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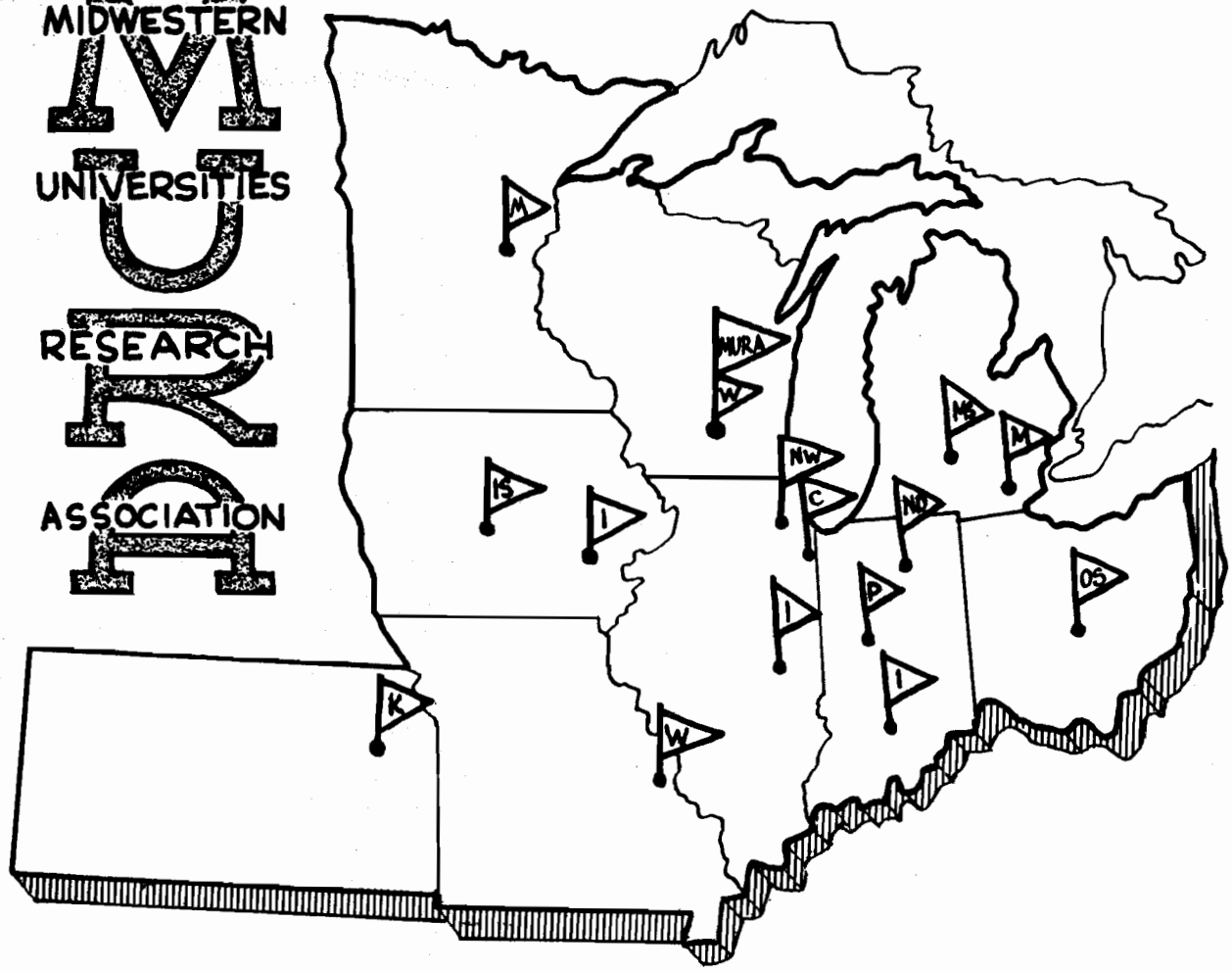

COMPUTATIONAL EXAMPLES OF SOLUTIONS TO DIFFERENTIAL EQUATIONS WHICH SIMULATE GROWTH OF AXIAL OSCILLATIONS IN AN FFAG ACCELERATOR OPERATED NEAR THE $\sigma_x = 2\sigma_y$ RESONANCE

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COMPUTATIONAL EXAMPLES OF SOLUTIONS TO DIFFERENTIAL EQUATIONS WHICH SIMULATE GROWTH OF AXIAL OSCILLATIONS IN AN FFAG ACCELERATOR OPERATED NEAR THE $\sigma_x = 2\sigma_y$ RESONANCE

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7 June, 1957

ABSTRACT

The effect of various coupling terms upon the ψ-growth phenomenon for an operating point near the $\sigma_\rho = 2\,\sigma_\psi$ resonance is examined computationally. Results are presented in tabular form and graphically. A tentative discussion of the results is also given.

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I. INTRODUCTION

The advent of the DUCK-ANSWER computational program[1] afforded an opportunity of studying the effect of various terms in producing the "turn-over" which can terminate the y-growth phenomenon exhibited by coupled differential equations operated near the $\nu_{x_0} = 2\nu_{y_0}$ resonance.[2] The present report presents the results of this investigation, with little theoretical discussion, for coupled differential equations which resemble the more elaborate FECKLESS-FIVE equations as previously applied to the operating point denoted as "Point θ" in an earlier report.[2]

The results of the investigation suggest certain questions, concerning the onset of true instability and in regard to the reliability of repeated turn-over as an indication of stability. These questions are mentioned in the Discussion [Section IV]. Individual runs did not exceed 200 sectors in length, an individual sector being characterized by $\Delta\tau = \pi$ = 32 Runge-Kutta steps.

II. THE DIFFERENTIAL EQUATIONS:

We list below the differential equations employed, a double prime denoting the second derivative with respect to τ.

(1) $\rho'' + (.184 + .8 \cos 2\tau)\rho = 0$

$$\psi'' - (.027 + .8 \cos 2\tau)\psi = .8 (\sin 2\tau)\rho\psi .$$

These non-Hamiltonian equations typify the equations employed to analyze y-growth in the manner suggested by Walkinshaw,[4, 2] in which the radial oscillation, uninfluenced by the axial motion (presumed to be initially small), is introduced as a <u>prescribed function</u> into the axial equation. For this system

of equations, y-growth, if present, will of course persist, but the equations are suitable for determining the threshold for y-growth and the initial lapse-rate characterizing y-growth when present.

$$(2) \quad \rho'' + (.184 + .8 \cos 2\tau)\,\rho = .4 \,(\sin 2\tau)\,\psi^2$$
$$\psi'' - (.027 + .8 \cos 2\tau)\,\psi = .8 \,(\sin 2\tau)\,\rho\psi .$$

The addition of the ψ^2 term to complete the ρ-equation here provides us with a Hamiltonian set of equations otherwise similar to equations (1) above. Effects of both the difference resonance $\nu_{x_0} - 2\,\nu_{y_0} = 0$ and the sum resonance $\nu_{x_0} + 2\,\nu_{y_0} = N$ (with $N = 2$ in the present instance) may be expected to arise with equations of this form.[5]

$$(3) \quad \rho'' + (.184 + .8 \cos 2\tau)\,\rho = 0$$
$$\psi'' - (.027 + .8 \cos 2\tau)\,\psi = .8 \,(\sin 2\tau)\,\rho\psi + (.4/3)\,(\cos 2\tau)\,\psi^3 .$$

This non-Hamiltonian system of equations affords an opportunity for a typical ψ^3 term to exhibit any influence which it may exert on growth of ψ-amplitude when the magnitude of ψ becomes appreciable.

$$(4) \quad \rho'' + (.184 + .8 \cos 2\tau)\,\rho = .4 \,(\sin 2\tau)\,\psi^2$$
$$\psi'' - (.027 + .8 \cos 2\tau)\,\psi = .8 \,(\sin 2\tau)\,\rho\psi + (.4/3)\,(\cos 2\tau)\,\psi^3 .$$

The inclusion of the ψ^2 term in the ρ-equation again provides a Hamiltonian system of equations, otherwise similar to (3). The presense of the ψ^3 term in the ψ-equation might be expected to introduce effects attributable to the $\sigma_{y_0} = \pi/2$ resonance ($\nu_{y_0} = N/4$) in equations of this form.

$$(5) \quad \rho'' + (.184 + .8 \cos 2\tau)\,\rho = .4 \,(\sin 2\tau)\,\psi^2 - .4 \,(\cos 2\tau)\,\rho\psi^2$$
$$\psi'' - (.027 + .8 \cos 2\tau)\,\psi = .8 \,(\sin 2\tau)\,\rho\psi - .4 \,(\cos 2\tau)\,\rho^2\psi .$$

This Hamiltonian system of equations includes the $\rho\psi^2$, $\rho^2\psi$ coupling terms which might be expected to introduce effects attributable to the sum resonance $2\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$ in equations of this form.

$$(6) \qquad \rho'' + (.184 + .8\cos 2\tau)\rho = .4(\sin 2\tau)\psi^2 - .4(\cos 2\tau)\rho\psi^2$$

$$\psi'' - (.027 + .8\cos 2\tau)\psi = .8(\sin 2\tau)\rho\psi - .4(\cos 2\tau)\rho^2\psi + (.4/3)(\cos 2\tau)\psi^3.$$

This Hamiltonian set of equations includes all the non-linear terms which appear in equations (2), (4), and (5).

It is noted that the foregoing equations contain A-G (τ-dependent) coefficients in both the linear and non-linear terms, computations in which alternate-gradient effects play a less prominent rôle being under study by Dr. Parzen.[6] We list below, however, two sets of non-AG coupled equations which are of interest for comparison of their performance with the results of the present investigation.

$$(7) \qquad \rho'' + .292\,356\,49\,\rho = .0542\,\psi^2$$

$$\psi'' + .056\,074\,24\,\psi = .1084\,\rho\psi.$$

These Hamiltonian equations are intended to have the same small-amplitude oscillation frequencies and a ρ-threshold for ψ-growth closely similar to the values applying to equations (1) - (6).

$$(8) \qquad \rho'' + .292\,356\,49\,\rho = -.08\,\rho\psi^2$$

$$\psi'' + .056\,074\,24\,\psi = -.08\,\rho^2\psi - .04\,\psi^3.$$

These equations are derivable from the Hamiltonian

$$H = (1/2)p_\rho^2 + (1/2)p_\psi^2 + (\nu_{x_0}^2/2)\rho^2 + (\nu_{y_0}^2/2)\Psi^2 + \frac{B^2}{4N^2}\rho^2\Psi^2 + \frac{B^2}{16N^2}\Psi^4,$$

with B = -.8 and N = 2, it having been suggested (K. R. S.) that this might in a sense be the smooth-approximation representation of the A-G problem (2).

