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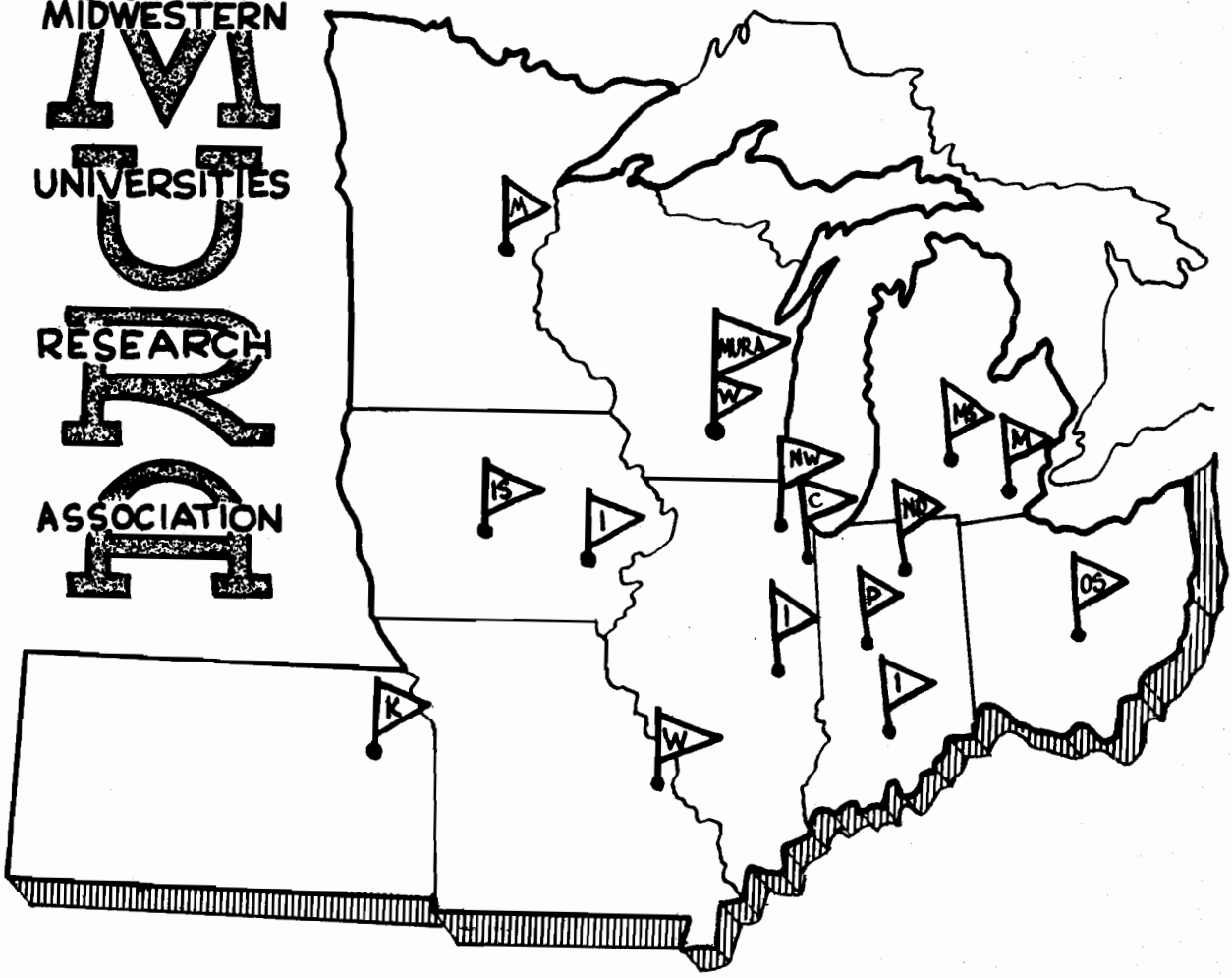

THE LINEAR EQUATIONS OF MOTION AND THE TUNE OF A FIXED-FIELD ACCELERATOR

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THE LINEAR EQUATIONS OF MOT'ON AND THE TUNE OF A FIXED-F ELD ACCELERAT R

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ABSTRACT: This p pe is a cont nuation of the report MURA 58. This paper gives a procedure f r obtaining the linear equa ions for the osc.llation of the particle about the equilibrium orbit an accelerator having an arbitrary magnetic field, and for s ving these linear equations for the tune and motion of the parti le. Results of the procedure are gi en for the general radial sector (Mark I) and the general spiral se tor (Mark V machines. The res s of the theory are compa ed with numerically computed results

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I. In roduction

This paper is a continuation of the report MURA-258, hereafter referred to as report I. In the previous report we treated the problem of finding the equilibrium orbit of a particle moving in an accelerator having an arbitrary magnetic field. The problem we will treat in this paper is that of obtaining the linear equations for the oscillation of the particle about the equilibrium orbit, and of solving these linear equations for the motion and tune of the particle.

