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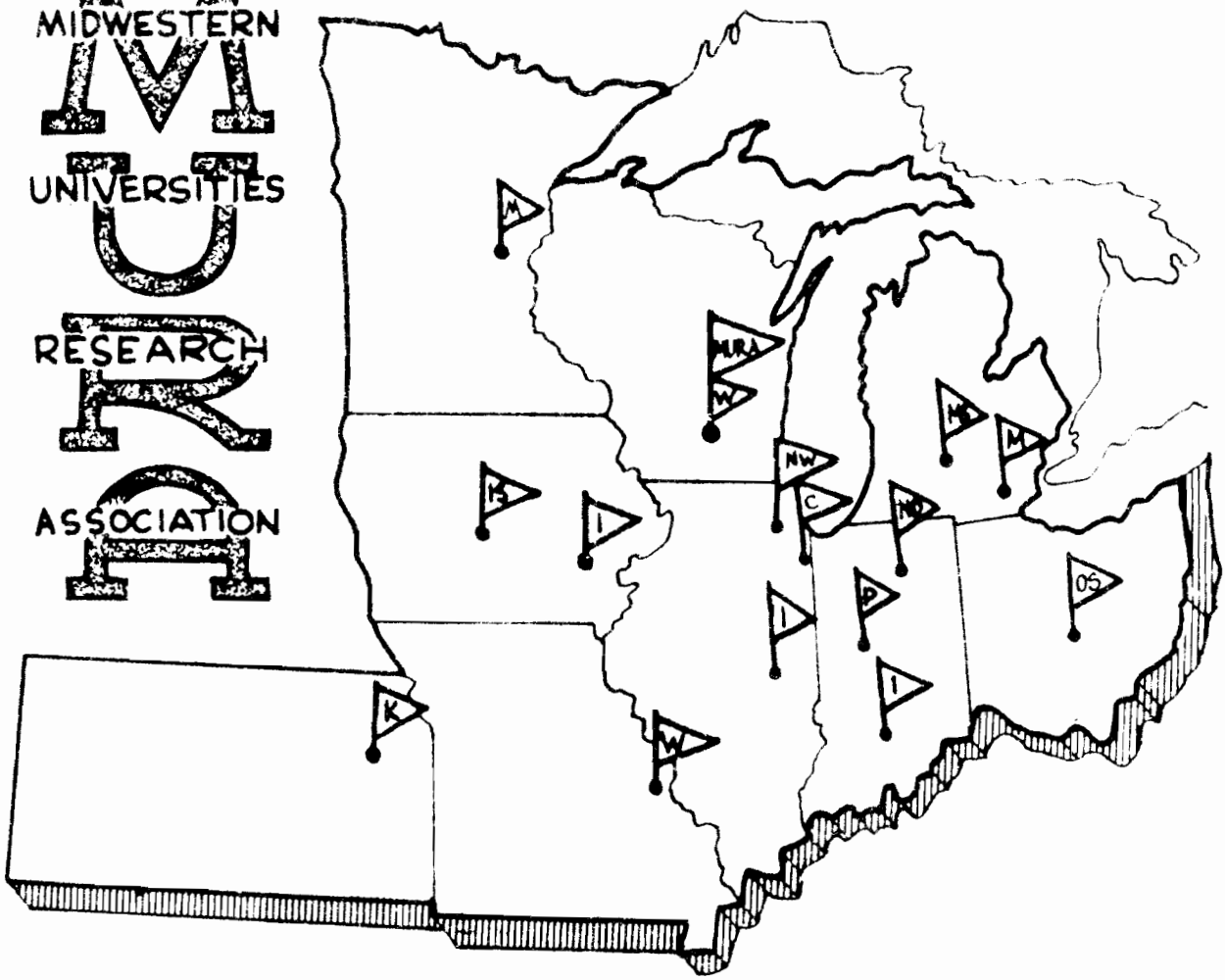

CONCERNING COUPLING RESONANCES IN THE SPIRALLY-RIDGED FFAG ACCELERATOR

L. Jackson Laslett and A. M. Sessler

REPORT **NUMBER** 263

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CONCERNING COUPLING RESONANCES IN THE SPIRALLY-RIDGED FFAG ACCELERATOR

L. Jackson Laslett* and A. M. Sessler**

May 6, 1957

ABSTRACT Theoretical and computational results are presented to illustrate the behavior of particle motion in spirally-ridged FFAG accelerators when operated in the neighborhood of the $\sigma_x - 2\sigma_y$, $\sigma_x - \sigma_y$, $\sigma_x + 2\sigma_y = 2\pi$ and $2\sigma_x + 2\sigma_y = 2\pi$ resonances. Save for the second of these resonances the computational examples pertain to structures with parameters characteristic of models (N=5).