III. RESULTS

A. Characteristics of Small-Amplitude Motion:

The oscillation frequencies for small-amplitude oscillations were determined in a preliminary run (#200) employing the linear uncoupled equations

$$\rho'' + (.184 + .8 \cos 2\tau)\rho = 0$$
$$\psi'' - (.027 + .8 \cos 2\tau)\psi = 0 .$$

In this way it was determined that

$$\nu_{x_0} \equiv \sigma_{x_0}/\pi = 0.5407$$

and

$$\nu_{y_0} \equiv \sigma_{y_0}/\pi = 0.2368,$$

N being equal to 2 in this work.

In the course of this orientation work the coefficients of quadratic forms which remain invariant as the motion progresses through successive sectors from one homologous point to the next were also determined. With

$$K_\rho^2 \equiv \xi_\rho \rho^2 + \eta_\rho \rho p_\rho + \zeta_\rho p_\rho^2$$

and

$$K_\psi^2 \equiv \xi_\psi \psi^2 + \eta_\psi \psi p_\psi + \zeta_\psi p_\psi^2,$$

we find

$$\xi_\rho = 1 \qquad \xi_\psi = 1$$
$$\eta_\rho = 0 \qquad \eta_\psi = 0$$
$$\zeta_\rho = 11.23121 \qquad \zeta_\psi = 8.2097551 .$$

A diagram indicating the location of the operating point with respect to various resonances of possible interest is given in Fig. 11.

B. Threshold for ψ - growth and Lapse-Rate:

(i) The growth of K_ψ, representing axial amplitude, is depicted in Figs. 1-7 and will be described in further detail later. The character of the initial ψ-motion, which develops from the small starting amplitude of 10^{-4}, is affected by the ρ-motion entirely or primarily through the coupling term which involves $\rho\psi$. The threshold for ψ growth, and the lapse-rate characterizing this growth when it occurs, can be estimated[4, 2] by introducing into the ψ-equation a solution $\rho(\tau)$ of the unperturbed ρ-equation. For equations of the form

$$\rho'' + (a_x + b \cos N\tau)\rho = (C/2)(\sin N\tau)\psi^2$$

$$\psi'' + (a_y - b \cos N\tau)\psi = C(\sin N\tau)\rho\psi,$$

the threshold for ψ-growth may thus be extimated[2] as

$$A_\rho\Big|_{thr.} = \frac{N^4}{8(\sigma_{x_0}/\pi)6C}\left|\left(\frac{\sigma_{x_0}}{\pi}\right)^2 - \left(2\frac{\sigma_{y_0}}{\pi}\right)^2\right|$$

$$= \frac{16}{8 \times 0.5407 \times 0.8 \times 0.8}\left|(.5407)^2 - (.4736)^2\right| = 0.39 \;;$$

this result may be compared with the empirical initial amplitude for the threshold,

$\rho_{0\,thr} = 0.315$ (Figs. 1 & 2).

Above the threshold, the square of the lapse-rate is seen to vary <u>linearly</u> with the square of the ρ- amplitude (Figs. 1 & 2). The coefficient of proportionality in this relationship may be estimated[2] as

$$\frac{\mu^2}{A_\rho^2 - A_{\rho,\,thr.}^2} = \left(\frac{bC}{N^3}\right)^2$$

$$= \left[\frac{N}{8\,A_{\rho,\,thr.}} \; \frac{\left|\left(\frac{\sigma_{\rho_0}}{\pi}\right)^2 - \left(2\,\frac{\sigma_{\psi_0}}{\pi}\right)^2\right|}{\left(\frac{\sigma_{\rho_0}}{\pi}\right)}\right]^2 \quad \text{for } \mu \text{ in nepers/radian}$$

$$= \left[\frac{2.73}{8A_{\rho,\,thr.}} \; \frac{\left|\left(\frac{\sigma_{\rho_0}}{\pi}\right)^2 - \left(2\,\frac{\sigma_{\psi_0}}{\pi}\right)^2\right|}{\left(\frac{\sigma_{\rho_0}}{\pi}\right)}\right]^2 \quad \text{for } \mu \text{ in decades/sector}$$

$$= \left[\frac{2.73}{8\times 0.315} \; \frac{(.5407)^2 - (.4736)^2}{.54}\right] = 0.018_6$$

for μ in decades/sector,

in which we have substituted the observed threshold value of ρ for $A_{\rho,\,thr.}$; this result may be compared with the initial slope 0.018_5 of the empirical μ^2 vs. ρ_0^2 curve (Fig. 1).

[The threshold and lapse-rates found here for the DUCK-ANSWER runs are similar to those found from μ^2 vs. u_0^2 plots of FECKLESS-FIVE results[2] obtained for operating points near the $\sigma_{x_0} = 2\,\sigma_{y_0}$ resonance (as for "Point θ"), if ρ is identified with u/w.]

(ii) The non-AG equations (7) are of the form

$$\rho'' + \nu_{\rho_0}^2\,\rho = (C_1/2)\,\psi^2$$

$$\psi'' + \nu_{\psi_0}^2\,\psi = C_1\,\rho\,\psi\,,$$

with $\nu_{\rho_0} = 0.5407$, $\nu_{\psi_0} = 0.2368$, and $C_1 = 0.1084$. The expected threshold for these equations may be readily obtained by substituting $\rho = A_\rho \cos \nu_{\rho_0}\tau$ into the ψ-equation. One obtains a linear equation of the Mathieu form with a threshold

$$A_\rho\Big|_{thr.} = \frac{1}{2C_1}\left|\nu_{\rho_0}^2 - (2\nu_{\psi_0})^2\right|$$
$$= 0.314,$$

in agreement with the observed threshold for equations (7). This threshold coincides (intentionally) with that found empirically for the alternate-gradient DUCK-ANSWER runs and the dependence of lapse-rate on ρ-amplitude above threshold is found, as expected, to be similar for the non-AG case (Fig. 7) and the AG cases.

C. Run Employing Reversed Integration:

As will be seen from Figs. 1-7, many of the DUCK-ANSWER runs exhibit a growth of ψ amplitude through some four orders of magnitude, followed by a decrease through one or two orders of magnitude. Since one may in a sense visualize the computations as launched with initial conditions corresponding to a mixture of exponentially ascending and descending solutions of the governing differential equation, it is of interest to inquire how accurately the computational program can follow an exponentially decreasing solution. To obtain information relevant to this question, one of the runs shown in Fig. 2 -- that with the initial condition $\rho_0 = -.400$ -- was reversed with initial values corresponding to those attained after traversal of 105 sectors in the forward integration. The result of this test is indicated by the dotted line in Fig. 2 and suggests that reasonable accuracy is attained when computing an exponentially decreasing solution through some two and a half decades.

D. Summary of Computational Results

The DUCK-ANSWER computations for the equations (1) - (7) are summarized in Table I, especially as they pertain to ψ- growth and turn-over. Information concerning ultimate instability is presumably least significant for runs based on non-Hamiltonian equations. For the various Hamiltonian systems of equations it is of interest to note that the onset of instability seems to become apparent at smaller initial ρ- amplitudes as additional coupling terms are introduced and as corresponding additional resonances become effective. There is evident a certain similarity between the AG results and those obtained for the non-AG equations (7). There is less similarity with the non-AG equations (8), for which the threshold for ψ- growth occurs at considerably higher initial ψ amplitudes (as an analytic study of these equations would in fact suggest).

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TABLE I.
SUMMARY OF DUCK-ANSWER COMPUTATIONS FOR EQUATIONS (1) - (8)

Equations	Run No.	ρ_0	Growth, Turn-Over, and Stability Characteristics
1 (Non Hamiltonian)	201	-.20	No growth, Stable through 200 sectors
	202	-.25	" " " " " "
	203	-.30	" " " " " "
	204	-.35	Exponential Growth
	205	-.40	" "
	206	-.45	" "
	207	-.50	" "
	208	-.55	" "
	209	-.60	" "
2 (Hamiltonian)	211	-.20	No growth, Stable through 200 sectors
	212	-.25	" " " " " "
	213	-.30	" " " " " "
	214	-.35	Growth, Turn-Over, Stable through 200 sectors
	215	-.40	" " " " " " "
	216	-.45	" " " " " " "
	217	-.50	" with variable turn-over, tho stable through 200 sectors
	218	-.55	" with very erratic turn-over, tho stable through 200 sectors
	219	-.60	" with variable turn-over, tho stable through 200 sectors
	254	-.70	" with slight turn-over, but overflow after 75 sectors
	253	Reversal of 215, after 105 Sectors of Forward Integration	(Retraced motion of Run 215 through some 2 1/2 decades)
	291-294	100-sector ψ -search	ψ_0=.75 unstable after 13 1/2 sectors; ψ_0=.70 Stable through 100 sectors

3 (Non-Hamiltonian)	223	-.30	No growth, Stable through 200 sectors
	225	-.40	Growth, regular Turn-Over, Stable through 200 sectors
	227	-.50	Growth irregular & erratic Turn-Over, Overflow after 128 sectors
	229	-.60	" " " " " " 101 "
4 (Hamiltonian)	233	-.30	No growth, Stable
	235	-.40	Growth, Irregular Turn-Over with questionable stability, tho held on through 200 sectors
	237	-.50	" Overflow after 164 sectors
	239	-.60	" " " 166 "
5 (Hamiltonian)	241	-.20	No growth, Stable
	243	-.30	No growth (just below threshold), Stable
	245	-.40	Growth, with regular Turn-Over, Stable through 200 sectors
	247	-.50	" erratic Turn-Over, Overflow after 169 sectors
	249	-.60	" no marked turn-over, Overflow after 79 sectors
6 (Hamiltonian)	321	-.20	No growth, Stable
	323	-.30	" " Stable through 200 sectors
	324	-.35	Growth, with Turn-Over, Stable through 200 sectors
	325	-.40	" " " " " " " "
	326	-.45	" with little turn-over, Overflow after 97 sectors
	327	-.50	" " " " " " " 89 "
	329	-.60	" " " " " " " 77 "

7 (Non-AG, Hamiltonian)	258	-.30	No Growth, Stable
	257	-.35	Slow Growth, Turn-Over, Stable through 200 sectors
	256	-.40	Growth " " " " " "
	255	-.60	" with fairly well-behaved Turn-Over (tho of diminishing amplitude excursions) through 200 sectors
	280	-.70	" with Variable Turn-Over, Overflow after 141 sectors
	281	-1.00	" with some Turn-Over, " " 79 "
	282	-1.30	" with early Overflow after 31 sectors
	300-302	100-sector ψ-search	ψ_0=1.7 unstable after 9 1/2 sectors; ψ_0 = 1.6 Stable through 100 sectors
8 (Non-AG Hamiltonian)	270	-.60	No Growth, Stable
	271	-.70	" " "
	272	-1.00	" " "
	273	-1.30	" " "
	276	-2.20	Growth, initially at $\cong$ .11 dec./sect.; Repeated Turn-Over at $K_\psi \cong 2$; Stable through 200 sectors