The linear motion problem has been treated previously by T. Ohkawa for a Mark I machine and by F. T. Cole[1] for a Mark V machine having a small flutter. The more general machine is being treated at present by F. T. Cole. Our procedure and results differ from theirs in that we will attempt to obtain an approximate solution which has an accuracy of 10 to 20%. This will allow us to make approximations which will simplify the equations and will give us a simple general solution.

II. Summary of Results

In this section we will simply list the results for the linear equations and tune for the general spiral sector accelerator (Mark V). The results for the general radial sector machine (Mark I) can be obtained from these by putting $1/w = 0$. We will give the derivations in Section III and IV. In some cases, for the sake of simplicity, the results listed are not quite as general as those found in Sections III and IV.

The median plane field for the general spiral sector machine is defined by

$$H_z = -H\left(\frac{r}{r_1}\right)^k \sum_n h_n e^{i\omega_n \varphi} , \tag{2.1}$$

where $\omega_n = nN \quad n = 0, \pm 1, \ldots$ and

$$\varphi = \theta - \frac{1}{\omega N} \ln \frac{r}{r_1} . \tag{2.2}$$

The r-tune is given by

$$\left(\frac{\nu_x}{N}\right)^2 = b h_0 \frac{k+1}{N^2} + 4 b^2 \sum_{n \geq 1} \frac{(k+1)(k+2)}{N^4} \frac{|h_n|^2}{n^2} \tag{2.3}$$

where b is defined in I as

$$b = \frac{1}{\beta}\left(\sqrt{h_0^2 + 2\beta} - h_0\right) , \tag{2.4}$$

and $\quad \beta = 4(k+2) \sum_{n \geq 1} \left[|h_n|^2 / \omega_n^2 \right] .$

The z-tune is given by

$$\left(\frac{\nu_z}{N}\right)^2 = \frac{2b^2}{N^2} \sum_{n \geq 1} |h_n|^2 - b h_0 \frac{k}{N^2} + 4 b^2 \sum_{n \geq 1} \frac{1}{N^2} |h_n|^2 . \tag{2.5}$$

The above tune results are obtained by using the smooth approximation.

The linear equation of the r-motion is given by

$$u'' + (E_0 - g(\theta))u = 0 , \tag{2.6a}$$

$$E_0 = b h_0 \frac{K+1}{N^2} + 2b^2 \sum_{n \geq 1} \frac{K(K+1) - n^2/w^2}{N^4} \frac{|h_n|^2}{n^2} , \tag{2.6b}$$

$$\frac{g_n}{N^2} = -b \frac{(K+1) - i\, n/w}{N^2} \bar{h}_n , \tag{2.6c}$$

where $\bar{h}_n = h_n \exp[-i(n/w) \ln(R/r_1)]$. Note that $\bar{h}_n = h_n$ when $1/w = 0$.

The linear equation for the Z -motion is given by

$$y'' + (E_0' - f(\theta))y = 0 , \tag{2.7a}$$

$$\frac{E_0'}{N^2} = \frac{2b^2}{N^2} \sum_{n \geq 1} |h_n|^2 - b h_0 \frac{K}{N^2} - 2b^2 \sum_{n \geq 1} \frac{K^2 - n^2/w^2}{N^4} \frac{|h_n|^2}{n^2} , \tag{2.7b}$$

$$\frac{f_n}{N^2} = b \frac{K - i\, n/w}{N^2} \bar{h}_n . \tag{2.7c}$$

The r-motion is given by a linear combination of the solutions u_ν and u_ν^* of Eq. (3.6) and u_ν is given by

$$u_\nu(\theta) = e^{i\nu\theta}\left\{1 - \sum_{n\neq 0} \frac{g_n}{\omega_n^2}\,\frac{1}{1+\frac{2\nu}{\omega_n}}\,e^{i\omega_n\theta}\right\}. \tag{2.8}$$

Similarly, the floquet solutions of Eq. (3.7) are given by

$$y_\nu(\theta) = e^{i\nu\theta}\left\{1 - \sum_{n\neq 0} \frac{f_n}{\omega_n^2}\,\frac{1}{1+\frac{2\nu}{\omega_n}}\,e^{i\omega_n\theta}\right\}. \tag{2.9}$$

The floquet solutions given are believed to be good to 10 or 20% for $(\nu/N) \lesssim .4$

III. Derivation of the Linear Equations

For the sake of simplicity, we will carry through the derivation for the general radial sector machine (Mark I). It will be clear that any other magnetic field can be treated in the same way, without much change in the procedure.

