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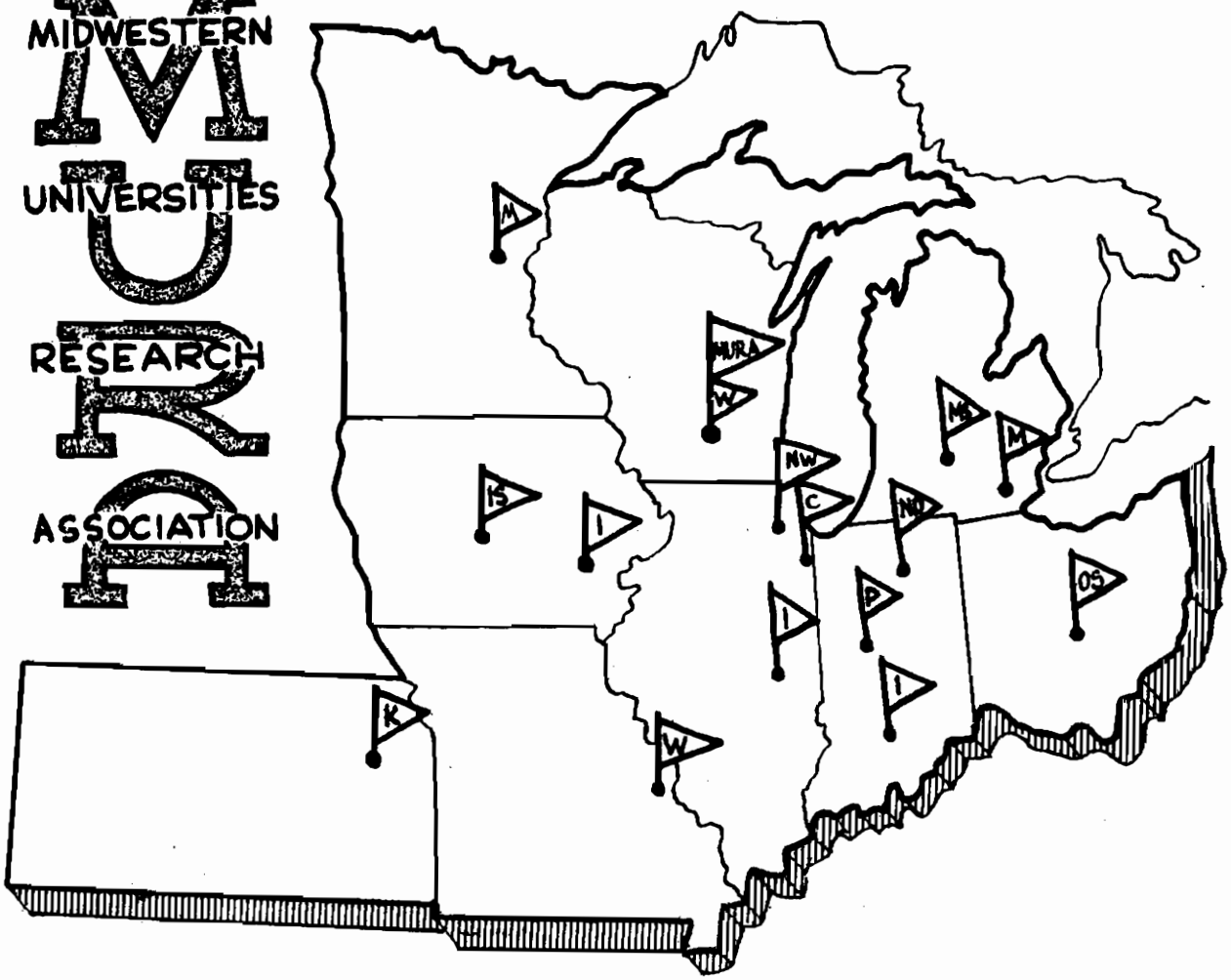

SOLUTIONS AND APPROXIMATE SOLUTIONS TO A HILL'S EQUATION AND THE MATHIEU EQUATION

By Roger E. Mills

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By Roger E. Mills

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ABSTRACT: A method, like that of Hill presented by Whittaker and Watson, is described for the solution of a Hill's equation whose periodic function may be of the type

$$f(N\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos nN\theta + \sum_{n=1}^{\infty} b_n \sin nN\theta \quad .$$

Approximate solutions are described. The special case of the Mathieu equation is treated as an example.

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I. INTRODUCTION

The author wishes to state at first that he makes no claim to the originality of the material included in this paper. The problem is treated in many standard references, such as that of Whittaker and Watson, "Modern Analysis", which is followed closely here+. In addition several papers by members of the MURA staff* have dealt with the problem. Work has been done on this problem by workers in several fields besides accelerator work**.

This paper is written in the desire to invite attention to the availability of algebraic methods of constructing solutions and approximate solutions of equations of the type

$$y'' + f(N\theta)y = 0 \qquad (1)$$

where

$$f(N\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos nN\theta + \sum_{n=1}^{\infty} b_n \sin nN\theta$$

(resolved into Fourier components).

For simple cases, such as that of the Mathieu equation, the formulas developed are quite easy to apply and give good accuracy. For more complicated cases (where $f(N\theta)$ contains more harmonics) the application becomes more difficult.

\+ 1, page 413 et seq

* Bibliography, particularly 4

** 6, Page 8

Section II describes the general solution. Section III describes the approximate solutions and their validity, and includes as an example, the Mathieu equation.

II. THE SOLUTION OF A HILL'S EQUATION

A class of Hill's equations is of the form

$$\frac{d^2y}{d\theta^2} + f(N\theta)y = 0 \qquad , N = 1, 2, 3, \ldots \quad (1)$$

where $f(N\theta)$ may be expanded by a Fourier analysis to the form

$$f(N\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n N\theta + \sum_{n=1}^{\infty} b_n \sin nN\theta \qquad (2)$$

where the series is to be absolutely and uniformly convergent, and is periodic with period $2\pi/N$.