During the course of a computation in which the ψ - amplitude, as measured by K_ψ, exhibits pronounced growth followed by turn-over, the ρ-amplitude, as measured by K_ρ, exhibits a noticeable complementary decrease and subsequent increase. This behaviour in the neighborhood of the difference resonance $\sigma_{\rho_0} - 2\sigma_{\psi_0} = 0$ was less immediately evident in the corresponding FECKLESS-FIVE runs,[2] but is suggestive of coupled motion governed by an energy integral.[7] The relationship between the ρ and ψ amplitudes, for a couple of examples based on the AG Hamiltonian equations (2), is illustrated in Figs. 8 and 9 by linear plots of amplitude vs. the indendent variable τ. Figure 10 illustrates the relationship in such cases by a plot of ψ-amplitude vs. ρ- amplitude. The trend indicated in Fig. 10 suggests a slope $d(A_\psi^2) / d(A_\rho^2) \cong -3/2$. This value may be somewhat misleading because the amplitudes employed here are determined at the center of a radially-focusing region, where ρ is large and ψ small. For points at which the Floquet amplitudes assume maximum or minimum values, however, Symon has been able to employ Moser theory[8] to obtain a simple expression relating the changes of squared amplitudes to the ratio of the "phase function" derivatives[12] at the point in question:

$$\frac{d(A_\psi^2)_0}{d(A_\rho^2)_0} = -2 \frac{[\varphi'_\rho]_0}{[\varphi'_\psi]_0} .$$

For a non-AG case near the $\sigma_\rho = 2\sigma_\psi$ resonance, this formula would give a slope $-2\nu_\rho/\nu_\psi = -4$ in agreement with the result $-\nu_\rho^2/\nu_\psi^2 = -4$ expected from the constancy of the energy; in the present instance φ' may be obtained conveniently from tables[13a, b] and we estimate $[\varphi'_\rho]_{t=0} \cong 0.30$,

$\left[\phi'_\psi\right]_{t=0} \cong 0.35$, to obtain $d(A^2_\psi)/d(A^2_\rho) = -1.7$ in reasonable agreement with the computational results.

IV. DISCUSSION OF STABILITY LIMITS:

Before attempting to comment quantitatively on the computational results pertaining to stability, it may be helpful to summarize the properties of the non-AG equations (7) as determined by the energy integral which exists in that case [Section II (7)]. The potential-energy contours for this problem are sketched in Fig. 12. With $\psi_0 = 0$, motion with $|\rho_0| > 0.5166$ is energetically able to exhibit instability, as is motion with $\rho_0 = 0$ and $|\psi_0| > 1.18$. For motion which commences with ψ_0 and ψ_0' equal to zero, $\rho_{0,\,thr.} = 0.314$ represents the approximate threshold for ψ- growth, below which stability might be expected to obtain without appeal to the energy integral.

The computational results for the equations (7) do not, in the course of a 200-sector run, afford any clear evidence for instability of motion commencing with $\rho_0 = -0.60$, nor does a 100-sector ψ- search commencing with $\rho_0 = 0$ and $\psi_0 = 1.6$ reveal instability [Table I]. These observations may lend some support to the view that computational runs of this duration are inadequate to reveal eventual unstable behavior of a conservative system which, it might be expected, would have access to all portions of phase-space which are energetically permitted.

With respect to the Hamiltonian AG equations (2), the computations begin to indicate a somewhat variable turn-over of ψ- growth for ρ_0 between -0.45 and -0.50, although instability is not definitely revealed until ρ_0 exceeds -0.60

in absolute value and is given the value -0.70. A 100-sector ψ-search indicates the onset of instability for ψ_0 between 0.70 and 0.75 [Table I and Fig. 2]. The values cited here apply, it may be noted, to the coordinates as observed at $2\tau = 0$, mod. 2π -- i.e., at the center of a radially-focusing region where the ρ- amplitudes may be expected to be a maximum and the ψ-amplitudes a minimum.

It is tempting to imagine that the stability limits for the AG equations (2) can be simply related to those applying to the non-AG equations (7) having the same small-amplitude oscillation frequencies and the same threshold. If the coefficient of the coupling term in the AG equations had a constant component, or even if the coefficient contained a component in phase with the cos θ term which appears in the linear portion of the equations, there might be some basis to support this contention. In the study of cases involving a non-AG coefficient in the coupling term Dr. Parzen has observed a decrease of oscillation frequency as the stability-limit is approached ("$\sigma = 0$ resonance"), a characteristic which would be expected for motion in the non-AG case when traversing the "pass" in the potential-energy surface.

In the present case, however, it is observed that the oscillation frequencies of solutions to the AG equations (2) do not decrease as the stability limit is approached, but in fact appear to increase. The presence of the additional resonance $\sigma_{\rho_0} + 2\sigma_{\psi_0} = 2\pi$ (sum resonance) would suggest a mechanism whereby true instability might arise, but direct substitution into a simple formula[2] for the threshold amplitude attributable to this somewhat remote resonance leads to the rather large amplitude 2.2. An interesting possibility

suggested by the nature of the computational results is, however, that σ_ρ and σ_ψ are "pulled" together, in the ratio 2:1 towards the values π and $\pi/2$ which represent the inter•section of the $\sigma_\rho - 2\sigma_\psi = 0$ and $\sigma_\rho + 2\sigma_\psi = 2\pi$ resonances.

One method of estimating the effect of the intersecting sum and difference resonances in producing instability has, accordingly, been suggested by Parzen. In this method the simple formula[2, 6] for instability arising from the sum resonance,

$$A_{\rho,\,lim.} = 4\,\frac{\left|\left(\frac{\sigma_{y_s}}{2\pi}\right)^2 - \left(\frac{\sigma_{y_0}}{2\pi}\right)^2\right|}{C}\,N^2 \quad ,$$

is employed and σ_{y_s} assigned the value $\pi/2$. For the equations (2) with which we are concerned here, this procedure leads to

$$A_{\rho,\,lim.} = 4\,\frac{\left|(.25)^2 - (.1184)^2\right|}{0.8}\,4 = 0.97 \quad ;$$

an approximate equivalent to this formula, based on the supposition that $\sigma_{y_0} \ll \sigma_{y_s} = \pi/2$ is

$$A_{\rho,\,lim} \cong \frac{N^2}{4C} = \frac{4}{4\times 0.8} = 1.25 .$$

Alternatively, what may be basically a similar approach is afforded by the observation that solution of the equations (2) is approximately achieved by harmonic balance with expressions of the form

$$\rho \cong A\left[\sin\frac{N\theta}{2} + \ldots\right]$$

$$\psi \cong B\left[\begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{N\theta}{4}\right) + \ldots\right] .$$

Specifically, with the trial expression for Ψ

$$\Psi = B_1 \sin \frac{N\theta}{4} + B_2 \sin \frac{3N\theta}{4},$$

one obtains

$$A_{\rho,\,lim.} = 1.35$$

and the approximate formula

$$A_{\rho,\,lim.} \cong \frac{9N^2}{32C} = \frac{9 \times 4}{32 \times 0.8} = 1.4.$$

These results for estimating the stability limit for equations of the form (2), while showing fair agreement between themselves, appear to overestimate somewhat the stability limit indicated by the computations. [It may be noted, from Section III B, that the analytic estimate of the threshold is also high (by some 20 percent).] It may be, however, that further computational studies will bear out the approximate validity of the general form suggested here for the stability limit arising from the $\sigma_{x_0} - 2\sigma_{y_0} \doteq 0$ resonance in the presence of the $\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$ sum resonance. The simple formula $A_{\rho,\,lim.} \cong N^2/(4C)$, when expressed[3] in terms of parameters characterizing a spirally ridged accelerator, would read $A_{\rho,\,lim.} = \dfrac{w}{4\,f/(wN^2)}$.

In closing this report the writer would like to emphasise that DUCK-ANSWER runs in which integration of coupled differential equations is carried through a few hundred sectors may provide a rather poor estimate of stability limits. This view is suggested firstly by the failure of runs made with the non-AG equations (7) to show instability for starting values as small as those

which would suffice energetically and, secondly, by the variable or erratic turnover exhibited by solutions of the AG equations (2) which none the less persist through the number of sectors prescribed for the computation. It accordingly may be informative to examine phenomena akin to those with which we have been concerned here by means of algebraic transformations, as is possible with aid of the ALGYTEE program.[9, 10] It may be noted that preliminary ALGYTEE computations for a problem similar to that described by our equations (2) have shown[11] in the course of 100,000 iteration runs a very marked erratic fluctuation in the duration of runs made with initial values near what may be supposed to represent a stability limit.

V. REFERENCES AND NOTES:

1. J. N. Snyder, Internal MURA Reports (I.B.M. Programs 75 & 77) 237 and 238 (Feb.-March, 1957).

2. L. Jackson Laslett and A. M. Sessler, MURA Report 263.

The correspondence between the variables ρ, ψ and τ employed in the DUCK-ANSWER program and the variables $u \equiv x - x_f$, y, and θ of the FECKLESS-FIVE program is obtained by identifying

$$\rho = u/w,$$
$$\psi = y/w, \text{ and}$$
$$N\theta = 2\tau.$$

The parameters of the FECKLESS-FIVE runs for "Point θ" were $k = .6880$, $1/w = 19.60$, $f = 1/4$, $N = 5$, $\sigma_{x_0} = 0.5400\,\pi$, and $\sigma_{y_0} = 0.2365\,\pi$, with $f/(wN^2) = 0.1960 \cong 0.2$. For approximate equations for motion about the equilibrium orbit of a spirally-ridged FFAG accelerator, see F. T. Cole [Ref. 3].

A portion of the work contained in the present report was described briefly at the General MURA Technical Meeting in Madison, March 1 and 2, 1957.

3. F. T. Cole, MURA/FTC-3.

4. W. Walkinshaw, "A Spiral Ridged Bevatron", A.E.R.E., Harwell (1956).

5. To emphasise that the resonances with which we are concerned are <u>inherent</u> rather than imperfection resonances, it might be preferable to express them in terms of σ $(= 2\pi\nu/N)$; thus for the resonances mentioned when introducing equations (2) we might preferably write $\sigma_{x_0} - 2\sigma_{y_0} = 0$ and $\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$.

6. Cf., f. ex., G. Parzen, MURA Reports 217 and 250 (Int.).

7. Cf. P. A. Sturrock, Harwell Report A.E.R.E. X/R 1771 (October, 1955)

8. J. Moser, Nachr. Akad. Wiss. Göttingen, Math.-Physik Kl. IIa, No. 6, 87 (1955); J. Moser, Commun. Pure and Appl. Math. 8, 409 (1955).

9. M.R. Storm, MURA Internal Report (IBM Program) 233 (March, 1957).

10. L. J. Laslett, MURA Internal Report 246 (March 11, 1957), with correction dtd. March 25, 1957; MURA Internal Report 247 (March 18, 1957).

11. L.J. Laslett, Oral presentation at MURA Technical Group Staff Meeting #25 (March 25, 1957).

12. The phase function φ occurs when Floquet solutions are represented in the form w(t) exp $[\pm i\varphi(t)]$ and $\langle\varphi'\rangle$ then equals ν -- see E. D. Courant and H. S. Snyder, Brookhaven report EDC-15 (September 2, 1954).