The exponential growth of y-amplitude which may be observed to occur over many orders of magnitude when operating near these resonances is generally concordant, semi-quantitatively, with the results of a theoretical treatment in which the equation for axial motion is treated as linear and the radial motion independently prescribed

+ Supported by Contract AEC No. AT(11-1)-384.

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INTRODUCTION

A. General

The present report is concerned with certain phenomena pertaining to particle motion with two degrees of freedom in a spirally-ridged FFAG accelerator, with the computational results briefly obtained from the "FECKLESS FIVE" ILLIAC program[1] during the spring and summer of 1956.* Briefly, the behavior to which we direct our attention is an exponential growth of the amplitude of axial oscillations, from very small initial amplitudes, when the structure is such that the oscillation frequencies lie in the neighborhood of certain "coupling resonances". This "y-growth" appears to be the more rapid the greater the amplitude of the radial motion, above a certain threshold, and more pronounced when the operating point is near the resonance in question.

When the controlling resonance is that for which $\sigma_x \cong 2\sigma_y$, for example, the exponential growth may be found ultimately to terminate, at relatively large y-amplitudes, if the amplitude of the radial oscillation is not too great. Despite the possible termination, or "turn-over", of the exponential growth in certain cases, however, the growth suggests a certain ergodic aspect of the motion and deserves serious recognition by the machine designer due to the possibility that this growth may lead to ultimate instability through the mechanism of other inherent or imperfection resonances.

* The greater part of the work reported here was performed while the authors were associated with the MURA group at the University of Illinois, Urbana, Illinois, with joint assistance from the National Science Foundation and Office of Naval Research. Use of the ILLIAC was made possible through the cooperation of the Graduate College of the University of Illinois.

B. Purpose

It is the purpose of this report to present computational results pertaining to these inherent (sector) coupling resonances, with emphasis on the amplitude range where whatever growth is present does not terminate, and to describe the results of theoretical analysis intended to account for such results The experimental study reported here was substantially confined to structures for which the flutter factor f, measuring the amount of sinusoidal spatial modulation of the median-plane field, was 1/4. Through historical accident the computational work evolved so as to concentrate on model-sized structures with N=5, although it might have been preferable to have employed the "cleaner" equations characteristic of motion in large-scale structures. *

C. Methods

Analysis of the results was aided by computing, once per sector, the quantity K_y, which is the square root of a quadratic form which remains invariant for linear uncoupled motion.[2] This quantity was taken to be

$$K_y \equiv \left(\xi y^2 + \eta y p_y + \zeta p_y^2\right)^{1/2}$$

where, in terms of the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which carries a particle through successive sectors from one homologous point to the next,[2]

$$\xi = \frac{-C}{\sin^2\sigma} B, \quad \eta = \frac{A-D}{\sin^2\sigma} B, \quad \zeta = \frac{B^2}{\sin^2\sigma}.$$

The coefficients ξ, η, ζ as well as the oscillation frequencies, were determined by preliminary short small-amplitude runs Physically, for linear uncoupled motion, K_y represents the maximum value which y can attain at

* As Parzen has emphasised,[15] the use of low values of f (such as 1/4) is also helpful in suppressing "non-scaling" terms

those homologous points for which the invariant K_y applies and so represents the amplitude of the motion at such points.