We derive a formal solution to (1) following the method of Hill as presented by Whittaker and Watson.*

First, we introduce complex notation for the trigonometric functions in (2). Then

$$f(N\theta) = a_0 + \tfrac{1}{2}\sum_{n=1}^{\infty} \alpha_n^* e^{in N\theta} + \tfrac{1}{2}\sum_{n=1}^{\infty} \alpha_n e^{-in N\theta} \qquad (3)$$

where

$$\alpha_n = a_n + ib_n \text{ and } \alpha_n^* = a_n - ib_n.$$

According to Floquet's Theorem**, we know that (1) has a solution of the form

$$y = e^{-i\nu\theta}\varphi(\theta) \qquad (4)$$

where $\varphi(\theta)$ is periodic. We substitute (4) into (1) and obtain

* 1. page 413-417 ** 1 page 412

$$\frac{d^2\varphi}{d\theta^2} - 2i\nu\frac{d\varphi}{d\theta} + \left[f(N\theta)-\nu\right]^2\varphi = 0 \quad . \tag{5}$$

We assume that φ may be written as

$$\varphi = \sum_{m=-\infty}^{\infty} \varphi_m e^{imN\theta} \quad , \tag{6}$$

where the φ_m's are complex constants. The series is to be absolutely and uniformly convergent.

Substitution of (6) in (5) leads to

$$-\sum_{m=-\infty}^{\infty}(\nu-mN)^2\varphi_m e^{imN\theta} + \left\{a_0 + \tfrac{1}{2}\sum_{n=1}^{\infty}\alpha_n^* e^{inN\theta} + \tfrac{1}{2}\sum_{n=1}^{\infty}\alpha_n e^{-inN\theta}\right\}\sum_{m=-\infty}^{\infty}\varphi_m e^{imN\theta} = 0.$$

Since the series involved are absolutely convergent, we may change the orders of summation to obtain

$$\sum_{m=-\infty}^{\infty}\left\{\left[a_0 - (\nu - mN)^2\right]\varphi_m + \sum_{n=1}^{\infty}\frac{\alpha_n^*}{2}\varphi_{m-n} + \sum_{n=1}^{\infty}\frac{\alpha_n}{2}\varphi_{m+n}\right\} e^{imN\theta} = 0. \tag{7}$$

Since the series (3) and (6) converge uniformly for both $f(N\theta)$ and $\varphi(N\theta)$, series (7) does also, and since the $e^{imN\theta}$ are linearly independent functions the coefficients of the $e^{imN\theta}$ in (7) must be zero.

Then we have the set of denumerably infinite number of unknowns

$$
\begin{aligned}
\cdots + \left[a_0 - (\nu+2N)^2\right]\phi_{-2} + \frac{\alpha_1}{2}\phi_{-1} + \frac{\alpha_2}{2}\phi_0 + \frac{\alpha_3}{2}\phi_1 + \frac{\alpha_4}{2}\phi_2 + \cdots &= 0 \\
\cdots + \frac{\alpha_1^*}{2}\phi_{-2} + \left[a_0 - (\nu+N)^2\right]\phi_{-1} + \frac{\alpha_1}{2}\phi_0 + \frac{\alpha_2}{2}\phi_1 + \frac{\alpha_3}{2}\phi_2 + \cdots &= 0 \\
\cdots + \frac{\alpha_2^*}{2}\phi_{-2} + \frac{\alpha_1^*}{2}\phi_{-1} + \left[a_0 - \nu^2\right]\phi_0 + \frac{\alpha_1}{2}\phi_1 + \frac{\alpha_2}{2}\phi_2 + \cdots &= 0 \\
\cdots + \frac{\alpha_3^*}{2}\phi_{-2} + \frac{\alpha_2^*}{2}\phi_{-1} + \frac{\alpha_1^*}{2}\phi_0 + \left[a_0 - (\nu-N)^2\right]\phi_1 + \frac{\alpha_1}{2}\phi_2 + \cdots &= 0 \\
\cdots + \frac{\alpha_4^*}{2}\phi_{-2} + \frac{\alpha_3^*}{2}\phi_{-1} + \frac{\alpha_2^*}{2}\phi_0 + \frac{\alpha_1^*}{2}\phi_1 + \left[a_0 - (\nu-2N)^2\right]\phi_2 + \cdots &= 0 \\
\cdots
\end{aligned}
\tag{8}
$$

In (8), we divide through the equation in which $a_0-(\nu-mN)^2$ appears by $a_0-m^2N^2$. Then this system of simultaneous linear equations has a non-trivial solution if and only if the determinant of the coefficients is zero:

$$\Delta(\nu) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{a_0-(\nu+2N)^2}{a_0-4N^2} & \frac{\alpha_1}{2(a_0-4N^2)} & \frac{\alpha_2}{2(a_0-4N^2)} & \frac{\alpha_3}{2(a_0-4N^2)} & \frac{\alpha_4}{2(a_0-4N^2)} & \cdots \\ \cdots & \frac{\alpha_1^*}{2(a_0-N^2)} & \frac{a_0-(\nu+N)^2}{a_0-N^2} & \frac{\alpha_1}{2(a_0-N^2)} & \frac{\alpha_2}{2(a_0-N^2)} & \frac{\alpha_3}{2(a_0-N^2)} & \cdots \\ \cdots & \frac{\alpha_2^*}{2a_0} & \frac{\alpha_1^*}{2a_0} & \frac{a_0-\nu^2}{a_0} & \frac{\alpha_1}{2a_0} & \frac{\alpha_2}{2a_0} & \cdots \\ \cdots & \frac{\alpha_3^*}{2(a_0-N^2)} & \frac{\alpha_2^*}{2(a_0-N^2)} & \frac{\alpha_1^*}{2(a_0-N^2)} & \frac{a_0-(\nu-N)^2}{a_0-N^2} & \frac{\alpha_1}{2(a_0-N^2)} & \cdots \\ \cdots & \frac{\alpha_4^*}{2(a_0-4N^2)} & \frac{\alpha_3^*}{2(a_0-4N^2)} & \frac{\alpha_2^*}{2(a_0-4N^2)} & \frac{\alpha_1^*}{2(a_0-4N^2)} & \frac{a_0-(\nu-2N^2)}{a_0-4N^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \quad (9)$$

The exponent ν has been arbitrary to this point. By choosing it to satisfy the necessary condition (9), we find part of the Floquet solution.