13 (a) Belford, Laslett, and Snyder, "Table Pertaining to Solutions of a Hill Equation", MURA Notes (April 3, 1956);

(b) Laslett, Snyder, and Hutchinson, "Tables for the Determination of Stability Boundaries and Characteristic Exponents for a Hill's Equation Characterizing the Mark V FFAG Synchrotron", MURA Notes (20 April 1955).

FIG. 1.

TREND OF ψ AMPLITUDE

FOR THE NON-HAMILTONIAN COUPLED EQUATIONS

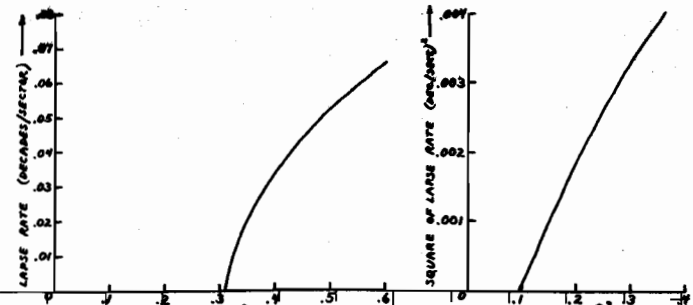
$$\rho'' = (-.184 - .8\cos 2\tau)\rho$$

$$\psi'' = (.027 + .8\cos 2\tau)\psi + .8(\sin 2\tau)\rho\psi,$$

with $\rho_0' = 0$.

$\sigma_{\rho_0} = .5407\pi \qquad \sigma_{\psi_0} = .2368\pi$

RESULTS OBTAINED WITH DUCK-ANSWER PROGRAM, WITH INVARIANTS – MURA LAB., MADISON



ρ_0	LAPSE RATE DECADES/SECT.	LAPSE RATE NEPERS/SECT.	THROB MAGNITUDE FACTOR	THROB MAGNITUDE SECTOR
−.600	.0635	.146	—	—
−.550	.0586	.135		
−.500	.0525	.121		
−.450	.0442−.0430	.102 − .099		
−.400	.0351−.0332	.081 − .076		
−.350	.021	.049		
−.300	—	—	8.10	2.09
−.250			3.78	1.33
−.200			2.55	0.94

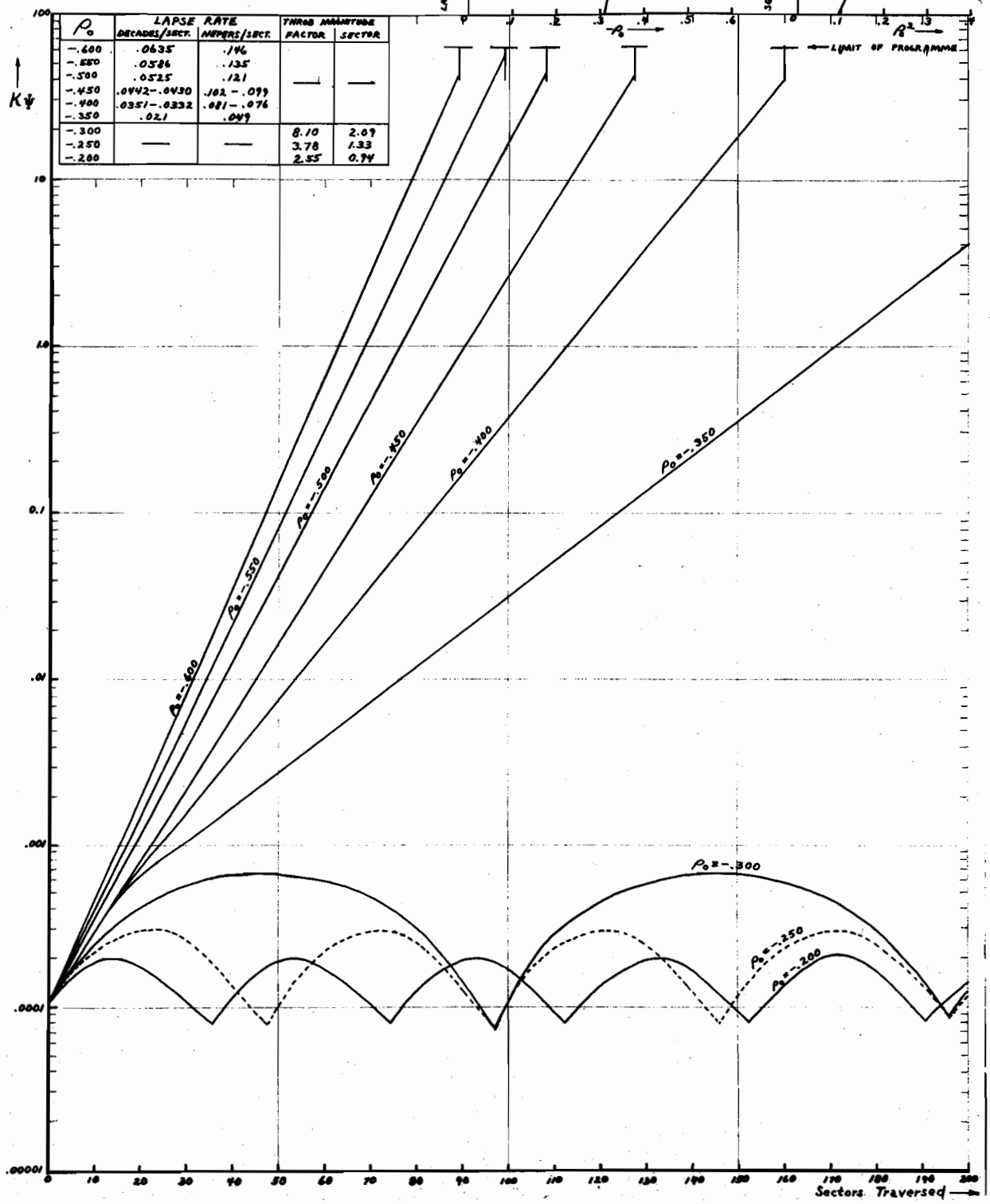


FIG. 2.

TREND OF Ψ AMPLITUDE

FOR THE HAMILTONIAN COUPLED EQUATIONS

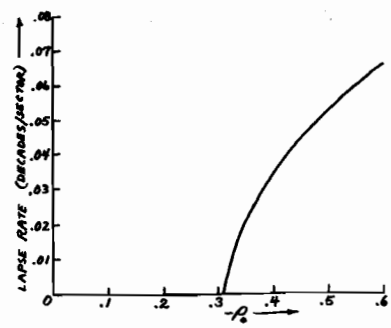
$$\rho'' = (-.184 - .8\cos 2\tau)\rho + .4(\sin 2\tau)\Psi^2$$

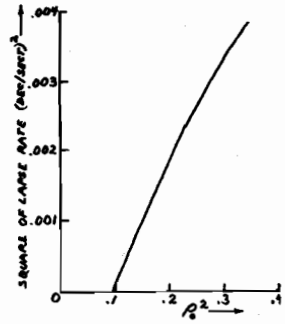
$$\Psi'' = (.027 + .8\cos 2\tau)\Psi + .8(\sin 2\tau)\rho\Psi,$$

with $\rho_0' = 0$.

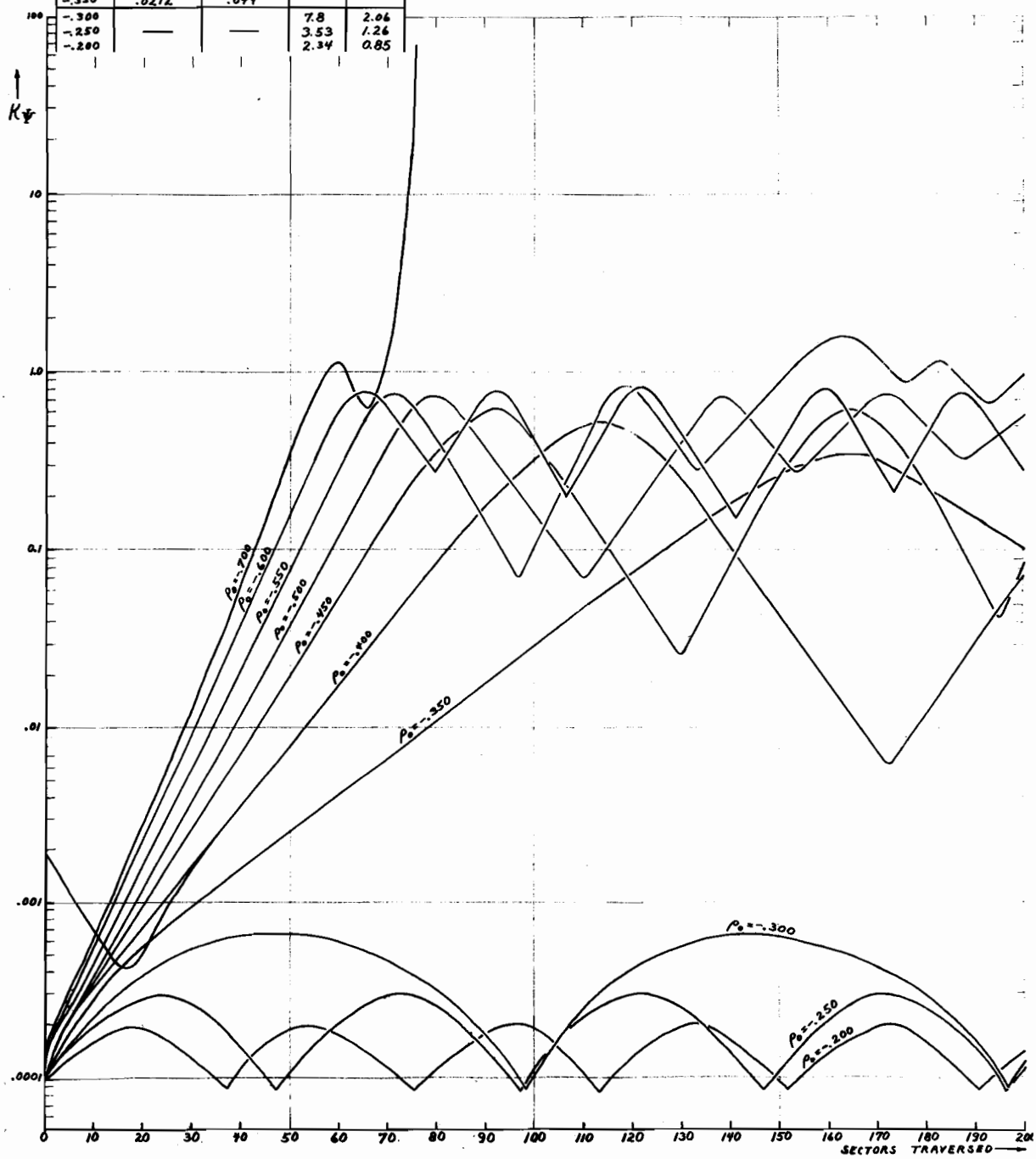
$\sigma_{\rho_0} = .5407\pi \qquad \sigma_{\Psi_0} = .2368\pi$