In the theoretical work attention is directed to appropriate coupling terms in the differential equation[3] for y which are linear in the dependent variable y but involve the radial coordinate u measured with respect to the stable equilibrium orbit. Suitable solutions of an approximate (usually linear) differential equation for u, obtained on the supposition that $y \equiv 0$, are introduced into the coupling terms of the y equation to obtain a linear differential equation for y with coefficients involving both the period of the structure and that of the radial oscillations. This introduction in a non-Hamiltonian way of what is taken in effect to be a prescribed u motion was originally suggested by Walkinshaw[4] and appears to be entirely defensible when the y-amplitudes are as small as those obtaining in the greater part of the present work. Since coupling terms are actually also present in the differential equation for u, it must be acknowledged that the development of a large amplitude y oscillation will "react back" on the u motion, in special cases where operation is substantially exactly on the $\sigma_x = 2\sigma_y$ resonance and y-growth accordingly can be achieved with small-amplitude radial oscillations, the growth of K_y has been seen to be ultimately accompanied by a noticeable decrease of the corresponding K_x. This last-named complication, however, is ignored in the remainder of this report and the results are taken to apply to y growth in its initial stages where the amplitude can be small.

Subsequently, considerable attention has also been given to the question of stability limits by Parzen[16], employing perturbation methods similar to those applied in solid-state physics, and by Symon, using the mathematical methods developed by Moser[17] and applied by him to problems of planetary dynamics.

The study of algebraic transformations[18,19] which also can exhibit the phenomena treated in this report and which permit following the particle for extended periods has also received computational attention recently and Dr. Symon has had some success in applying Moser's methods[17] to account for their behavior.

In Walkinshaw's analysis[4] of a linear y-equation similar to that considered in the present report, containing the prescribed u-solution as well as coefficients with the period of the magnet structure, a type of "smooth approximation" is made to eliminate the terms of comparatively high frequency which involve the structure and to obtain thereby a Mathieu equation with coefficients having the period of the radial oscillations. We, however, suggest an alternative method of analysis, imagining the two periods to be commensurate in some very large interval and examining questions of stability by methods generally applicable to a Hill equation. The basic mathematical results which are required for this latter approach are treated in Appendices and have been subjected to some separate computational checks which may form the subject of a later report. Within the main body of the present report separate coupling resonances are treated specifically in turn, with accompanying emperical data from the computational studies.[5]

D. The Equations Employed:

The differential equations which are employed are taken from Cole's report[3] cited above. The radial displacement (in units of a convenient reference radius) is written

$$x = x_f + u,$$

where x_f represents the forced motion resulting in the (periodic) equilibrium orbit. The free radial oscillation satisfies, in linear terms, the following differential

equation (if terms in $\cos 2N\theta$, etc. are neglected):

$$u'' + \left[a_x + b_x \cos N\theta\right] u = 0$$

or, through quadratic terms in u,

$$u'' + \left[a_x + b_x \cos N\theta\right] u = (1/2) b_1 u^2,$$

where $$a_x = k + 1 - \frac{f^2}{2w^2\left[N^2-(k+1)\right]} \cong k+1-\frac{f^2}{2w^2N^2}$$

$$b_x = f/w$$

and the dominant term in b_1 is considered to be

$$b_1 \doteq -(f/w^2) \sin N\theta.^*$$

Similarly, the y equation governing the axial motion is taken to be

$$y'' + \left[a_y + b_y \cos N\theta\right] y = b_5 u y + (c_{10}/2) u^2 y.$$

where

$$a_y = -k + \frac{f^2}{2w^2\left[N^2-(k+1)\right]} \cong -k + \frac{f^2}{2w^2N^2}$$

$$b_y = -f/w$$

and the dominant terms of the coupling coefficients are

$$b_5 \doteq (f/w^2) \sin N\theta$$

$$c_{10} \doteq -(f/w^3) \cos N\theta.$$

E. The Scaling Properties of the Solutions:

The foregoing simplified equations, taken to represent the trajectories of particles in a spirally-ridged FFAG accelerator with a pure sinusoidal variation of the median-plane field and containing coefficients which indeed appear to be the major ones when typical parameters for low values of f are chosen, may be shown to require a simple scaling feature in any of the results which follow

* It will be noted that the sign of b_1 is deliberately chosen to differ from that in Cole's report,[3] in order to correct what we believe to have been an inadvertent error [cf, for example, the report by Judd, MURA/DLJ-2 (1955)].