Equation (9) is somewhat difficult to solve for ν. However, the difficulties may be reduced somewhat by solving instead the equation

$$\sin^2\left(\frac{\pi}{2}\nu\right) = \Delta(0)\ \sin^2\left(\frac{\pi}{2}\sqrt{a_0}\right) \quad , \qquad (10)$$

which follows from (9)*.

We discuss now the convergence of $\Delta(0)$. It may be shown** that an infinite determinant converges if

1) The product of the diagonal elements converges absolutely, and

2) the sum of the non-diagonal elements converges absolutely.

$\Delta(0)$ may be written explicitely as

$$\Delta(0) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \Big| \frac{\alpha_1}{2(a_0-4N^2)} & \frac{\alpha_2}{2(a_0-4N^2)} & \frac{\alpha_3}{2(a_0-4N^2)} & \frac{\alpha_4}{2(a_0-4N^2)} & \cdots \\ \cdots & \frac{\alpha_1^*}{2(a_0-N^2)} & \Big| \frac{\alpha_1}{2(a_0-N^2)} & \frac{\alpha_2}{2(a_0-N^2)} & \frac{\alpha_3}{2(a_0-N^2)} & \cdots \\ \cdots & \frac{\alpha_2^*}{2(a_0)} & \frac{\alpha_1^*}{2a_0} & \Big| \frac{\alpha_1}{2a_0} & \frac{\alpha_2}{2a_0} & \cdots \\ \cdots & \frac{\alpha_3^*}{2(a_0-N^2)} & \frac{\alpha_2^*}{2(a_0-N^2)} & \frac{\alpha_1^*}{2(a_0-N^2)} & \Big| \frac{\alpha_1}{2(a_0-N^2)} & \cdots \\ \cdots & \frac{\alpha_4^*}{2(a_0-4N^2)} & \frac{\alpha_3^*}{2(a_0-4N^2)} & \frac{\alpha_2^*}{2(a_0-4N^2)} & \frac{\alpha_1^*}{2(a_0-4N^2)} & \Big| \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (11)$$

* 1, p. 415,416. This proof, since it does not depend on the character of the numerators of the off-diagonal elements, but only on the dependence of Δ on ν (W&W's $i\mu$), follows just as given there and so is not reproduced here.

** 1, p. 36,37

Then $\Delta(0)$ has singularities at

$$a_0 = m^2N^2 \quad , \; |m| = 0,1,2,\cdots .$$

The singularity $a_0 = 0$ may be removed from (10) by a limiting process, If mN is even, these may be removed from (10) in a similar way. Assume that we do not approach any of the singularities. Then the product of the diagonal elements of $\Delta(0)$ is one. That the sum of the non-diagonal elements converges absolutely follows also. For we may write this sum as

$$\sum_{\substack{i,j \\ i \neq j}} \left| \frac{\alpha_i}{(a_0 - j^2N^2)} \right| = \sum_{j=-\infty}^{\infty} \left| \frac{1}{(a_0 - j^2N^2)} \right| \sum_{i=1}^{\infty} |\alpha_i| \quad .$$

The sum over i converges since the series for $f(N\theta)$ is absolutely and uniformly convergent*. Call this limit I. Then we have

$$\sum_{\substack{i,j \\ i \neq j}} \left| \frac{\alpha_i}{a_0 - j^2N^2} \right| = I \sum_{j=-n}^{n} \left| \frac{1}{a_0 - j^2N^2} \right| + I \sum_{j=-(n+1)}^{-\infty} \left| \frac{1}{a_0 - j^2N^2} \right| + I \sum_{j=n+1}^{\infty} \left| \frac{1}{a_0 - j^2N^2} \right| \tag{12}$$

If a_0 is negative, comparison with

$$\sum_j \frac{1}{j^2}$$

shows that the series is absolutely convergent. If a_0 is

* 1, p. 158, Corollary

positive, we pick n to be such that

$$\left|\sqrt{a_0} - nN\right| \geq 1. \tag{13}$$

Then the first sum, being a finite sum and having no singularities, by definition of a_0, converges. The third sum

$$I\sum_{j=n+1}^{\infty}\left|\frac{1}{a_0 - j^2N^2}\right| = I\sum_{j=n+1}^{\infty}\left|\frac{1}{(\sqrt{a_0} - jN)(\sqrt{a_0} + jN)}\right| \leq I\sum_{j=n+1}^{\infty}\frac{1}{(\sqrt{a_0} - jN)^2}$$

But by (13), the terms of the sum on the right are less, term by term, than the terms of the sum

$$I\sum_{m=1}^{\infty}\frac{1}{m^2}$$

which is absolutely convergent. Hence the third sum converges. The second sum may be shown to converge in a similar manner. Then the original sum converges absolutely.

Therefore $\Delta(0)$ converges for values of a_0 which are not equal to m^2N^2, $m = 0,1,2,\cdots$. Equation (10) may be used for computation whenever the singularities of $\Delta(0)$ may be removed by a limiting process.