RESULTS OBTAINED WITH DUCK-ANSWER PROGRAM, WITH INVARIANTS – MURA LAB., MADISON





ρ_0	LAPSE RATE DECADES/SECT.	LAPSE RATE NEPERS/SECT.	THRU MAGNITUDE FACTOR	THRU MAGNITUDE NEPERS
-.600	.062	.142		
-.550	.0560	.129		
-.500	.0502	.116	—	—
-.450	.0439	.101		
-.400	.0345	.079		
-.350	.0212	.049		
-.300			7.8	2.06
-.250	—	—	3.53	1.26
-.200			2.34	0.85



TREND OF Ψ AMPLITUDE

FOR THE NON-HAMILTONIAN COUPLED EQUATIONS

FIG. 3.

$$\rho'' = (-.184 - .8\cos 2\tau)\rho$$

$$\Psi'' = (.027 + .8\cos 2\tau)\Psi + .8(\sin 2\tau)\rho\Psi + .133333\ldots(\cos 2\tau)\Psi^3,$$

with $\rho' = 0$

$\sigma_{\rho_0} = .5407\pi$ $\quad \sigma_{\Psi_0} = .2368\pi$

RESULTS OBTAINED WITH DUCK-ANSWER PROGRAM, WITH INVARIANTS — MURA LAB., MADISON

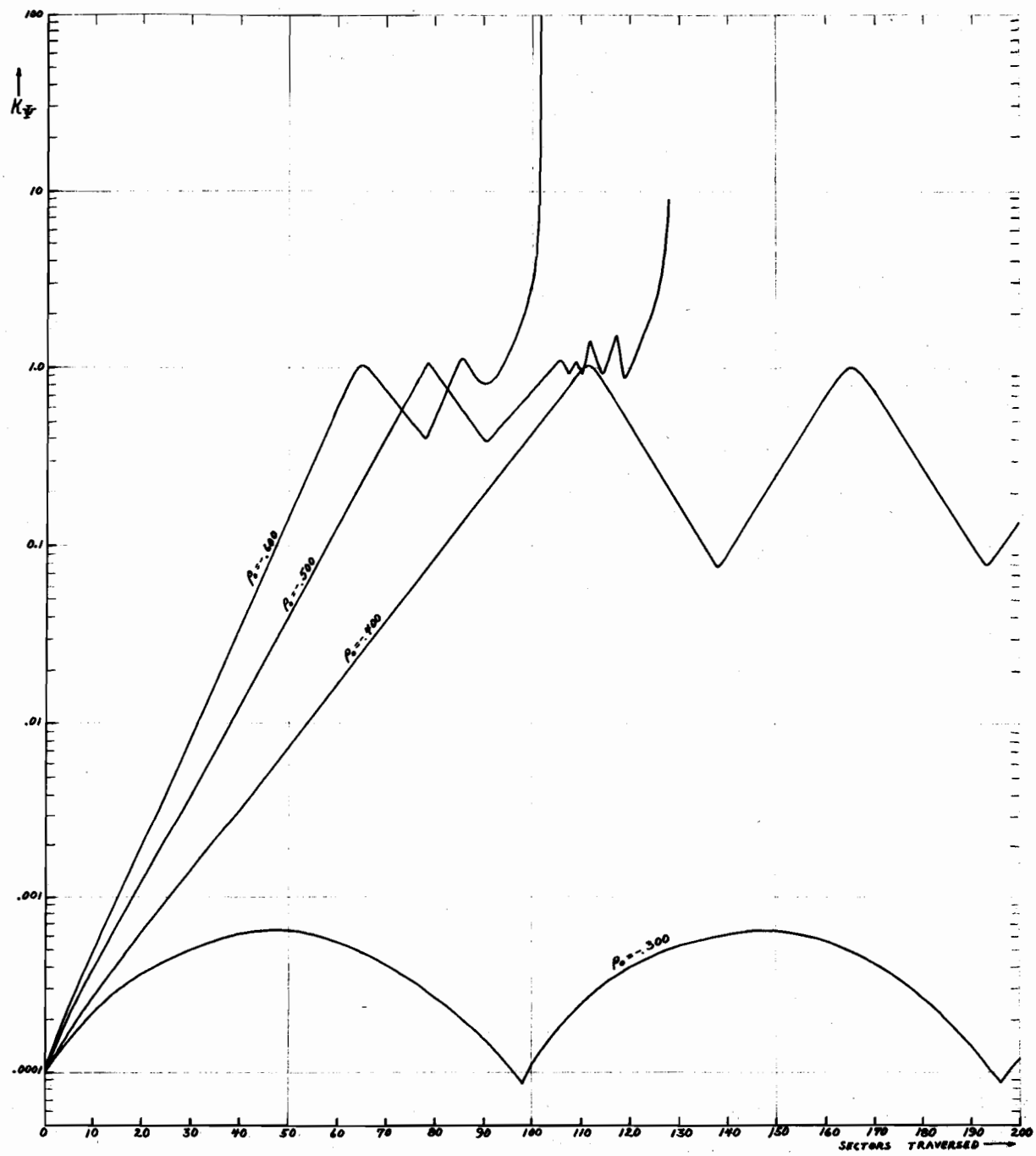


FIG. 4.

TREND OF Ψ AMPLITUDE

FOR THE HAMILTONIAN COUPLED EQUATIONS

$$\frac{d^2\rho}{d\tau^2} = (-.184 - .8\cos 2\tau)\rho + .4(\sin 2\tau)\Psi^2$$

$$\frac{d^2\Psi}{d\tau^2} = (.027 + .8\cos 2\tau)\Psi + .8(\sin 2\tau)\rho\Psi + (4/30)(\cos 2\tau)\Psi^3$$

with $\rho_0' = 0$

$\sigma_{\rho_0} = .5407\pi \qquad \sigma_{\Psi_0} = .2368\pi$

RESULTS OBTAINED WITH DUCK-ANSWER PROGRAM WITH INVARIANTS - MURA LAB., MADISON

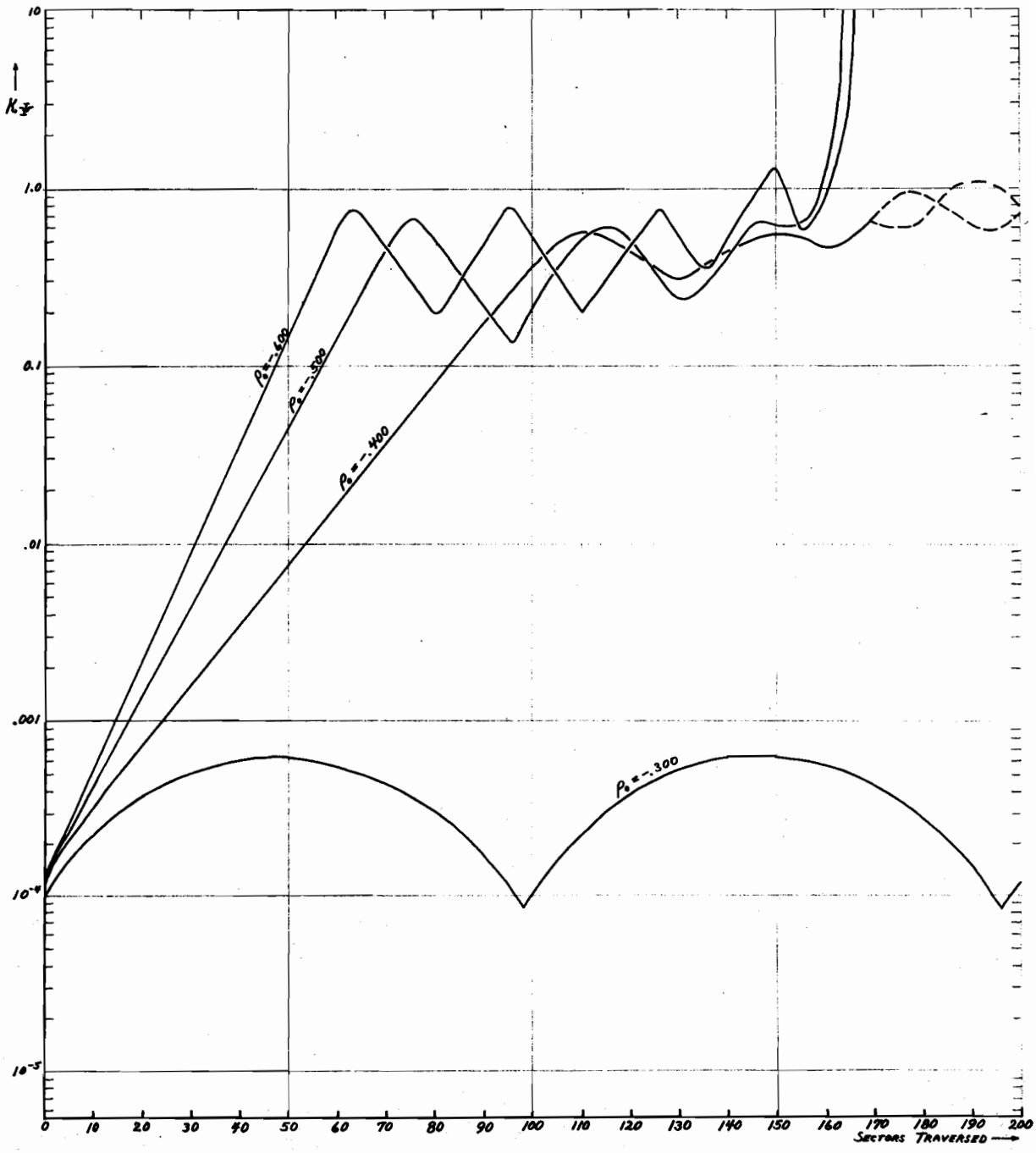


FIG. 5.