therefrom.[5] The equations employed are

$$u'' + \left[a_x + (f/w)\cos N\theta\right] u = \frac{f}{2w^2}(\sin N\theta)\, u^2 \qquad \text{and}$$

$$y'' + \left[a_y - (f/w)\cos N\theta\right] y = \frac{f}{w^2}(\sin N\theta)\, uy - \frac{f}{2w^3}(\cos N\theta)\, u^2 y;$$

if the dependent variables are measured in units of w and the independent variable is transformed by $N\theta = 2t$, there results

$$\frac{d^2(u/w)}{dt^2} + 4\left[\frac{a_x}{N^2} + \frac{f}{wN^2}\cos 2t\right](u/w) = 2\frac{f}{wN^2}(\sin 2t)(u/w)^2$$

and

$$\frac{d^2(y/w)}{dt^2} + 4\left[\frac{a_y}{N^2} - \frac{f}{wN^2}\cos 2t\right](y/w) = 4\frac{f}{wN^2}(\sin 2t)(u/w)(y/w) - 2\frac{f}{wN^2}(\cos 2t)(u/w)^2(y/w).$$

The σ_{x_0} and σ_{y_0} which characterize small-amplitude, uncoupled oscillations, are determined by the quantities a_x/N^2, a_y/N^2 and $f/(wN^2)$. With respect to larger amplitude oscillations, with coupling, it is moreover now apparent that solutions (u and y) may correctly be scaled together in direct proportion to w if the parameters are changed so as to keep σ_{x_0}, σ_{y_0}, and $f/(wN^2)$ constant. Thus, to apply this result to the features of the motion which are of particular interest in the present report, the threshold u-amplitude for engendering y-growth must be given by w times a function of σ_{x_0}, σ_{y_0}, and $f/(wN^2)$; i.e., by w times a function of ν_{x_0}/N, ν_{y_0}/N, and $f/(wN^2)$. Likewise the characteristic exponent characterizing the exponential growth of unstable solutions to these equations <u>per period of the structure</u> must be expressible as a function of these three quantities and of $|A|/w$, A representing the amplitude of the radial oscillation. The results obtained in this report are consistent with this scaling property.

The theoretical equations which will be obtained may frequently be expressed in terms of the oscillation frequencies (oscillations per circumference) for convenience only. It is emphasised, however, that the phenomena with which

we are concerned are strictly sector resonances prevailing in machines which may be free of imperfections, and could appropriately be expressed in terms of σ_x and σ_{y_e}.

II. THE $\sigma_x = 2\sigma_y$ RESONANCE

A. Theory:

For the analysis of axial motion as affected by the relatively strong coupling resonance which prevails when σ_x lies in the neighborhood of $2\sigma_{y_0}$, it is sufficient to characterize the radial motion by a linear equation in u and to represent the coupling by inclusion of the term $b_5 uy$ in the axial equation. The equations considered then are

$$u'' + \left[a_x + (f/w) \cos N\theta \right] u = 0$$

$$y'' + \left[a_y - (f/w) \cos N\theta - (f/w^2) (\sin N\theta) u \right] y = 0.$$

The solution to the u-equation is then given by the results of Appendix II, if we identify the coefficient b of the Appendix with f/w and drop the phase-shift for convenience, as

$$u = A \left[\sin \nu_x \theta + \frac{f}{wN^2} \sin \nu_x \theta \cos N\theta - 2 \frac{f \nu_x}{wN^3} \cos \nu_x \theta \sin N\theta \right].$$

Substitution of this expression for u into the y-equation and neglect of terms in 2Nθ then leads to

$$y'' + \left[a_y - (f/w) \cos N\theta - \frac{Af}{w^2} \sin \nu_x \theta \sin N\theta + \frac{Af^2 \nu_x}{w^3 N^3} \cos \nu_x \theta \right] y = 0.$$

This equation is of the form of that considered in the first part of Appendix III (and of Appendix V) with the lower sign, viz.:

$$y'' + \left[a + b \cos N\theta + c \sin \nu_x \theta \sin N\theta + d \cos \nu_x \theta \right] y = 0,$$

for which the stability boundaries are given by

$$\left| \nu_x^2 - (2\nu_{y_0})^2 \right| = 2 \left| \frac{\nu_x bc}{N^3} + d \right|.$$

