-200