TREND OF Ψ AMPLITUDE

FOR THE HAMILTONIAN COUPLED EQUATIONS

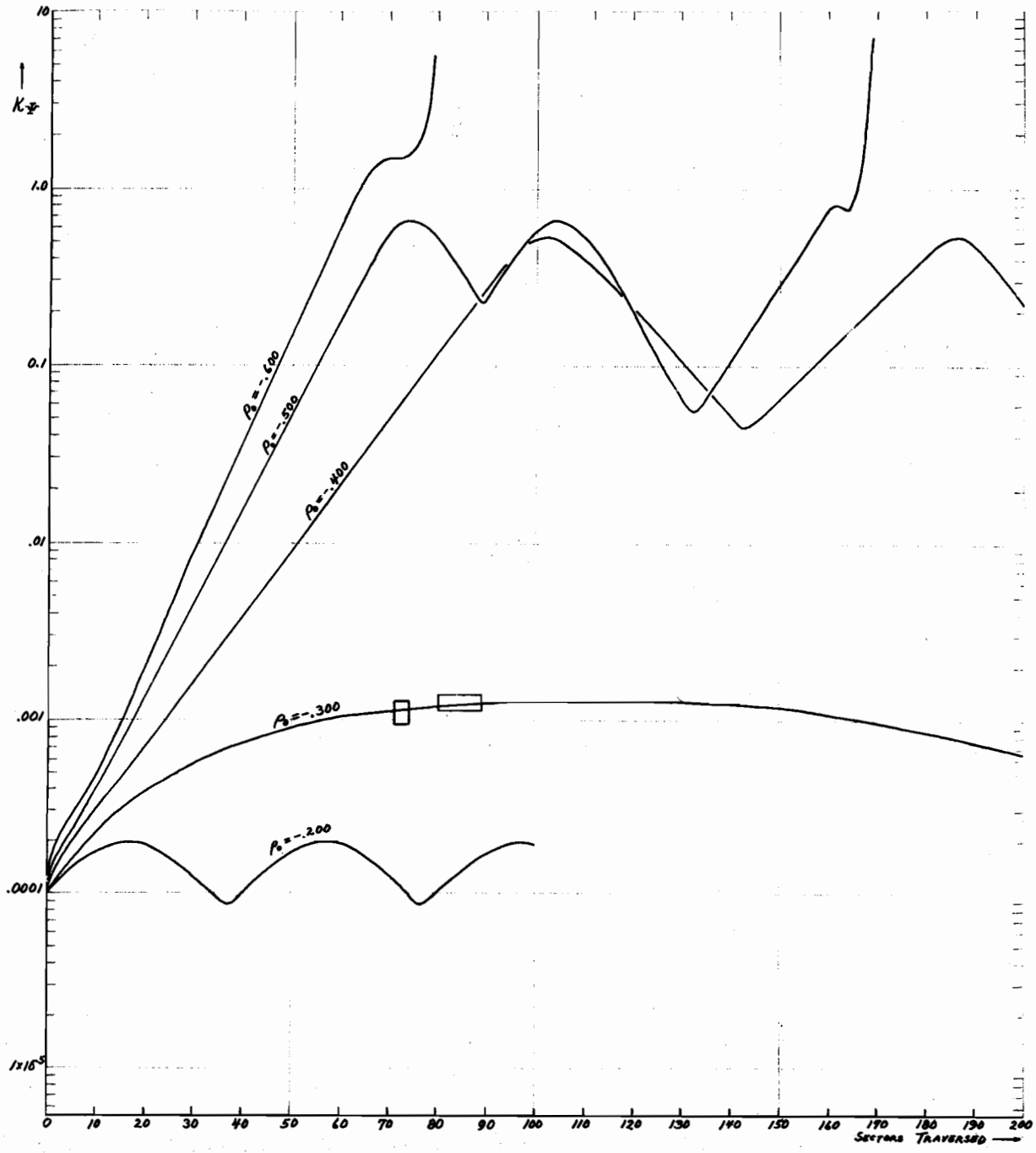
$$\frac{d^2\rho}{d\tau^2} = (-.184 - .8\cos 2\tau)\rho + .4(\sin 2\tau)\Psi^2 - .4(\cos 2\tau)\rho\Psi^2$$

$$\frac{d^2\Psi}{d\tau^2} = (.027 + .8\cos 2\tau)\Psi + .8(\sin 2\tau)\rho\Psi - .4(\cos 2\tau)\rho^2\Psi$$

with $\rho_0' = 0$

$\sigma_{\rho_0} = .5407\pi \qquad \sigma_{\psi_0} = .2368\pi$

RESULTS OBTAINED WITH DUCK-ANSWER PROGRAM, WITH INVARIANTS - MURA LAB., MADISON



TREND OF Ψ AMPLITUDE

FOR THE HAMILTONIAN COUPLED EQUATIONS

FIG. 6.

$$\frac{d^2\rho}{d\tau^2} = (-.184 - .8\cos 2\tau)\rho + .4(\sin 2\tau)\Psi^2 - .4(\cos 2\tau)\rho\Psi^2$$

$$\frac{d^2\Psi}{d\tau^2} = (.027 + .8\cos 2\tau)\Psi + .8(\sin 2\tau)\rho\Psi - .4(\cos 2\tau)\rho^2\Psi + (4/30)(\cos 2\tau)\Psi^3$$

with $\rho_0' = 0$

$\sigma_{\rho_0} = .5407\pi \qquad \sigma_{\Psi_0} = .2368\pi$

RESULTS OBTAINED WITH DUCK-ANSWER PROGRAM, WITH INVARIANTS — MURA LAB., MADISON

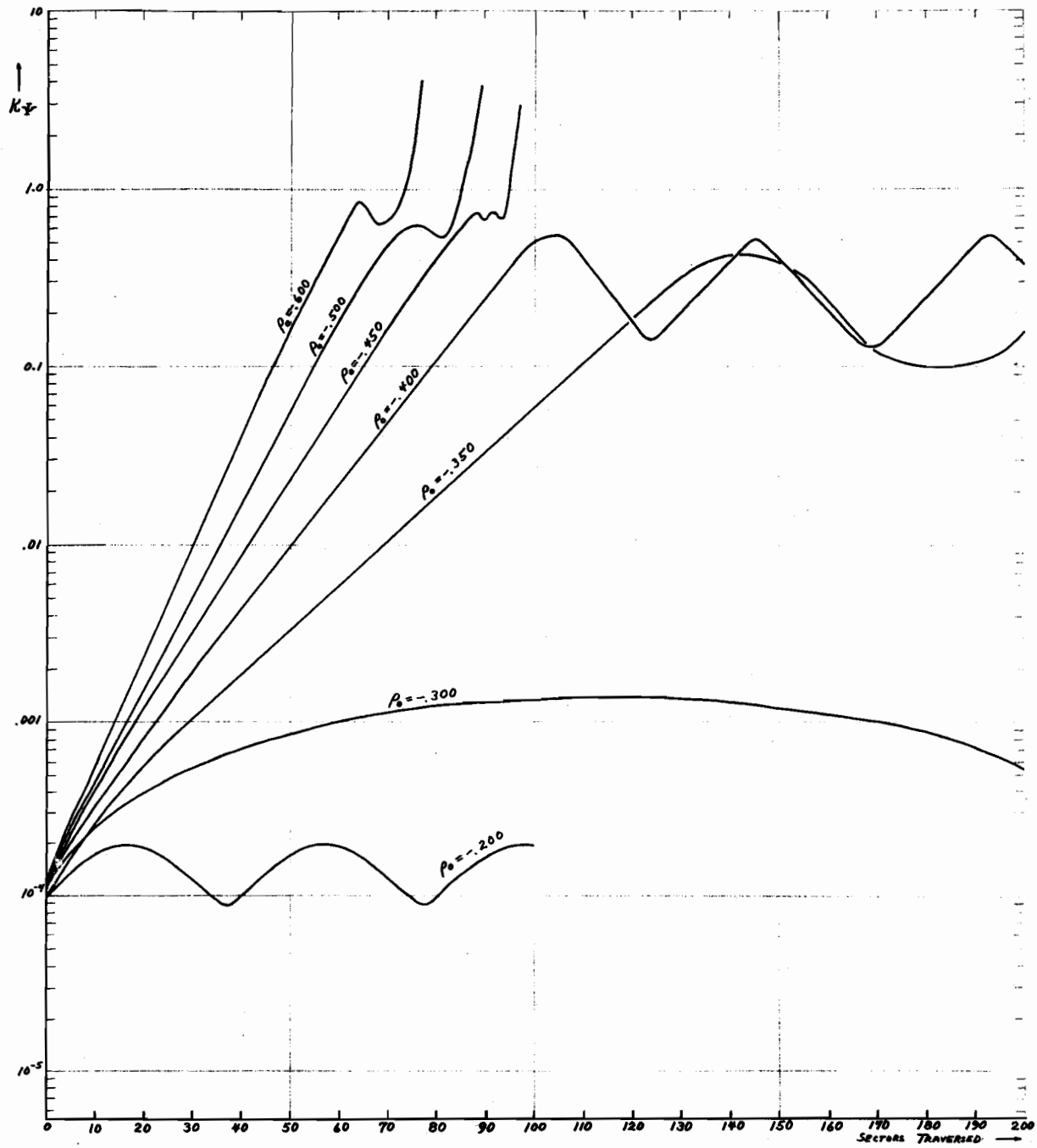


FIG. 7.