Thus, formally at least, we may find a ν such that (10) is satisfied. By construction of suitable cofactors of the determinant of the coefficients of (9), we may multiply the respective equations, add them and thus solve for the ratios of the φ_n,s. If we have a criterion for normalization, we may obtain values for all the φ_n's. Then we must test to be sure that the series (6) is absolutely and uniformly convergent as required.

III. APPROXIMATE SOLUTIONS

1. In General

Since in practical work we could not calculate the $\Delta(0)$ precisely, or all the ϕ_n, it is necessary to find ways of approximating $\Delta(0)$ and ϕ sufficiently closely for the problem at hand.

A way of approximating $\Delta(0)$ which is immediately obvious is to substitute for it in (10) the determinant $\Delta_m(0)$ which is the central sub-determinant of $\Delta(0)$ such that Δ_m has 2m+1 rows and columns. To estimate how good an approximation this will be, we make use of a discussion on the convergence of infinite determinants*, making some small changes in notation.

If we write $\Delta(0)$ as

$$\Delta(0) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & 1+a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & 1+a_{0,0} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \quad ,$$

we see that

$$a_{ik} = \frac{\alpha_{k-i}}{2(a_0 - i^2 N^2)}$$

where

$$\alpha_{k-i} = \begin{cases} \alpha_n & \text{if} \quad k-i = n > 0 \\ 0 & \text{if} \quad k-i = 0 \\ \alpha^*_{-n} & \text{if} \quad k-i = n < 0 \end{cases} \tag{14}$$

* 1, p. 37

The product $\overline{P}$ is defined as

$$\overline{P} \equiv \prod_{i=-\infty}^{\infty} \left(1 + \sum_{k=-\infty}^{\infty} |a_{ik}|\right) \tag{15}$$

which converges since $\sum_{i,k=-\infty}^{\infty} a_{ik}$ is convergent.

We also define

$$\overline{P}_m = \prod_{i=-m}^{m} \left(1 + \sum_{k=-m}^{m} |a_{ik}|\right) . \tag{16}$$

Then

$$\overline{P} = \lim_{m \to \infty} \overline{P}_m .$$

An inequality is derived:

$$\left|\Delta_{m+p}(0) - \Delta_m(0)\right| \leq \overline{P}_{m+p} - \overline{P}_m .$$

But since $\lim_{m \to \infty} \overline{P}_m$ exists, given an ϵ greater than zero, there exists an N such that

$$\left|\overline{P}_{m+p} - \overline{P}_m\right| < \epsilon \quad \text{for all } p > N.$$

Then

$$\left|\Delta_{m+p}(0) - \Delta_m(0)\right| < \epsilon \quad \text{for all } p > N.$$

Therefore $\lim_{m \to \infty} \Delta_m(0)$ exists. Taking the limit as p goes to infinity, we have

$$\left|\Delta(0) - \Delta_m(0)\right| \leq \overline{P} - \overline{P}_m \tag{17}$$

Examining $\overline{P}$ more closely, we have

$$\overline{P} = \prod_{i=-\infty}^{\infty} \left[1 + \sum_{k=-\infty}^{\infty} \left|\frac{\alpha_{k-i}}{2(a_0 - i^2 N^2)}\right|\right]$$

$$= \prod_{i=-\infty}^{\infty} \left[1 + \left|\frac{1}{a_0 - i^2 N^2}\right| \sum_{n=1}^{\infty} |\alpha_n|\right] . \tag{18}$$

Since $f(N\theta)$ converges absolutely and uniformly, the sum in (18) converges. If we call this sum F, we have

$$\bar{P} = \prod_{i=-\infty}^{\infty} \left[1 + \frac{F}{|a_0 - i^2 N^2|} \right] . \tag{19}$$

If we similarly define

$$F_m = \sum_{n=1}^{m} |\alpha_n| ,$$

we have

$$\bar{P}_m = \prod_{i=-m}^{m} \left[1 + \frac{F_m}{|a_0 - i^2 N^2|} \right] . \tag{20}$$

From (17), (19) and (20), we obtain

$$\left| \Delta(0) - \Delta_m(0) \right| \leq \prod_{i=-\infty}^{\infty} \left[1 + \frac{F}{|a_0 - i^2 N^2|} \right] - \prod_{i=-m}^{m} \left[1 + \frac{F_m}{|a_0 - i^2 N^2|} \right] . \tag{21}$$

This inequality may be of help in estimating the error involved in approximating $\Delta(0)$ by $\Delta_m(0)$ if we can calculate the infinite product with sufficient accuracy.

Having thus used Δ_m to calculate the ν, we proceed to obtain an approximation to φ. Since we desire to keep only a finite number of terms for φ, we assume that the φ_n converge sufficiently fast that above a certain n we may ignore them. Formally, we assume

$$\varphi_n \equiv 0 \quad \text{if } |n| \geq p+1 \tag{22}$$

Then φ will have 2p+1 (p+ the constant term) components.

Applying (22) to (8), we obtain an infinite set of equations in the 2p+1 non-zero φ_n's $(|n| \leq p)$. We assume further that the φ_n $(|n| \leq p)$ and the α_n and α_n^* are such

that we can ignore all but the 2p+1 central equations of this infinite set. Then we are left with 2p+1 equations in 2p+1 unknowns. We proceed to solve for the ratios of these unknowns. If we have a normalization condition, we can apply it to get a complete set of the φ_n ($|n| \leq p$). We must use this and the α_n and α_n* to be sure that the equations which we have ignored are satisfied sufficiently well that approximation has been justified.

2. $f(N\theta)$ With Only a Finite Number of Components

If the $f(N\theta)$ in (1) has only a finite number of components, it may be possible to find easier ways of finding the φ_n ($|n| \leq p$) than by solving several (2p+1)-order determinants.