With the identification

$$b = -\frac{f}{w}, \quad c = -\frac{Af}{w^2}, \quad d = \frac{Af^2 \nu_x}{w^3 N^3},$$

the stability boundaries accordingly are given by

$$\left| \nu_x^2 - (2\nu_{y_0})^2 \right| = 4 \frac{f^2 \nu_x}{w^3 N^3} |A|$$

and the "threshold" amplitude for radial motion, above which y-growth may occur, correspondingly by

$$|A|_{thr.} = \frac{w^3N^3}{4f^2\nu_x}\left|\nu_x^2 - (2\nu_{y_0})^2\right|$$

$$= \frac{1}{16}w\left(\frac{wN^2}{f}\right)^2\frac{N}{\nu_x}\left|\left(\frac{\sigma_x}{\pi}\right)^2 - \left(2\frac{\sigma_{y_0}}{\pi}\right)^2\right|$$

An estimate for the lapse-rate characterizing exponential growth in the unstable region is likewise obtainable directly from the result of Appendix IV:

$$\mu^2 = -\frac{(a-a_1)(a_2-a)\langle c^2\rangle\langle s^2\rangle}{4\langle sc'\rangle\langle cs'\rangle}$$

Since differences of "a" are identical to differences of $\nu_{y_0}^2$ and, for the present purpose, the functions c and s may be taken (Appendix III) as proportional to the cosine and sine of $\nu_x\theta/2$,

$$\mu = \left\{\left[\nu_{y_0}^2 - (\nu_{y_0})_1^2\right]\left[(\nu_{y_0})_2^2 - \nu_{y_0}^2\right]/\nu_x^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left[\frac{f^2\nu_x}{w^3N^3}A + \left(\nu_{y_0}^2 - \frac{\nu_x^2}{4}\right)\right]\left[\frac{f^2\nu_x}{w^3N^3}A - \left(\nu_{y_0}^2 - \frac{\nu_x^2}{4}\right)\right]/\nu_x^2\right\}^{\frac{1}{2}}$$

$$= \frac{f^2}{w^3N^3}\sqrt{A^2 - A_{thr}^2} \qquad \text{nepers/radian}$$

$$= \frac{2\pi f^2}{w^3N^4}\sqrt{A^2 - A_{thr}^2} \qquad \text{nepers/sector}$$

$$= 2.73\left(\frac{f}{wN^2}\right)^2\frac{\sqrt{A^2 - A_{thr}^2}}{w} \qquad \text{decades/sector.}$$

In particular the maximum lapse-rate, for a given amplitude A, is given by

$$\mu_{max} = 2.73\left(\frac{f}{wN^2}\right)^2\frac{A}{w} \quad \text{decades/sector.}$$

B. Computational Results

A large number of ILLIAC runs were made, in the manner described in the Introduction, to examine y-growth quantitatively in the neighborhood of $\sigma_y = 2\sigma_{y_0}$ for spirally-ridged machines with parameters characteristic of models. In all the cases, $f = 1/4$, $N = 5$, the initial conditions were characterized by $u' = 0$, $y = 10^{-6}$, $y' = 0$, and the duration of a typical run was 80 sectors. Results were depicted on a semi-logarithmic plot of K_y vs. the number of sectors traversed by the particle and, although high-frequency wiggles were present in such plots, the lapse-rate could readily be determined.

The results of this series of runs are summarized in Table I. In the initial evaluation of the data, a linear plot of lapse-rate against u was made for each structure and extrapolated to zero to find the threshold. From the theoretical analysis it appears that it would be preferable to plot the squares of these quantities against each other, but in practice this distinction was of no importance for most of the cases studied. Figure 1 presents the graphically-estimated thresholds, plotted against $\nu_{y_0} - 2\nu_{y_0}$.

For comparison with theory it might be supposed that one should identify A not with $|u_0|$ itself, but with $|u_0| \Big/ \left[1 + \frac{f}{wN^2}\right]$ in accordance with the solution for u taken from Appendix II (when $\cos\epsilon$ is set equal to zero to assure that u_0' will vanish). In the present work the factor $1 + \frac{f}{wN^2}$ is not greatly different from unity -- being, for example, 1.2082 for Point ϵ -- and its inclusion does not improve the agreement between the theory and the computational results. Without dwelling further on this possible distinction, we give below in Table II the values of $|u|_{thr.}$ which were plotted on Fig. 1 for points C, ϵ, D, θ, J, and K and the theoretical value of $|A|_{thr}$ for each of these structures.

TABLE I.