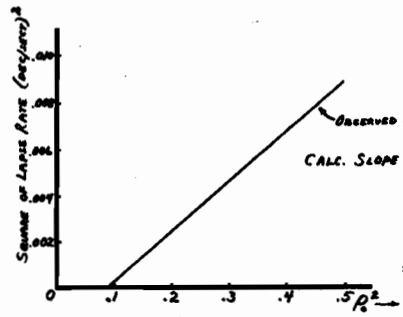
TREND OF Ψ AMPLITUDE

FOR THE NON-AG COUPLED HAMILTONIAN EQUATIONS

$$\rho'' = -(.5407)^2\rho + .0542\Psi^2$$

$$\Psi'' = -(.2368)^2\Psi + 2(.0542)\rho\Psi$$

with $\rho_0' = 0$

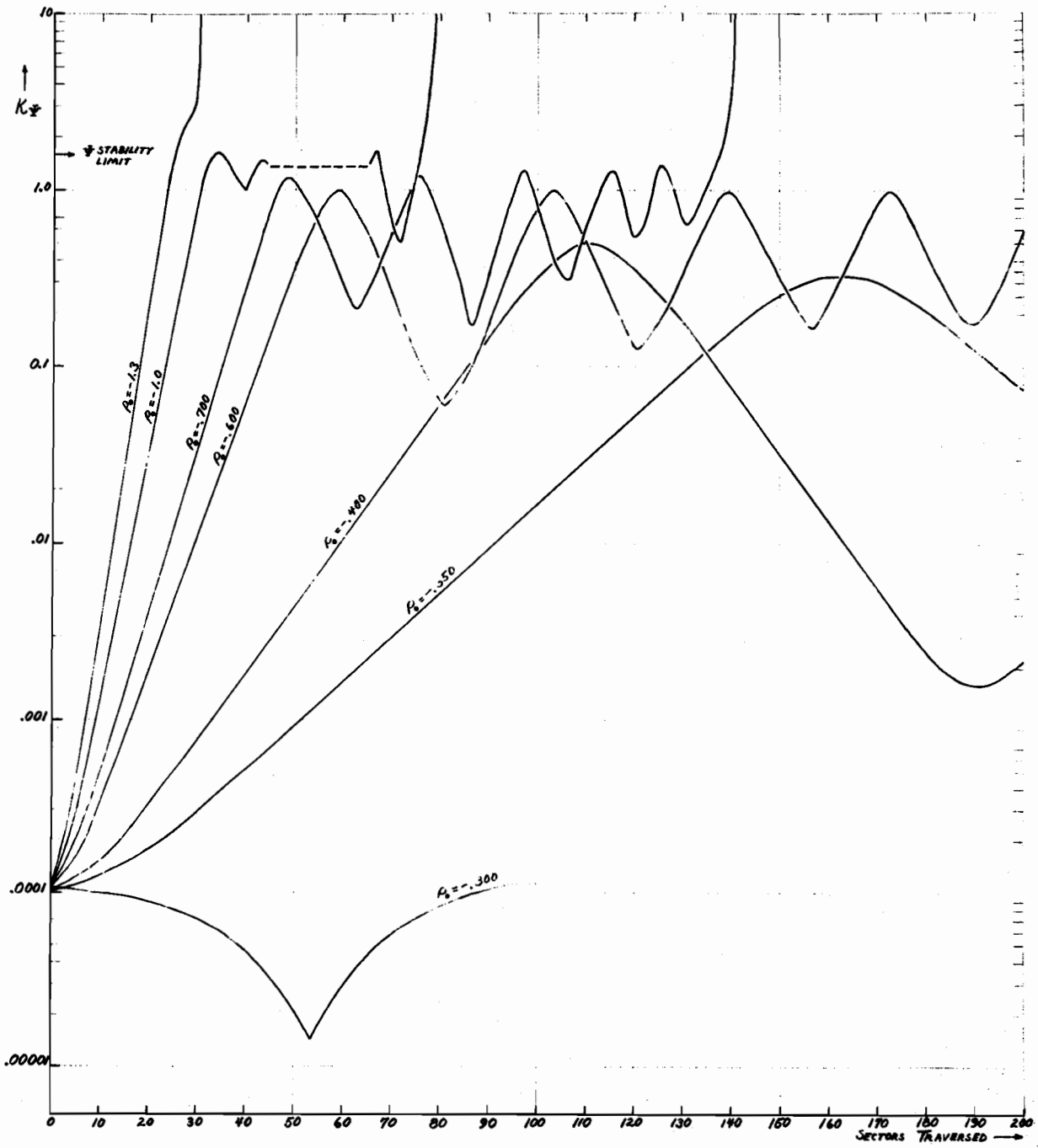


OBSERVED SLOPE $\cong 0.022$

$$\text{CALC. SLOPE} = \left[\frac{2.73}{16\rho_{0_{THR.}}}\,\frac{\left|(\sigma_{\rho_0}/\pi)^2-(2\sigma_{\psi_0}/\pi)^2\right|}{(\sigma_{\psi_0}/\pi)}\right]^2$$

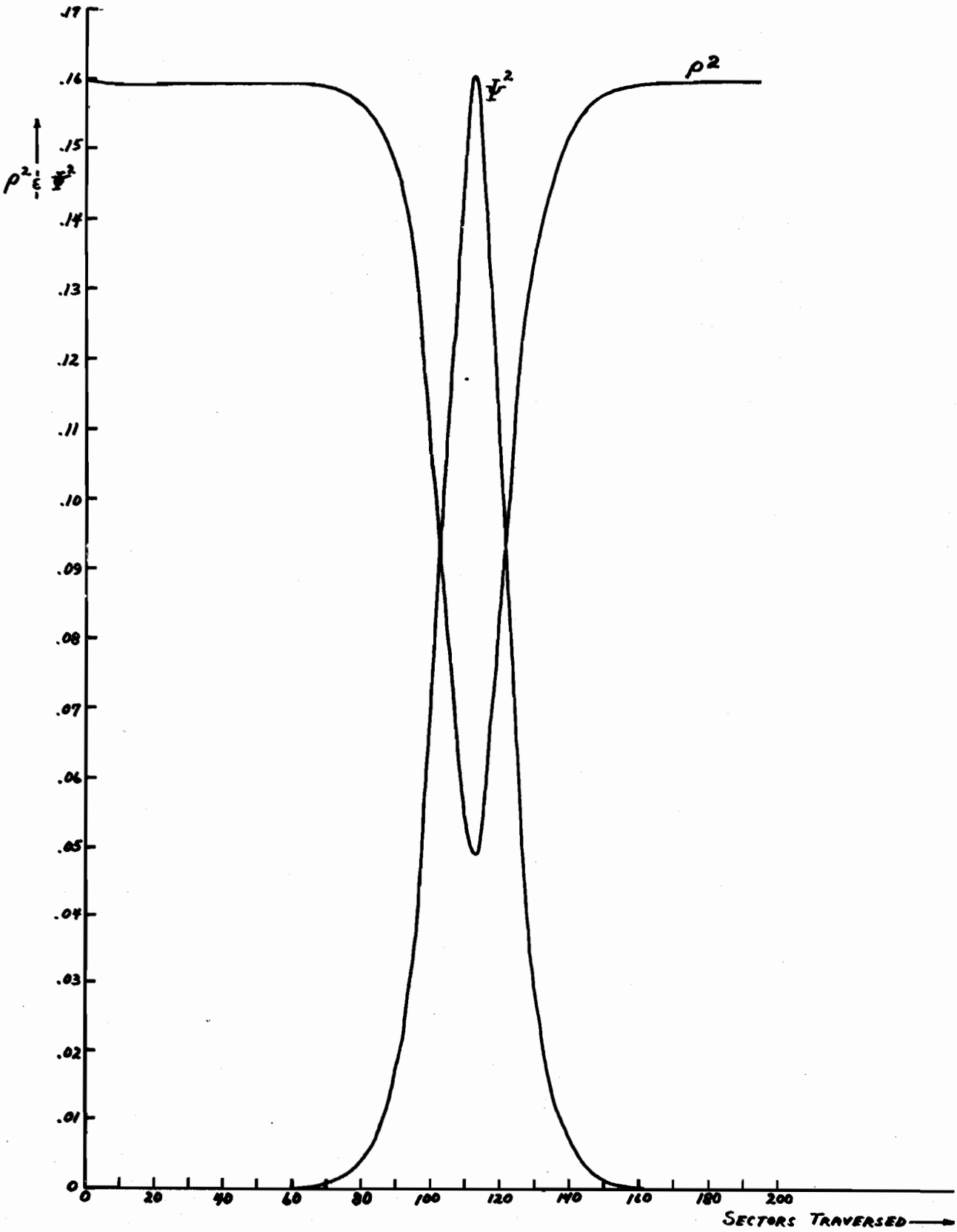
$$= \left[\frac{2.73}{16\times 0.314}\,\frac{(.5407)^2-(.4736)^2}{.2368}\right]^2$$

$$= 0.024$$



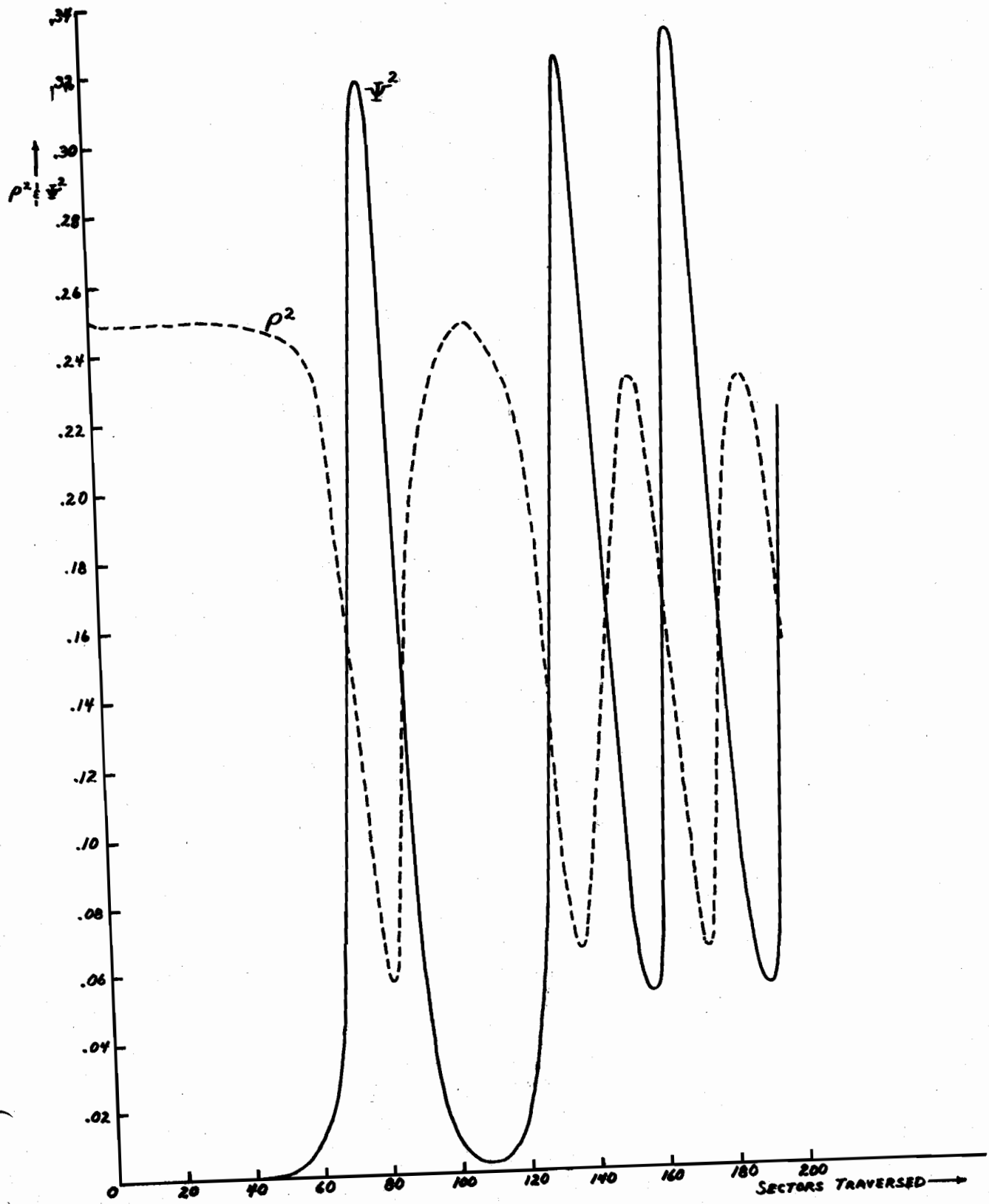
DUCK-ANSWER RUN 215
($\rho_0 = -.400$)

FIG. 8.