As an example of this, we consider an equation similar to the Mathieu equation*,

$$\frac{d^2y}{d\theta^2} + (a_0 + a_1 \cos N\theta + b_1 \sin N\theta)\, y = 0. \tag{23}$$

Then the equations (8) with the division by $(a_0 - mN^2)$ as discussed on page 5, become

* 1, p.405

$$
\begin{aligned}
&\cdots + \frac{\alpha_1}{2(a_0-9N^2)}\phi_{-2} + 0 + 0 + \cdots = 0 \\
&\cdots + \frac{a_0-(\nu+2N)^2}{a_0-4N^2}\phi_{-2} + \frac{\alpha_1}{2(a_0-4N^2)}\phi_{-1} + 0 + 0 + \cdots = 0 \\
&\cdots + \frac{\alpha_1^*}{2(a_0-N^2)}\phi_{-2} + \frac{a_0-(\nu+N)^2}{(a_0-N^2)}\phi_{-1} + \frac{\alpha_1}{2(a_0-N^2)}\phi_0 + 0 + 0 + \cdots = 0 \\
&\cdots 0 + \frac{\alpha_1^*}{2a_0}\phi_{-1} + \frac{a_0-\nu^2}{a_0}\phi_0 + \frac{\alpha_1}{2}\phi_1 + 0 + \cdots = 0 \\
&\cdots 0 + 0 + \frac{\alpha_1^*}{2(a_0-N^2)}\phi_0 + \frac{a_0-(\nu-N)^2}{a_0-N^2}\phi_1 + \frac{\alpha_1}{2(a_0-N^2)}\phi_2 + \cdots = 0 \\
&\cdots \quad 0 + 0 + \frac{\alpha_1^*}{2(a_0-4N^2)}\phi_1 + \frac{a_0-(\nu-2N)^2}{2(a_0-4N^2)}\phi_2 + \cdots = 0 \\
&\cdots \quad + 0 + 0 + \frac{\alpha_1^*}{2(a_0-9N^2)}\phi_2 + \cdots = 0 \\
&\cdots
\end{aligned}
\tag{24}
$$

If we assume that the ϕ_n for $|n| \leq 3$ are zero, then formally we would have that <u>all</u> the ϕ_n are zero. We ignore this for the moment. If we make this assumption and consider only the central five equations, we find that

$$\varphi_{-1} = \frac{-\alpha_1 \varphi_0}{2[a_0-(\nu+N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu+2N)^2]}}, \quad \varphi_1 = \frac{-\alpha_1^* \varphi_0}{2[a_0-(\nu-N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu-2N)^2]}} \tag{25a}$$

$$\varphi_{-2} = \frac{-\alpha_1 \varphi_{-1}}{2[a_0-(\nu+2N)^2]}, \quad \varphi_2 = \frac{-\alpha_1^* \varphi_1}{2[a_0-(\nu-2N)^2]} \tag{25b}$$

If we consider $\varphi_n = 0$ if $|n| \leq 4$, and consider the seven central five equations of (8), we obtain

$$\varphi_{-1} = \frac{-\alpha_1 \varphi_0}{2[a_0-(\nu+N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu+2N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu+3N)^2]}}}, \quad \varphi_1 = \frac{-\alpha_1^* \varphi_0}{2[a_0-(\nu-N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu-2N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu-3N)^2]}}} \tag{26a}$$

$$\varphi_{-2} = \frac{-\alpha_1 \varphi_{-1}}{2[a_0-(\nu+2N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu+3N)^2]}} \quad , \quad \varphi_{+2} = \frac{-\alpha_1^* \varphi_1}{2[a_0-(\nu-2N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu-3N)^2]}} \tag{26b}$$

$$\varphi_{-3} = \frac{-\alpha_1 \varphi_{-2}}{2[a_0-(\nu+3N)^2]} \quad , \quad \varphi_3 = \frac{-\alpha_1^* \varphi_2}{2[a_0-(\nu-3N)^2]} \tag{26c}$$

If we consider the 2n+1 central equations in (8), and assume that $\varphi_{n+1}, \varphi_{n+2}, \ldots$ are zero, we obtain

$$\varphi_{-n} = \frac{-\alpha_1 \varphi_{-n+1}}{2[a_0-(\nu+nN)^2]} \quad , \quad \varphi_n = \frac{-\alpha_1^* \varphi_{n-1}}{2[a_0-(\nu-nN)^2]}$$

$$\varphi_{-(n-1)} = \frac{-\alpha_1 \varphi_{-(n-2)}}{2[a_0-(\nu+(n-1)N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu+nN)^2]}} \quad , \quad \varphi_{n-1} = \frac{-\alpha_1^* \varphi_{n-2}}{2[a_0-(\nu-(n-1)N)^2] - \dfrac{|\alpha_1|^2}{2[a_0-(\nu-nN)^2]}}$$

and so forth until we have reached $\varphi_{\pm 1}$ in terms of φ_0. It appears that if we would continue this, allowing n to become ever greater, we would find the continued fractions adding terms in a regular manner, and that we would obtain recursive formulas for the $\varphi_{\pm n}$,

$$\varphi_n = \cfrac{-\alpha_1^* \varphi_{n-1}}{2\{a_0-[\nu-nN]^2\} - \cfrac{|\alpha_1|^2}{2\{a_0-[\nu-(n+1)N]^2\} - \cfrac{|\alpha_1|^2}{2\{a_0-[\nu-(n+2)N]^2\} - \underline{|\alpha_1|^2}}}} \tag{27a}$$

$$\varphi_{-n} = \cfrac{-\alpha_1 \varphi_{-(n-1)}}{2\{a_0-[\nu+nN]^2\} - \cfrac{|\alpha_1|^2}{2\{a_0-[\nu+(n+1)N]^2\} - \cfrac{|\alpha_1|^2}{2\{a_0-[\nu+(n+2)N]^2\} - \underline{|\alpha_1|^2}}}} \tag{27b}$$

To accept these as true, it would be necessary to establish them with some kind of proof. (This the author is unable to construct. However, neither is he able to foresee a case in which the formulas (27) will not hold.) Since the $a_0 - [\nu + nN]^2$, $a_0 - [\nu \pm (n+1)N]^2, \cdots$ in (27 b) and the similar sequence in (27 a) diverges approximately sequence as m^2, it is expected that the continued fractions will converge.