CHARACTER OF AXIAL MOTION NEAR $\sigma_{x_0} = 2\sigma_{y_0}$

$f = 1/4$ $N = 5$

Point	PARAMETERS k	$1/\omega$	σ_{x_0}/π	σ_{y_0}/π	ν_{x_0}	ν_{y_0}	$\nu_{x_0} - 2\nu_{y_0}$	GROWTH CHARACTERISTICS Init. u Ampl.	Decades/Sector	Length of Run
γ	.6278	21.61	.5380	3147	1.345	0.787	-.229	-.0375	.029	$\nearrow$80 sectors
								-.0300	.021	$\nearrow$80
								-.0225	const.	$\nearrow$80
ϵ	.6486	20.82	.5388	.2855	1.347	0.714	-.081	-.0375	.101	U.S. after 55
								-.0300	.080	$\nearrow$80
								-.0225	.055	$\nearrow$80
								-.0150	.029	$\nearrow$80
								-.0075	const.	$\nearrow$80
ζ	.6522	20.39	.5392	.2689	1.348	0.672	+.004	-.0375	.115	U.S. after 56
								-.0300	.09	$\nearrow$80
								-.0225	.06	$\nearrow$150
								-.0150	.039	$\nearrow$150
								-.0075	.02	$\nearrow$350
								-.0050	.01	$\nearrow$350
								-.0010	.0023	$\nearrow$350
								-.0001	const.	$\nearrow$350
θ	.6580	19.60	.5400	2365	1.350	0.591	+.168	-.0375	.115	U.S. after 49
								-.0300	.090	" " 64
								-.0250	.067	" " 79
								-.0225	.055	$\nearrow$80
								-.0200	.043	$\nearrow$200
								-.0175	.029	$\nearrow$200
								-.0150	Ampl. $< 9\times10^{-6}$	$\nearrow$200
								-.0125	" $< 3\times10^{-6}$	$\nearrow$200
								-.0075	$< 1.6\times10^{-6}$	$\nearrow$200

λ									
	)4	22.93	5559	.3328	1.390	)32	-.0375	.12	U.. a. er 40
							- 0300	.022	≥ 80
					- 274		-.0225	const.	≥ 80
A	.766	23.8	5676	3456	1.419	0.864	-.0375	.26 - .14	U.S. after 28
					-.309		-.0300	.02	≥ 80
							-.0225	const.	≥ 80
B	.776	23.4	.5692	.3306	1.423	0.827	- 0375	.125	U.S. after 36
					-.230		-.0300	.05	≥ 80
							-.0225	Ampl. levels off	
C	.790	22.87	.5696	.3095	1.424	0.774	-.0375	.135	U.S. after 50
							-.0300	103	" " 46
					-.124		-.0225	.070	≥ 80
							-.0150	.036 - .031	≥ 80
							-.0075	const.	≥ 80
D	.802	22.4	.5695	.2902	1.424	0.726	-.0375	.16	U.S. after 29
							-.0300	.13	" " 47
					-.027		-.0225	.08	≥ 80
							-.0150	.05	≥ 80
							-.0075	.02	≥ 80
							-.005	.01	≥ 80
J	.836	21.6	.5733	.2508	1.433	0.627	-.0375	.19	U.S. after 29
							-.0300	.14	" " 38
					+.179		-.0225	.09	≥ 80
							-.0150	.045	≥ 80
							-.0075	const.	≥ 80
K	.832	21.2	.5714	.2368	1.429	0.592	- 0375	.19	U.S. after 39
							-.0300	.14	" " 39
					+.245		- 0225	.08	" " 60
							-.0150	.01	≥ 80
							-.0075	const.	≥ 80
L	.842	20.8	.5677	.2173	1.419	0.543	- 0375	18	U.S. after 30
							-.0300	.13	" " 49
					+.333		- 0225	.06	≥ 80
							-.0150	Ampl. < 2.6×10^{-6}	≥ 80

TABLE II.

OBSERVED AND THEORETICAL THRESHOLD AMPLITUDES
FOR OPERATION NEAR $\sigma_{x_0} = 2\sigma_{y_0}$

Point	$\lvert u_0 \rvert_{thr.}$	$\lvert A \rvert_{thr.}$ (theoretical)
C	0.010	0.0108
ϵ	0.007_5	0.0093
D	0.003	0.0024
θ	0.010 - 0.013	0.0210
J	0.008	0.0167
K	0.013	0.0235

Point ζ is, of course, virtually right on the resonance and one would expect A_{thr} to be very small -- about 0.0005

With respect to the lapse-rate in the region of y-growth, we give in Table III two examples of a comparison between the observed value of μ and that estimated theoretically from the observed threshold. It will be noted that the lapse-rate for Point ζ should correspond to $\mu_{max.}$, since for Point ζ the resonant condition is almost exactly satisfied.