First, let us consider the radial oscillation. The equation of motion in the radial direction is given by

$$\frac{d}{d\theta}\,\frac{r'}{\sqrt{r^2+r'^2+z'^2}} - \frac{r}{\sqrt{r^2+r'^2+z'^2}} = F_r\,, \tag{3.1a}$$

where

$$F_r = \frac{e}{pc}\left(r H_z - z' H_\theta\right). \tag{3.1b}$$

We wish to expand these equations about the equilibrium orbit. Let $r = r_s(\theta)$ represent the equilibrium orbit. $r_s(\theta)$ has been found in I. Let $r_s = R(1+x)$ then x is given by Eq. (3.14) in I as

$$x = b \sum_{n \neq 0} \frac{h_n}{\omega_n^2} e^{i\omega_n \theta} . \tag{3.2}$$

Now let us introduce the variable ρ which will give the displacement of the particle relative to the equilibrium orbit. ρ is defined by

$$r = r_s + \rho . \tag{3.3}$$

We wish to expand both sides of Eq. (3.1a) in powers of ρ and in this paper we are interested only in terms linear in ρ.

The expansion of F_r in powers of ρ up to linear terms in ρ is given by,

$$F_r = \frac{e}{pc} \left\{ r H_z + (H_z + r H_{z,r}) \rho + \cdots \right\}, \tag{3.4}$$

where the coefficients rH_z and $(H_z + rH_{z,r})$ are evaluated on the orbit where $r = r_s(\theta) = R(1+x)$. $H_{z,r}$ means $\partial H_z / \partial r$ (See Appendix A for the derivation of the expansion of F_r.)

To expand the left side it is convenient to rewrite Eq. (3.1a) as

$$\frac{d}{d\theta} \frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = F_r \tag{3.5}$$

where

$$L = \sqrt{r^2 + r'^2 + z'^2} \quad . \tag{3.6}$$

Let us expand L in powers of ρ and ρ'. The expansion of L up to quadratic terms is[1]

$$L = \frac{1}{2} a_1 \rho^2 + a_2 \rho \rho' + \frac{1}{2} a_3 \rho'^2 + \frac{1}{2} a_6 z'^2 + \cdots , \tag{3.7}$$

where

$$a_1 = r'^2 / L^3 , \tag{3.8a}$$

$$a_2 = -r r' / L^3 , \tag{3.8b}$$

$$a_3 = r^2 / L^3 , \tag{3.8c}$$

$$a_6 = 1 / L . \tag{3.8d}$$

The coefficients a_1, a_2, a_3, a_6 are all to be evaluated on the orbit which means we put $r = r_s(\theta)$, $r' = r_s'(\theta)$, $z = 0$. The linear terms have been dropped in Eq. (3.7) for L as they do not contribute to the equation for ρ.

Keeping up to linear terms in ρ, we find that Eq. (3.5) becomes

$$a_3 \rho'' + a_3' \rho' + (a_2' - a_1)\rho = \frac{e}{pc}\left(H_z + r H_{z,r}\right)\rho. \tag{3.9}$$

Eq. (3.9) is the linear equation for the radial motion and is true for any magnetic field. We will now specialize to the radial sector field,

$H_z = -H(r/r_1)^K \sum_n h_n \exp(i\omega_n \theta)$ in the median plane.

For the radial sector field, we find that

$$\begin{aligned} H_z + r H_{z\,r} &= -H\left(\frac{R}{r_1}\right)^K \sum_n (1+x)^K (K+1)\, h_n\, e^{i\omega_n\theta} \\ &= -H\left(\frac{R}{r_1}\right)^K \sum_n (1+Kx)(K+1)\, h_n\, e^{i\omega_n\theta}, \end{aligned} \tag{3.10}$$

In Eq. (3.10) we have used the condition assumed in I, that $Kx \ll 1$.

The terms in the left side of Eq. (3.9) involving ρ and ρ' turn out to be small. (See Appendix B). The coefficient a_3 may be written as $a_3 \simeq 1/r \simeq 1/R$ by using Eq. (3.8c) and remembering that $x' \ll 1$ and $x \ll 1$.

Thus Eq. (3.9) becomes

$$u'' + n_x(\theta)\, u = 0, \tag{3.10a}$$

where $u = \rho/R$ and

$$n_x(\theta) = b \sum_n (1+Kx)(K+1)\, h_n\, e^{i\omega_n\theta}. \tag{3.10b}$$

We can also write Eqs.(3.10) as

$$u'' + \left(E_0 - g(\theta)\right) u = 0, \tag{3.11a}$$

$$\frac{E_0}{N^2} = b h_0 \frac{(K+1)}{N^2} + b^2 \sum_{n \neq 0} \frac{K(K+1)}{N^4} \frac{|h_n|^2}{n^2} , \tag{3.11b}$$

$$\frac{g_n}{N^2} = - b h_n \frac{(K+1)}{N^2} - b^2 \sum_{m \neq 0, n} \frac{K(K+1)}{N^4} \frac{h_m h_{n-m}}{m^2} . \tag{3.11c}$$

The g_n are the coefficients in the expansion $g(\theta) = \sum_n g_n \exp(i \omega_n \theta)$.