To use the equations (27), it is necessary to find a value of ν. We use equation (10) and approximate $\Delta(0)$ by $\Delta m(0)$. In this case, $\Delta(0)$ is

$$\Delta(0) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \frac{\alpha_1}{2(a_0-4N^2)} & 0 & 0 & 0 & \cdots \\ \cdots & \frac{\alpha_1^*}{2(a_0-N^2)} & 1 & \frac{\alpha_1}{2(a_0-N^2)} & 0 & 0 & \cdots \\ \cdots & 0 & \frac{\alpha_1^*}{2a_0} & 1 & \frac{\alpha_1}{2a_0} & 0 & \cdots \\ \cdots & 0 & 0 & \frac{\alpha_1^*}{2(a_0-N^2)} & 1 & \frac{\alpha_1}{2(a_0-N^2)} & \cdots \\ \cdots & 0 & 0 & 0 & \frac{\alpha_1^*}{2(a_0-4N^2)} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{28}$$

Expanding central sub-determinants of (28) leads to the sequence

$$\Delta_0(0) = 1$$

$$\Delta_1(0) = \Delta_0 - \frac{|\alpha_1|^2}{2a_0(a_0 - N^2)} \tag{29}$$

$$\Delta_2(0) = \Delta_1 - \frac{|\alpha_1|^2}{2(a_0 - N^2)(a_0 - 4N^2)}\left[\Delta_0 - \frac{|\alpha_1|^2}{4a_0(a_0 - N^2)}\right]$$

$$\Delta_3(0) = \Delta_2 - \frac{|\alpha_1|^2}{2(a_0 - 4N^2)(a_0 - 9N^2)}\left\{\Delta_1 - \frac{|\alpha_1|^2}{4(a_0 - N^2)(a_0 - 9N^2)}\left[\Delta_0 - \frac{|\alpha_1|^2}{4a_0(a_0 - N^2)}\right]\right\}$$

Thus each approximation to $\Delta(0)$ is equal to the previous approximation plus or minus a small correction factor (depending on a_0). Eventually, when m is large enough, these corrections will all be negative, and successively smaller. It can be seen that to achieve a given absolute accuracy, we will need to use higher approximations whenever $|\alpha_1|$ or a_0 is increased.

It appears that we have a singularity at $a_0 = 0$. However, when the Δ_m is used in (10) and the limit taken as a_0 approaches zero, this offers no difficulty. For example, with Δ_1, (10) becomes

$$\sin^2\frac{\pi\nu}{2} = \left[1 - \frac{|\alpha_1|^2}{2a_0(a_0 - N^2)}\right]\sin^2\frac{\pi\sqrt{a_0}}{2}$$

$$= \sin^2\frac{\pi\sqrt{a_0}}{2} - \frac{|\alpha_1|^2}{2(a_0 - N^2)}\left[\frac{\sin\frac{\pi}{2}\sqrt{a_0}}{\sqrt{a_0}}\right]^2 . \tag{30}$$

Taking the limit,

$$\lim_{a_0 \to 0} \sin^2 \frac{\pi\nu}{2} \doteq \frac{\pi^2 |\alpha_1|^2}{8N^2} \tag{31}$$

Remark: if ν and a_0 are both small, (30) gives

$$\left(\frac{\pi}{2}\nu\right)^2 \doteq \frac{\pi^2}{2^2} a_0 - \frac{|\alpha_1|^2}{2(a_0 - N^2)} \frac{\pi^2}{2^2}$$

or

$$\nu^2 \doteq a_0 - \frac{|\alpha_1|^2}{2(a_0 - N^2)} \quad . \tag{32}$$

Equation (32) agrees substantially with the "smooth approximation"* result

$$\nu^2 \doteq a_0 + \frac{|\alpha_1|^2}{2N^2} \quad .$$

Some special examples of this case are of interest. Laslett and Sessler** have previously considered the equation

$$y'' + (A + B \cos 2\theta)\, y = 0. \tag{33}$$

Their method differs somewhat in that coefficients of a trial solution are determined by harmonic balance of the trial solution in (33). The resulting coefficients are the same as (25), except for the algebraic form in which they are presented. For a numerical comparison, equation (33) was integrated on the MURA IBM 704 computer and the results subjected to Fourier analysis. The agreement was found to be good. The $\cos \pi\nu$ was also calculated for various val-

* 2; 7, p. 69 and bibliography
** 6

ues of A and two values of B. Table I below presents a comparison of these values of $\cos \pi\nu$ * with $\cos \pi\nu$ calculated using Δ_1 and Δ_2 in (10).

Table I.

.1782	-.978291	-1.00968	-.983939	.450	-.965918	$-.98419_0$	$-.96836_8$
.1648	-.940199	-.970545	-.945728	.430	-.930942	$-.94869_6$	$-.93331_2$
.1514	-.900950	-.930215	-.906340	.411	-.895869	$-.91313_3$	$-.89818_7$
.1216	-.809434	-.836244	-.814517	.368	-.809637	$-.82579_5$	$-.81191_0$
.0904	-.707195	-.731465	-.712063	.323	-.708788	$-.72459_7$	$-.71184_3$
.0327	-.500067	-.523887	-.508957	.244	-.503814	$-.51627_6$	$-.50573_0$
0352	-.224636	-.237319	-.228117	.180	-.309811	$-.31923_9$	$-.31061_8$
-.0850	+.0004918	-.007162	-.002389	.092	$+.001299_4$	$-.00567_0$	$+.00006_6$
-.1470	.309801	.308848	+.307787	.017	.310321	$+.30639_2$	$+.30944_1$
-.1823	.501004	.504065	.499529	+.025	$.502277_5$	$.50026_1$	$+.50172_2$
-.2170	$.700100_5$	.707271	.699183	+.065	.698389	$.69884_9$	$+.69875_1$
-.2330	$.795928_4$	.804829	.794928	+.086	.806719	$.80800_3$	$+.80701_0$