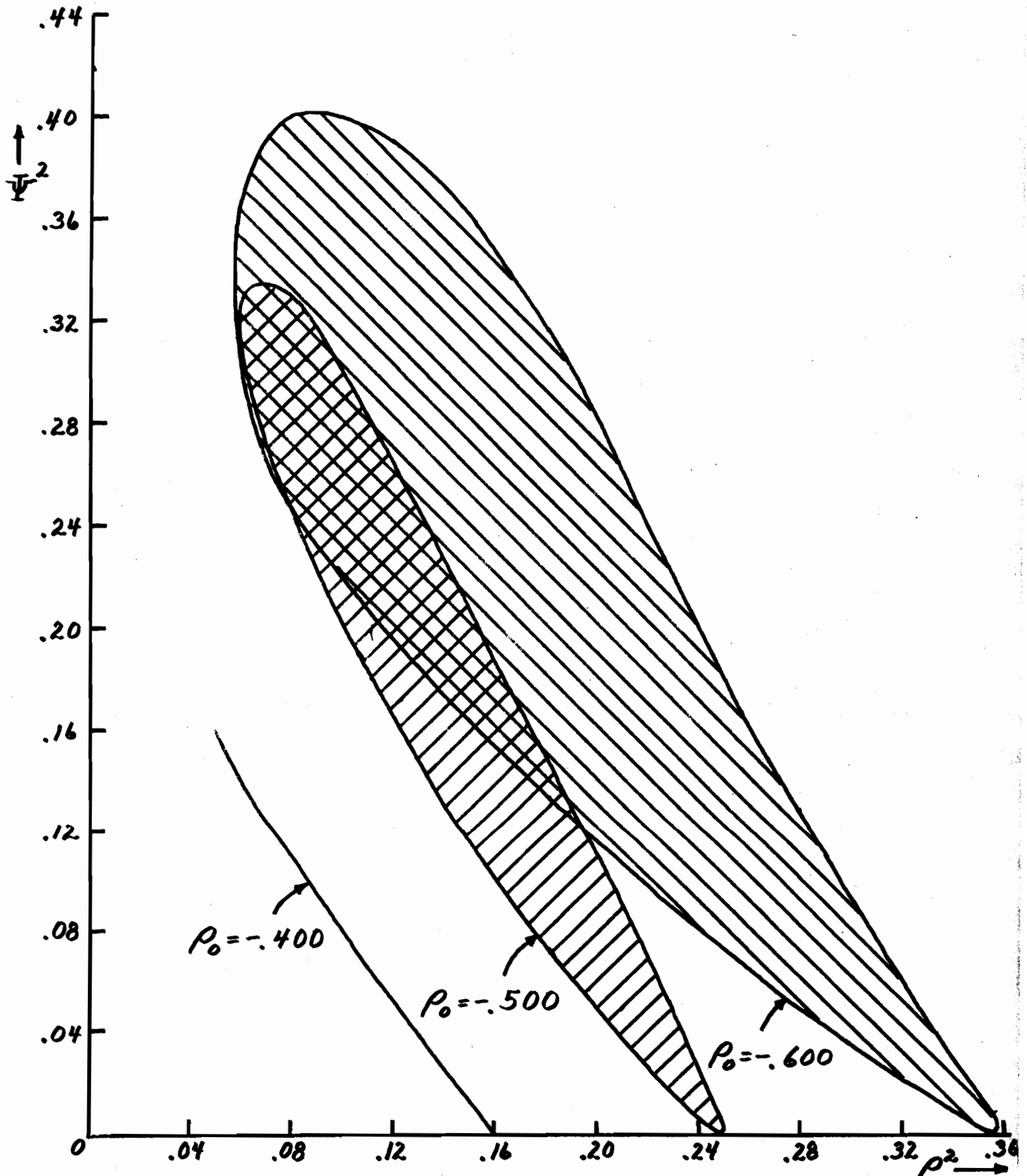
DUCK-ANSWER RUN 217
(ρ_0 = -.500)

FIG. 9.



DUCK-ANSWER RUNS 215, 217, 219

FIG. 10.



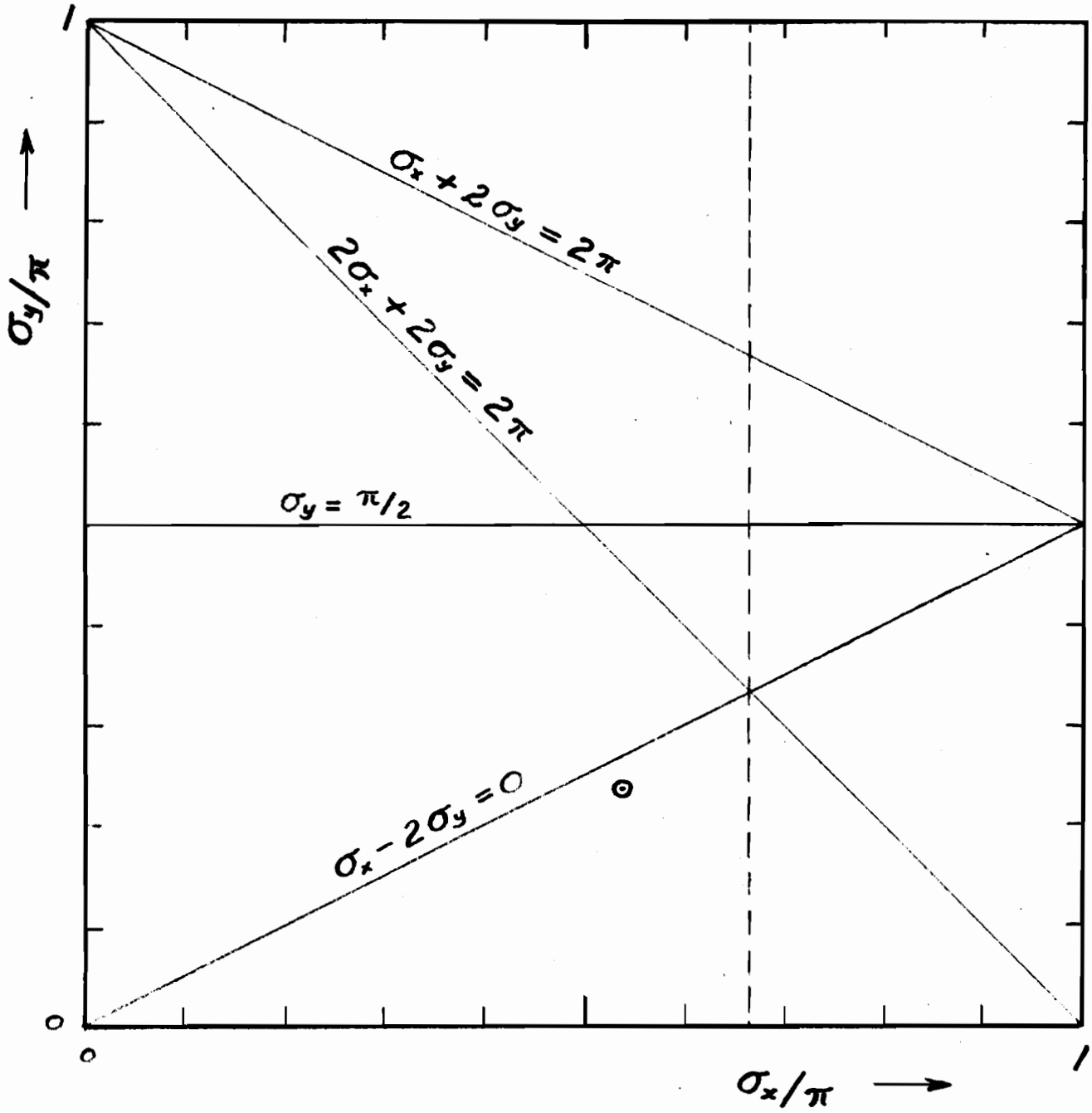


FIG. 11. LOCATION OF OPERATING POINT.

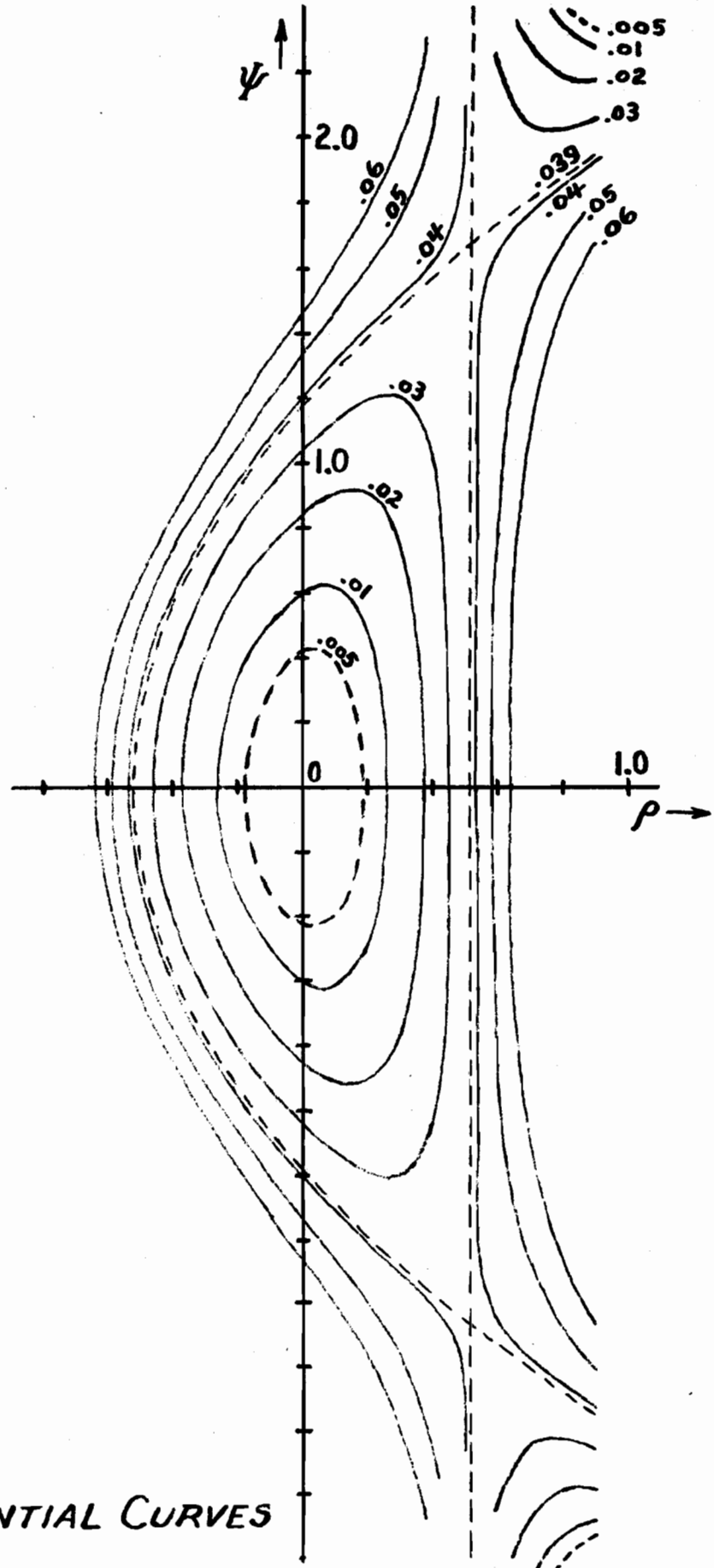


FIG. 12.

EQUIPOTENTIAL CURVES
for
$V = .146\,\rho^2 + .028\,\psi^2 - .0542\,\rho\psi^2$