Eqs. (3.11) are the equations giving the linear radial motion. They can be solved to give the r-tune and the r-motion. The second term in Eq. (3.11c) for g_n / N^2 can usually be neglected.

We can find the linear equation for the z -motion in the same way. The equation of motion in z -direction is

$$\frac{d}{d\theta} \frac{z'}{\sqrt{r^2 + r'^2 + z'^2}} = F_z , \tag{3.12a}$$

where

$$F_z = \frac{e}{pc} (r' H_\theta - r H_r) . \tag{3.12b}$$

We wish to expand Eq. (3.12) in powers of z . The expansion of F_z in powers of z up to linear terms is given by (see Appendix A)

$$F_z = \frac{e}{pc} \left\{ \frac{r'}{r} H_{z,\theta} - r H_{z,r} \right\} z + \ldots , \tag{3.13}$$

where the coefficient of z is to be evaluated on the orbit where $r = r_s(\theta)$ and $z = 0$.

To expand th left side of Eq. (3.12) we rewrite it as

$$\frac{d}{d\theta}\frac{\partial}{\partial z'} L = F_z \tag{3.14}$$

and use the expansion Eq. (3.7) for L. Thus we get for the linear equation of the z-motion

$$a_6 z'' + a_6' z' = \frac{e}{pc}\left(\frac{r'}{r} H_{z,\theta} - r H_{z,r}\right) z \, . \tag{3.15}$$

Eq. (3.15) will hold for any magnetic field. For the radial sector field we find that

$$\frac{r'}{r} H_{z,\theta} - r H_{z,r} = -H\left(\frac{R}{r}\right)^K \sum_n x'(1+x)^{K-1} i\omega_n h_n e^{i\omega_n\theta} \tag{3.16}$$

$$+ H\left(\frac{R}{r}\right)^K \sum_n K(1+x)^K h_n e^{i\omega_n\theta} \, .$$

The term on the left side of Eq. (.135) which is linear in z' may be neglected (see Appendix B). The coefficient $a_6 \simeq 1/r \simeq 1/R$.

Thus Eq. (3.15) becomes

$$y'' + n_y(\theta) y = 0 \tag{3.17a}$$

where $y = z/R$ and

$$h_y(\theta) = b \sum_n \omega_n x' [1 + (K-1)x] h_n e^{\omega_n \theta} - b \sum_n K(1 + Kx) h_n e^{\omega_n \theta} \tag{3.17b}$$

We can also write Eq. (3.17) as

$$y'' + (E_0' - f(\theta))\, y = 0, \tag{3.18a}$$

where

$$\frac{E_0'}{N^2} = \frac{b^2}{N} \sum_{m \neq 0} |h_m|^2 - b^3 \sum_{\substack{m+m'+m''=0 \\ m, m', m'' \neq 0}} \frac{K-1}{N^2} \frac{\omega_{m'}}{\omega_m} \frac{h_m h_{m'} h_{m''}}{\omega_{m''}^2} - b h_0 \frac{K}{N^2} - b^2 \sum_{m \neq} \frac{|h_m|^2}{\omega_m^2} \frac{K^2}{N^4} \tag{3.18b}$$

$$\frac{f_n}{N^2} = \frac{b}{N^2} \sum_{\substack{m \neq 0 \\ m \neq n}} \frac{n-m}{m} h_m h_{n-m} + b^3 \sum_{\substack{m+m'+m''=n \\ m, m', m'' \neq 0, n}} \frac{K-1}{\omega_m^2} \frac{\omega_{m'}}{\omega_m} \frac{h_m h_{m'} h_{m''}}{\omega_{m''}^2} + b h_n \frac{K}{N^2} + b^2 \sum_{\substack{m \neq 0 \\ m \neq n}} \frac{h_m h_{n-m}}{m^2} \frac{K^2}{N^4}. \tag{3.18c}$$

The f_n are the coefficients in the expansion $f(\theta) = \sum_n f_n \exp(\omega_n \theta)$.