The agreement of these values is good. The absolute error is nearly the same over the range $.2 \leq \pi\nu \leq 1$. We note there are two cases where the higher Δ_m's must be used. The first is when $|\cos \pi\nu| \doteq 1$, and the second when $\cos \nu\pi \doteq 0$. In each case, the estimated absolute error should be less than

* 6, p. 9, by permission of Dr. A. M. Sessler

the absolute difference of $\cos \pi\nu$ and 1 or 0, as the case may be. In the first case (since N=2, $\pi\nu = \sigma$*), we may incorrectly take a stable solution for one which is unstable, or vice-versa. In the second case, we may use the incorrect sign of $\cos\pi\nu$ in further calculations.

As a second numerical example, we treat the case

$$\frac{d^2y}{d\theta^2} + (.2 + \cos 2\theta + \sin 2\theta)\, y = 0. \tag{34}$$

Then $a_0 = .2$ $a = 1$, $b = 1$, $N = 2$. Substituting in the various formulas gives

$$\begin{aligned} \nu &= .86138 \\ \varphi_1 &= .47868\,(1-i) \\ \varphi_{-1} &= .062741(1+i) \\ \varphi_2 &= -.049698i \\ \varphi_{-2} &= .002286i \end{aligned} \tag{35}$$

where we have normalized so that $\varphi_0 = +1$.

Then a solution of (34) is of the form

$$y = e^{-.86138\,\theta}\,\varphi \tag{36}$$

where

$$\varphi = 1 + .47868(1-i)e^{2i\theta} + (.062741)(1+i)e^{-2i\theta} - .049698ie^{4i\theta} + \\ + .002286ie^{-4i\theta} + \ldots$$

y may be separated into real and imaginary parts:

$$\mathrm{Re}(y) = \cos .86138\theta + .47868(\cos 1.13862\theta + \sin 1.13862\theta) + .062741 \times \tag{37 a}$$
$$\times(\cos 2.86138\theta \sin 2.86138\theta) + .049698 \sin 3.13862\theta + .002286 \sin 4.86138\theta$$

* $N\sigma = 2\pi\nu$, and the solution is unstable if $|\cos\sigma| > 1$

$$\mathcal{J}_m(y) = -\sin .861380\theta + .47868(-\cos 1.138620\theta + \sin 1.138620\theta) + .062741 \times$$
$$\times(\cos 2.861380\theta - \sin 2.861380\theta) - .049698 \cos 3.138620\theta + .002286 \times \qquad (37b)$$
$$\times \cos 4.861380\theta$$

Equation (34) has been integrated on the MURA IBM 704 in order to obtain a comparison with (37). This comparison is presented in figure 1. The agreement appears to be good.

Substitution of (36) into (34) gives the result

$$\Big\{ -.000555 - .000002(1-i)e^{1.13862i\theta} - .002287(1+i)e^{-2.86138i\theta} +$$
$$+ .000952ie^{-3.13862i\theta} + .009288ie^{-4.86138i\theta} + .024849(1+i)e^{5.13862i\theta} -$$
$$.001143(1-i)e^{-6.86138i\theta}\Big\} \doteq 0 \qquad (38)$$

We do not discuss here the second condition for approximation stated on page (10) since it has not been used. Formulas (27), if correct, make this unnecessary. Since the $\varphi_{\pm n}$ are less in magnitude than $\varphi_{\pm}(n-1)$ by a factor which is approximately $1/n^2N^2$ when nN is large, the series will converge absolutely.

It can be seen that the remaining terms are small, and on the order of the terms of which have been neglected in (36).

Tables* have been set up to give values of ν for equations of the type

$$y'' + (A + B \cos 2\tau + C \cos 3\tau + D \cos 4\tau)\, y = 0.$$

$\frac{\sigma}{\pi} = \frac{2\nu}{N}$ has been computed for various values of A and B for C <u>or</u> D not zero.