TABLE III.

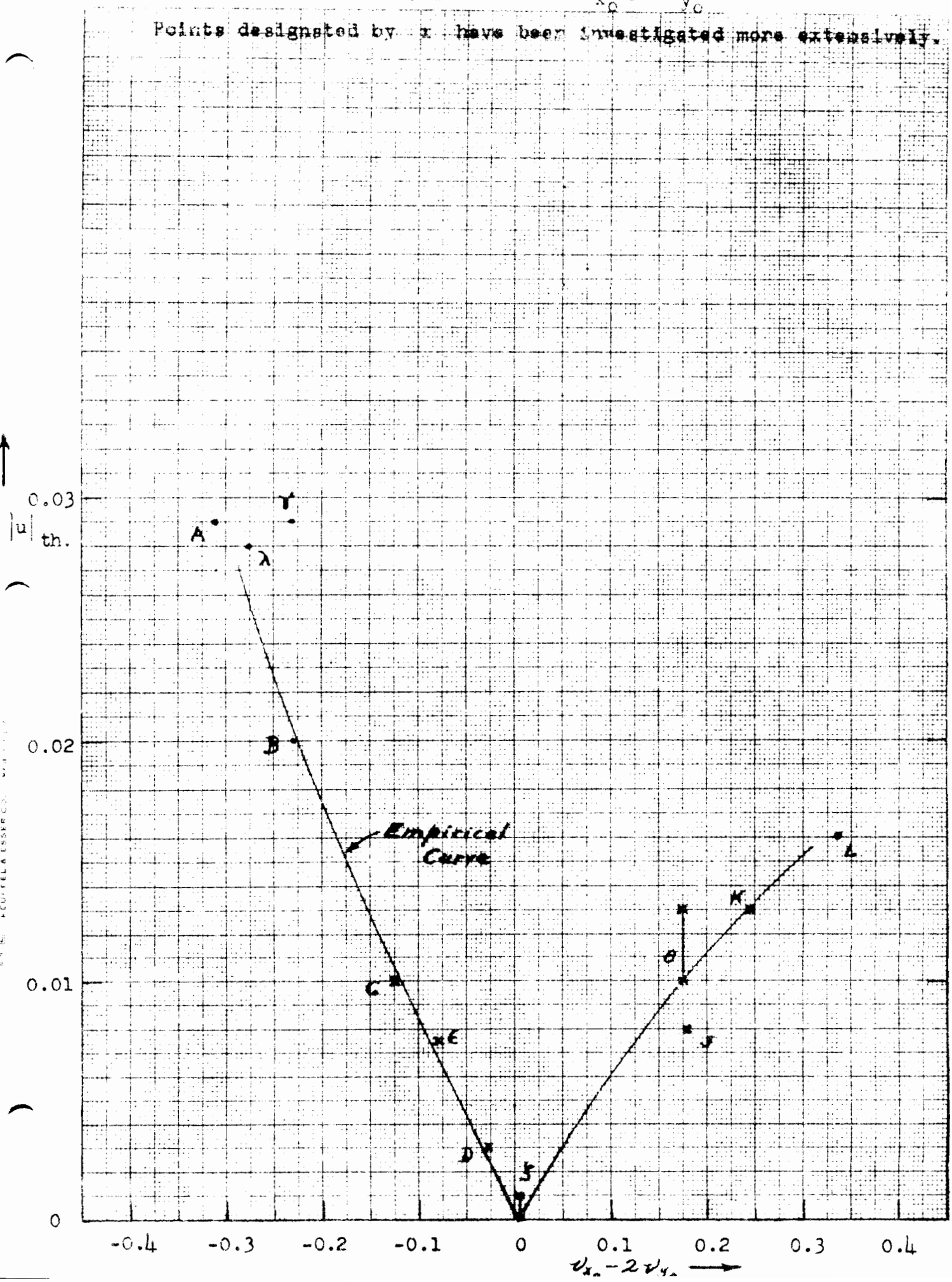
OBSERVED AND THEORETICAL LAPSE-RATES
FOR OPERATION NEAR $\sigma_{x_0} = 2\sigma_{y_0}$

Point	u_0	$\sqrt{\lvert u_0 \rvert^2 - \lvert u \rvert^2_{thr.,\,obs}}$	μ (decades/sector) observed	μ (decades/sector) theor. calc.
ϵ	-.0225	.0212	0.055	0.052
ζ	-.0150	.0150	0.039	0.035

The agreement may be felt to be reasonably satisfactory in these cases.