Some of the terms in E_0' and f_n are usually negligible. We can usually drop the third term in E_0' and all but the third term of f_n. If we drop these terms we can write,

$$\frac{E_0'}{N^2} = \frac{b^2}{N^2}\sum_{m\neq 0}|h_m|^2 - b h_0 \frac{K}{N^2} - b^2 \sum_{m\neq 0}\frac{|h_m|^2}{m^2}\frac{K^2}{N^4}, \tag{3.19a}$$

$$\frac{f_n}{N^2} = b h_n \frac{K}{N^2}. \tag{3.19b}$$

Tune and Floquet Solutions of a Radial Sector Accelerator

Eqs. (3.11) and (3.18) are the linear equations for the r-motion and the z-motion in a radial sector accelerator. We will now use these equations to find the tune and the floquet solutions.

First, let us find the tune of the radial motion, ν_x, which is determined by Eq. (3.11). The r-tune, ν_x, depends on E_0 and g_n according to the relation[2]

$$\alpha\left(\frac{\nu}{N}\right)^2 = \frac{E_0}{N^2} + 2\sum_{n\geq 1}\left|\frac{g_n}{N^2}\right|^2\frac{1}{n^2}, \tag{3.20}$$

where

$$\alpha = 1 - 8\sum_{n\geq 1}\left|\frac{g_n}{N^2}\right|^2\frac{1}{n^4}. \tag{3.21}$$

If we put $\alpha = 1$ in Eq. (3.20), then this is just the smooth approximation. The coefficient α gives about 10% correction to the smooth approximation. Eq. (3.2) should be valid for $\nu/N \lesssim .4$

If we substitute the values of E_0 and g_n given by Eqs. (3.11), we find for the r-tune

$$\alpha\left(\frac{\nu_x}{N}\right)^2 = b h_0 \frac{K+1}{N^2} + 4 b^2 \sum_{n \geq 1} \frac{(K+1)(K+\frac{1}{2})}{N^2} \frac{|h_n|^2}{n^2}, \tag{3.22a}$$

$$\alpha_x = 1 - 8 b^2 \sum_{n \geq 1} \frac{(K+1)^2}{N^4} \frac{|h_n|^2}{n^4}. \tag{3.22b}$$

In the same way we can find the z-tune. The z-tune is given by

$$\alpha_z \left(\frac{\nu_z}{N}\right)^2 = 2 \frac{b^2}{N^2} \sum_{n \geq 1} |h_n|^2 - b h_0 \frac{K}{N^2}, \tag{3.23a}$$

$$\alpha_z = 1 - 8 b^2 \sum_{n \geq 1} \frac{K^2}{N^4} \frac{|h_n|^2}{n^4}. \tag{3.23b}$$

The r-motion is given by a linear combination of the solutions u_ν and u_ν^* of Eq. (3.11), and u_ν is given by[2]

$$u_\nu = e^{i\nu\theta}\left\{1 - \sum_{n\neq 0} \frac{g_n}{\omega_n^2} \frac{1}{1+2\nu/\omega_n} e^{i\omega_n\theta}\right\}. \tag{3.24}$$

The z-motion is given by the solutions y_ν, y_ν^* of Eq. (3.18) and the y_ν are given by

$$y_\nu = e^{i\nu\theta}\left\{1 - \sum_{n\neq 0} \frac{f_n}{\omega_n^2} \frac{1}{1+2\nu/\omega_n} e^{i\omega_n\theta}\right\}. \tag{3.25}$$

IV. The Spiral Sector Accelerator

So far we have treated only the radial sector field (Mark I). The derivation given could be applied, without very much change, to any other magnetic field. We would like to indicate, in this section, the derivation of the linear equation results for the spiral sector field (Mark V) whose median plane magnetic field is given by

$$H_z = -H\left(\frac{r}{r_1}\right)^K \sum_n h_n e^{i\omega_n\varphi}, \tag{4.1a}$$

$$\varphi = \theta - \frac{1}{\tan\alpha} \ln\left(\frac{r}{r_1}\right). \tag{4.1b}$$

It turns out, that most of the results for the spiral sector field can be obtained from the results for the radial sector field by making a simple substitutuion in the radial sector results. This substitution is given by

$$K^s h_n \longrightarrow (K - in/w)^s \bar{h}_n , \tag{4.2a}$$

where

$$\bar{h}_n = h_n e^{-i(n/w)\ln(r/r_1)} \tag{4.2b}$$

and s is any integer 0, 1, 2 . . .

To see how the substitution appears, let us rewrite the spiral sector field as

$$H_z = -H\left(\frac{R}{r_1}\right)^K \sum_n (1+x)^{K-in/w} \bar{h}_n e^{i\omega_n \theta} . \tag{4.3}$$

In Eq. (4.3) we have put $r = R(1+x)$.

If we compare this field with the radial sector field given by

$$H_z = -H\left(\frac{R}{r_1}\right)^K \sum_n (1+x)^K e^{i\omega_n \theta} \tag{4.4}$$

we see that the latter can be obtained from the former by letting $K \rightarrow K - in/w$ $h_n \longrightarrow \bar{h}_n$ in the coefficient of exp ($i\omega_n \theta$). If one looks through the derivation given for the radial sector field, one can see that the sub-

stitution rule Eq. (4. 2) will hold; though not in all cases.

Using the substitution Eq. (4. 2), we can now find the linear equation for the spiral sector field.

From Eq. (3. 11), we find the r-motion equation

$$u'' + \left(E_0 - g(\theta)\right) u = 0 \,, \tag{4. 5a}$$

$$\frac{E_0}{N^2} = b h_0 \frac{(K+1)}{N^2} + b^2 \sum_{n \neq 0} \frac{K(K+1) - n^2/w^2}{N^4} \frac{|h_n|^2}{n^2} \,, \tag{4. 5b}$$

$$\frac{g_n}{N^2} = - b \bar{h}_n \frac{(K+1) - i n/w}{N^2} \,. \tag{4. 5c}$$

In Eq. (4. 5c) we dropped the second term from Eq. (3. 11c)

From Eqs. (3. 18) and (3. 19) we find the y-motion equation.

$$y'' + \left(E_0' - f(\theta)\right) y = 0 \tag{4. 6a}$$

$$\frac{E_0'}{N^2} = \frac{2b^2}{N^2}\sum_{n\geq 1}|h_n|^2 - b h_0 \frac{K}{N^2} - b^2 \sum_{n\neq 0} \frac{K^2 - n^2/w^2}{N^4} \frac{|h_n|^2}{n^2}, \tag{4.6b}$$

$$\frac{f_n}{N^2} = b\bar{h}_n \frac{K - i n/w}{N^2}. \tag{4.6c}$$

Now we apply Eq. (3.20) to the above linear equations to find the tune.

The r-tune is thus found to be

$$\alpha_x\left(\frac{\nu_x}{N}\right)^2 = b h_0 \frac{K+1}{N^2} + 4b^2 \sum_{n\geq 1} \frac{(K+1)(K+\frac{1}{2})}{N^4} \frac{|h_n|^2}{n^2}, \tag{4.7a}$$

$$\alpha_x = 1 - 8b^2 \sum_{n\geq 1} \frac{(K+1)^2 + n^2/w^2}{N^4} \frac{|h_n|^2}{n^4}. \tag{4.7b}$$

Note that the substitution rule Eq. (4.2) should not be used to find the tune.

The z-tune is found to be

$$\alpha_z\left(\frac{\nu_z}{N}\right)^2 = \frac{2b^2}{N^2}\sum_{n\geq 1}|h_n|^2 - b h_0 \frac{K}{N^2} + 4b^2 \sum_{n\geq 1} \frac{1}{w^2} |h_n|^2, \tag{4.8a}$$

$$\alpha_z = 1 - 8b^2 \sum_{n \geq 1} \frac{K^2 + n^2/\nu^2}{N^4} \frac{|h_n|^2}{n^4} . \qquad (4.8b)$$

The floquet solutions are given by Eqs. (3.24) and (3.25) except that the g_n and f_n are now given by Eqs. (4.5c) and (4.6c).

V. Comparison with Numerical Results

In this section we would like to indicate the accuracy that can be expected with our linear equation results. In report I, we found the equilibrium orbit to an accuracy of 10 to 20%. Since this result for the equilibrium orbit was used in this paper, we may expect the same sort of accuracy in the linear equations.

To compare the theoretical results with the numerical results, we will calculate the tune of two radial sector machines.

Machine I is defined by $N = 12$, $K = 8$, $h_0 = 1$ $h_1 = h_{-1} = 3.1$ and $1/\nu = 0$.

Machine II is defined by $N = 64$, $K = 159$, $h_0 = 0$ $h_1 = h_{-1} = 1/2$. and $1/\nu = 0$.

Comparison of theory and numerical results is given in Table I.

TABLE I

Machine		Theory	Numerical Results
I	ν_x/N	.306	.357
	ν_z/N	.118	122
II	ν_x/N	.302	.3205
	ν_z/N	.085	.0715

Eqs. (3.22) and (3.23) were used in calculating the tune. The α_x and α_z factors in this equation give a correction of about 5% to the smooth approximation results.

The numerical results for Machine II are due to T. Ohkawa

Appendix A

In this appendix we will develop the expansions of the force terms F_r and F_z in the equations of motion. Let us consider F_r first. F_r depends on the magnetic field according to

$$F_r = \frac{e}{pc}\left(r H_z - z' H_\theta\right), \tag{A. 1}$$

Thus to obtain the expansion for F_r, we need the expansion for H_z, and H_θ in powers of ρ and z.