* 9

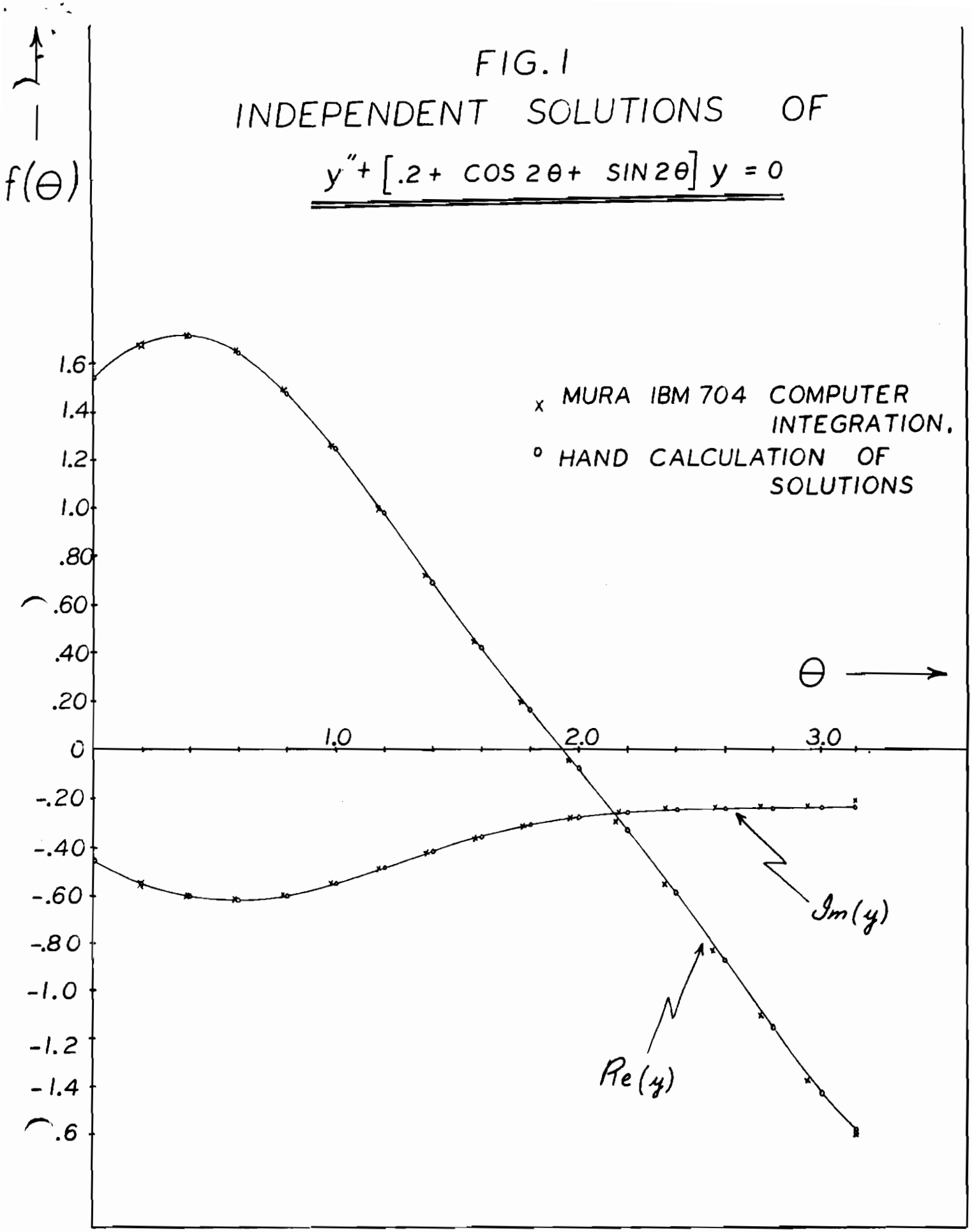
FIG. 1
INDEPENDENT SOLUTIONS OF
$y'' + [.2 + \cos 2\theta + \sin 2\theta]\, y = 0$
$f(\theta)$
1.6
1.4
1.2
1.0
.80
.60
.40
.20
0
-.20
-.40
-.60
-.80
-1.0
-1.2
-1.4
-.6
1.0
2.0
3.0
θ
x MURA IBM 704 COMPUTER INTEGRATION.
o HAND CALCULATION OF SOLUTIONS
$Im(y)$
$Re(y)$

To apply (10) to a more complicated case, we consider

$$\frac{d^2y}{d\theta^2} + (A + \cos 2\theta + \tfrac{1}{2}\cos 4\theta)\, y = 0 \tag{39}$$

Δ_1 is used in (10) to approximate Δ for various values of A. The results of these calculations and a comparison with the tables is presented in Table II.

Table II.

A	$\cos \pi\nu$		A	$\cos \pi\nu$	
	(10), Δ_1	tables		(10), Δ_1	tables
-.12	.971932	.883897	+.09	-.016223	-.081071
-.09	.808474	.723917	+.12	-.129570	-.191413
-.06	.652709	.571680	+.15	-.236595	-.295492
-.03	.504572	.426962	+.20	-.401377	-.455560
-.00	.36383	.289540	+.30	-.683268	-.728559
+ .03	.230246	.159201	+.40	-.907054	-.944154
+.06	.103444	.035733			

The comparison between the calculated and tabulated values is not as good in Table II as in Table I. However, we see that the absolute accuracy again improves as $\cos \pi\nu$ becomes smaller. (Note that when $\cos \pi\nu$ is near zero, the relative error may become quite large.) Similar conclusions to those reached before as suggested: better approximations must be used when $\cos \pi\nu$ is near one or zero; Δ_1 is not likely to be more than roughly adequate.

Calculation of the components of φ proceeds in a manner similar to that described above.* Algebraic expressions for the case of f (Nθ) with two or more harmonics become rather involved and so no attempt is made to include them here.

* p. 12

IV. CONCLUSIONS.

It is possible to obtain a formal solution to a Hill's equation whose periodic coefficient may be Fourier-analysed. Approximations to the formal solutions may be obtained. In the event that the periodic coefficient of the Hill's equation is simple, such as in the Mathieu case, these approximations take convenient forms for computation and become good rapidly. As the complexity of the periodic coefficient increases, higher approximations are necessary to calculate ν. Also it is more convenient to calculate the magnitudes of the components in each special case than to try to apply general formulas.

If the infinite product in equation (21) can be obtained in a closed form, then (21) will give a convenient way of telling how well Δ_m approximates Δ .

Dr. A. M. Sessler has proposed recently that newer, more extensive tables pertaining to the solutions of Hill's equations be compiled. If this is done perhaps the algebraic methods described herein will be easier to program and allow faster computation than the methods previously used.

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SOLUTIONS AND APPROXIMATE SOLUTIONS TO A HILL'S EQUATION AND THE MATHIEU EQUATION

R. E. Mills

August, 1957

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Table I on page 21 should have the following heading

B=1.5 A	$\cos\pi\nu$ Computer	$\cos\pi\nu$ (10), Δ_1	$\cos\pi\nu$ (10), Δ_2	B=1 A	$\cos\pi\nu$ computer	$\cos\pi\nu$ (10), Δ_1	$\cos\pi\nu$ (10), Δ_2

Also on page 24 - third line from the bottom of the page should read as follows

reached before are suggested:

not

reached before as suggested: