APPROVED BY
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IDENTIFICATION

Fixed Point LOGARITHM, Base e. MU LOG3

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PURPOSE

To compute the natural logarithm, in fixed point arithmetic, of 1 + y, with

$$-1/2 \leq y \leq 1 - 2^{-35}.$$

METHOD

The quantity 1 + y is scaled so that $1/2 \leq z = 2^m (1 + y) \leq 1 - 2^{-35}$, then

$$\ln (1 + y) = \ln z - m \ln 2$$

where ln z is computed from the approximation

$$\ln z \cong x + C_3x^3 + C_5x^5 + C_7x^7 + C_9x^9 - \ln \sqrt{2},$$

where

$$x = \frac{2 (z - 1/\sqrt{2})}{(z + 1/\sqrt{2})}$$

The power series for ln z has a maximum error less than 10^{-11} for $1/2 \leq z < 1 - 2^{-35}$.

(Cf. ILLIAC Subroutine Library - Program 42 - S1.)

Accuracy: The truncation error is less than 10^{-11} and is therefore negligible in comparison with the roundoff error.

The error introduced by roundoff in the computation of x does not exceed 2^{-33}. The maximum value of x for z in the range (1/2, 1) occurs for z = 1, thus

$$x_{max} = 2 \frac{(1 - 1/\sqrt{2})}{1 + 1/\sqrt{2}} = .343.$$

It follows that the maximum error in x^2 does not exceed 2^{-33}. The dominant contribution to the error in ln (1 + y) comes from the above initial

error in x. The contribution of the error in x to the error in ln (1 + y) is obtained from the relation

$$\text{error} \cong \delta_2 (C_3 + 2C_5x^2 + 3C_7x^4 + 4C_9x^6)$$
$$+ \delta_1 (1 + C_3x^2 + C_5x^4 + C_7x^6 + C_9x^8)$$

where δ_1 is the error in x and δ_2 is the error in x^2 and second order terms in the errors have been discarded. Recalling that x_{max} is about 0.343 it follows that a good estimate of the maximum error in ln (1 + y) is given by

$$\text{error in } \ln (1 + y) \cong C_3 \ \delta_2 + \delta_1$$

The constant C_3 is a little less than 1/10, hence the

$$\text{error in } \ln (1 + y) < 2^{-32}.$$

In general the error will be considerably less than the above upper bound, since, under the assumption that roundoff errors are distributed uniformly and randomly, the RMS error in ln (1 + y) is about 1.5×2^{-35}.

USAGE

Calling Sequence: Entry is made with y in the accumulator.

UA SAP		MURASS	
Loc.	Instruction	Loc.	Instruction
α	TSX LOG3, 4	α	BTSX A 4
α+ 1	return	α+ 1	return

The return from MU LOG3 is made with ln (1 + y) in the accumulator. Both y and ln (1 + y) are fixed point fractions. There is no error return, since it is the responsibility of the user to insure that $-1/2 \leq y \leq 1 - 2^{-35}$. Inaccurate results will be obtained for y's outside of this range and a DIVISION HALT is also possible when these limits are not observed.

Space Required:

	UA SAP	MURASS
41 words program at:	LOG3	A
3 words temporary at:	COMMON	T

CODING INFORMATION

Timing: 2.7 ms.

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```
      REM LOG3 MURA FIXED POINT LOGARITHM, BASE E                    MU  LOG3
LOG3  SXD LOG3+39,1      SAVE IR1                                   LOG3  0001
      TMI LOG3+7         TRANSFER IF Y LESS THAN 0                  LOG3  0002
      ADD LOG3+38        Y POS, Y+1-2 EXP-35                        LOG3  0003
      ARS 1              1/2(1+Y-2 EXP-35)=Z                        LOG3  0004
      STO COMMON+1       STORE Z                                    LOG3  0005
      CLA LOG3+33        1/2 LN 2                                   LOG3  0006
      TRA LOG3+10                                                   LOG3  0007
      ADD LOG3+38        Y NEG, Y+1-2 EXP-35=Z                      LOG3  0008
      STO COMMON+1       STORE Z                                    LOG3  0009
      CLS LOG3+33        -1/2 LN 2                                  LOG3  0010
      STO COMMON                                                    LOG3  0011
      CLA COMMON+1       Z INTO AC                                  LOG3  0012
      ADD LOG3+40        Z+2 EXP-1/2                                LOG3  0013
      ARS 1              1/2(Z+2 EXP-1/2)                           LOG3  0014
      STO COMMON+2                                                  LOG3  0015
      CLA COMMON+1       Z INTO AC                                  LOG3  0016
      SUB LOG3+40        Z-2 EXP-1/2                                LOG3  0017
      DVH COMMON+2       X=2(Z-2 EXP-1/2)/(Z+2 EXP-1/2)             LOG3  0018
      STQ COMMON+1       STORE X                                    LOG3  0019
      MPR COMMON+1       X EXP 2                                    LOG3  0020
      STO COMMON+2       STORE X EXP 2                              LOG3  0021
      LXA LOG3+39,1      SET IR1=4                                  LOG3  0022
      LDQ LOG3+34        C(9)                                       LOG3  0023
      MPR COMMON+2       FORM                                       LOG3  0024
      ADD LOG3+39,1      1+C(3)XEXP2+C(5)XEXP4+C(7)XEXP6            LOG3  0025
      LRS 35             +C(9) XEXP8                                LOG3  0026
      TIX LOG3+23,1,1    LOOP, OUT AFTER 4 PASSES                   LOG3  0027
      LRS 1                                                         LOG3  0028
      MPY COMMON+1       X TIMES SUM                                LOG3  0029
      LLS 1                                                         LOG3  0030
      ADD COMMON         PLUS OR MINUS 1/2 LN 2                     LOG3  0031
      LXD LOG3+39,1                                                 LOG3  0032
      TRA 1,4                                                       LOG3  0033
      OCT 130562057737   1/2 LN 2                                   LOG3  0034
      OCT 000074645376   C(9)                                       LOG3  0035
      OCT 000444064022   C(7)                                       LOG3  0036
      OCT 003146331215   C(5)                                       LOG3  0037
      OCT 025252525045   C(3)                                       LOG3  0038
      OCT 377777777777   1-2 EXP-35                                 LOG3  0039
      HTR 4              IR1 CONSTANT AND STORAGE                   LOG3  0040
      OCT 265011714640   2 EXP-1/2                                  LOG3  0041
```