The expansion of the magnetic field around the median plane, in powers of z, has been given by Akeley[3]. The expansion of H_z in powers of z is given by

$$H_z(x,y,z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots, \tag{A. 2}$$

where

$$C_0 = H_z, \tag{A. 3a}$$

$$C_1 = -\nabla_t \cdot \vec{H}_t \tag{A. 3b}$$

$$C_2 = -\frac{1}{2}\nabla_t^2 H_z, \tag{A.3c}$$

$$C_3 = \frac{1}{6}\nabla_t^2\left(\nabla_t \vec{H}_t\right). \tag{A. 3d}$$

In Eqs. (A. 3), the coefficients C_n are all evaluated at $z = 0$ and the C_n depend on x and y, or on r and θ. $\vec{H}_t$ is the component of the magnetic field perpendicular to the median plane, the $z = 0$ plane, ∇_t is the operator $\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y$. Since $\vec{H}_t = 0$ for $z = 0$ in our case, then $C_1 = C_3 = 0$.

The expansion of $\vec{H}_t(x, y, z)$ in powers of z is given by

$$\vec{H}_t(x,y,z) = \vec{a}_0 + \vec{a}_1 z + \vec{a}_2 z^2 + \vec{a}_3 z^3 + \ldots \tag{A. 4}$$

where

$$\vec{a}_0 = \vec{H}_t, \tag{A. 5a}$$

$$\vec{a}_1 = \nabla_t H_z, \tag{A. 5b}$$

$$\vec{a}_2 = -\frac{1}{2} \nabla_t^2 \vec{H}_t, \tag{A. 5c}$$

$$\vec{a}_3 = -\frac{1}{6} \nabla_t^2 (\nabla H_z). \tag{A. 5d}$$

In Eqs. (A. 5), the coefficients $\vec{a}_n$ are to be evaluated in the median plane at $z = 0$. Since $\vec{H}_t = 0$ for $z = 0$, $\vec{a}_0 = \vec{a}_2 = 0$. H_r and H_θ can be found from $\vec{H}_t$ by taking the radial and azimuthal components of $\vec{H}_t$. H_r and H_θ are given by.

$$H_r = a_{1r} z + a_{3r} z^3 + \ldots , \tag{A. 6}$$

$$H_\theta = a_{1\theta} z + a_{3\theta} z^3 + \ldots , \tag{A. 7}$$

where

$$a_{1r} = H_{z,r} \quad , \quad a_{1\theta} = (1/r) H_{z,\theta} , \tag{A. 8}$$

$$a_{3r} = -\frac{1}{6} \frac{\partial}{\partial r} \left(\nabla_t^2 H_z \right) , \tag{A. 9}$$

$$a_{3\theta} = -\frac{1}{6} (1/r) \frac{\partial}{\partial \theta} \left(\nabla_t^2 H_z \right) . \tag{A. 10}$$

So far we have expanded the magnetic field in powers of z . We must also expand around the equilibrium orbit $r = r_s(\theta)$. We get then for H_z, keeping only up to quadratic terms,

$$H_z = \left(c_0 + c_{0,r} \rho + \frac{1}{2} c_{0,rr} \rho^2 \right) + c_2 z^2 + \ldots \tag{A. 11}$$

$$= H_z + H_{z,r} \rho + \frac{1}{2} H_{z,rr} \rho^2 - \frac{1}{2} \nabla_t^2 H_z z^2 + \ldots$$

where the coefficients in the power series are to be evaluated on the orbit where $r = r_s(\theta)$, $z = 0$.

Similarly we find for H_r.

$$H_r = (a_{1r} + a_{1r,r}\,\rho)\,z + \cdots \tag{A. 12}$$

$$= H_{z,r}\,z + H_{z,rr}\,\rho z + \cdots$$

We find for H_θ.

$$H_\theta = (a_{1\theta} + a_{1\theta,r}\,\rho)\,z + \cdots \tag{A. 13}$$

$$= \frac{1}{r} H_{z,\theta}\,z + \frac{\partial}{\partial r}\left(\frac{1}{r} H_{z,\theta}\right)\rho z + \cdots$$

$$= \frac{1}{r} H_{z,\theta}\,z + \left(\frac{1}{r} H_{z,\theta r} - \frac{1}{r^2} H_{z,\theta}\right)\rho z + \cdots$$

We can now find the expansion for F_r, up to quadratic terms, by putting the expansions for H_z, H_r, H_θ into Eq. (A. 1). Thus we find for F,

$$F_r = \frac{e}{\beta c}\Big\{(r+\rho)\left(H_z + H_{z,r}\rho + \frac{1}{2} H_{z,rr}\rho^2 - \frac{1}{2}\nabla^2 H_t\, z^2\right)$$
$$- z'\left[\frac{1}{r} H_{z,\theta}\,z + \left(\frac{1}{r} H_{z,\theta r} - \frac{1}{r^2} H_{z,\theta}\right)\rho z\right] + \cdots\Big\}$$

$$F_r = \frac{e}{\beta c}\Big\{ r H_z + (H_z + H_{z,r})\,\rho \tag{A. 14}$$
$$+ \left(\frac{r}{2} H_{z,rr} + H_{z,r}\right)\rho^2 - \frac{r}{2}\nabla_t^2 H_z\, z^2$$
$$- \frac{1}{r} H_{z,\theta}\, z z' + \cdots \Big\}.$$

In the same way we can find $F_z = (e/pc)(r H_\theta - r H_r)$,

$$F_z = \frac{e}{pc}\Big\{(r'+\rho')\Big[\frac{1}{r} H_{z,\theta}\, z + \Big(\frac{1}{r} H_{z,\theta r} - \frac{1}{r^2} H_{z,\theta}\Big)\rho z\Big] - (r+\rho)\,(H_{z,r}\, z + H_{z,rr}\,\rho z) + \dots\Big\} \tag{A.15}$$

$$F_z = \frac{e}{pc}\Big\{\Big(\frac{r'}{r} H_{z,\theta} - r H_{z,r}\Big) z + \Big[\frac{r'}{r}\Big(H_{z,\theta r} - \frac{1}{r} H_{z,\theta}\Big) - r H_{z,rr} - H_{z,r}\Big]\rho z + \frac{1}{r} H_{z,\theta}\,\rho' z + \dots\Big\}$$

One should keep in mind that the coefficients in the above expansions for F_r and F_z in ρ and z should be evaluated on the equilibrium orbit. The above expansions are correct up to quadratic terms and will be used in discussing the quadratic non-linear resonances.

Appendix B

In this section we will discuss the neglect of the linear terms in ρ', ρ and z' in linear Eqs. (3.9) and (3.15) which arise from the expanding $L_0 = \{ r^2 + r'^2 + z'^2 \}^{\frac{1}{2}}$.

Let us consider the r-equation first. If we keep the ρ and ρ' terms in Eq. (3.9) then the equation becomes

$$u'' + (E_0 - g(\theta))u - g^{(1)}(\theta)u - g^{(2)}(\theta)u' = 0, \tag{B.1}$$

where

$$g^{(1)}(\theta) = \frac{1}{R} r_s'', \tag{B.2}$$

$$g^{(2)}(\theta) = \frac{1}{R}(r_s' + 3 r_s' r_s''). \tag{B.3}$$

$g^{(1)}(\theta)$ and $g^{(2)}(\theta)$ have the fourier coefficients

$$\begin{cases} g_0^{(1)}/N^2 = 0, \\ g_n^{(1)}/N^2 = -b\, h_n/N^2, \end{cases} \tag{B.4}$$

$$g_0^{(2)}/N^2 = 0,$$

$$g_n^{(2)}/N^2 = \frac{i}{N^3}\left\{ -3b^2 \sum_{\substack{m \neq 0, \\ m \neq n}} \frac{h_m h_{n-m}}{m} + \frac{b\, h_n}{n} \right\}. \tag{B.5}$$

It may be shown that all the results we found in the report for the solutions and tune of the linear r-equation will now hold if we make the substitution

$$g_n \longrightarrow g_n + g_n^{(1)} + i\,\nu_x\, g_n^{(2)} \,. \tag{B. 6}$$

This can be seen by repeating the derivation given in report MURA-200, including the new terms.

For small enough K or N, the above substitution may make a considerable difference in the results, but usually these terms contribute about 10% at most.

Similarly, if we keep the z' term in the z -equation Eq. (3.15), the z -equation becomes

$$y'' + (E_0' - f(\theta))\, y - f^{(1)}(\theta)\, y' = 0 \,, \tag{B. 7}$$

where

$$f^{(1)}(\theta) = \frac{1}{R}\left(r_s' + r_s' r_s''\right). \tag{B. 8}$$

$f^{(1)}(\theta)$ has the fourier coefficients

$$\begin{cases} f_0^{(1)}/N^2 = 0 \,, \\ f_n^{(1)}/N^2 = \dfrac{i}{N^3}\left\{ -b^2 \displaystyle\sum_{\substack{m\neq 0\\ m\neq n}} \frac{h_m h_{n-m}}{m} + \frac{b\, h_n}{n} \right\}. \end{cases} \tag{B. 9}$$

Again all our previous results for the solutions, and tune of the z -equation hold if we make the substitution.

$$f_n \longrightarrow f_n + i\,\nu_z\, f_n^{(1)} . \tag{B.10}$$

References

1 F. T. Cole - MURA/FTC- 3

2 G. Parzen - MURA-200

3 E. Akeley - MURA/ESA-1