FIG. 1. Threshold for y-growth near $\nu_{x_0} = 2\nu_{y_0}$.

Points designated by x have been investigated more extensively.



III. THE CASE $\sigma_x = \sigma_y$

A. Expected Performance.

A narrow zone of instability would be expected to arise from a $\sigma_{x_0} = \sigma_{y_0}$ resonance, in analogy to the second zone of instability for Mathieu's equation. Since, however, the resonance is second order in its dependence on the u-amplitude A^*, a consistent analysis of the problem requires consideration of (i) possible contributions from the $u^2 y$ term in the y-equation and (ii) supplementary terms, proportional to A^2, which will enter in uy when a solution to the non-linear u-equation is attempted. These features complicate the analytic work considerably, but we undertake an approximate treatment, taking f/wN^2 and ν_x^2/N^2 to be small and employing for convenience at one point the "smooth-approximation" method illustrated in Appendix V. [Curiously, retention of the $u^2 y$ term appears to affect noticeably the intermediate steps of the analysis but not, in the present approximation, the final result.]

The equations with which we commence are, accordingly, those of Section II plus supplementary terms and take the form specified in the Introduction (Section I D):

$$u'' + \left[a_x + b_x \cos N\theta\right] u = (1/2) b_1 u^2$$

$$y'' + \left[a_y + b_y \cos N\theta\right] y = b_5 uy + (c_{10}/2) u^2 y ,$$

where the coefficients are taken to be

$$b_x = f/w \qquad b_y = -f/w$$

$$b_1 = -(f/w^2) \sin N\theta \qquad b_5 = (f/w^2) \sin N\theta$$

$$c_{10} = -(f/w^3) \cos N\theta .$$

* To emphasise the second order nature of this resonance, it is sometimes designated $2\sigma_x = 2\sigma_{y_0}$.

The solution of the u-equation is now taken to be of the form previously taken from Appendix II for use in analysing the $\sigma_{x_0} = 2\sigma_{y_0}$ resonance (Section II), but supplemented by additional terms, proportional to A^2, obtained therefrom by a perturbation procedure:

$$u = A\left[\sin \nu_x\theta + \frac{f}{WN^2}\sin \nu_x \cos N\theta - 2\frac{f\nu_x}{WN^3}\cos \nu_x\theta \sin N\theta\right]$$
$$+ \frac{fA^2}{4W^2N^2}\left[\sin N\theta - \sin N\theta \cos 2\nu_x\theta + 4\frac{\nu_x}{N}\cos N\theta \sin 2\nu_x\theta\right].$$

For the purpose at hand we also take, then,

$$u^2 \doteq A^2\left[\sin^2\nu_x\theta + 2\frac{f}{WN^2}\sin^2\nu_x\theta\cos N\theta - 4\frac{f\nu_x}{WN^3}\sin\nu_x\theta\cos\nu_x\theta\sin N\theta\right].$$

In forming the quantities b_5u and $(c_{10}/2)\,u^2$ we drop terms involving the sine or cosine of $2N\theta$ to obtain

$$b_5u \cong \frac{f}{W^2}\left[A\sin\nu_x\theta\sin N\theta - \frac{f\nu_xA}{WN^3}\cos\nu_x\theta + \frac{fA^2}{8W^2N^2} - \frac{fA^2}{8W^2N^2}\cos 2\nu_x\theta\right]$$

and

$$\left(\frac{c_{10}}{2}\right)u^2 \cong -\frac{1}{2}\frac{fA^2}{W^3}\left[\sin^2\nu_x\theta\cos N\theta + \frac{f}{WN^2}\sin^2\nu_x\theta\right]$$
$$= -\frac{1}{2}\frac{fA^2}{W^3}\left[\sin^2\nu_x\theta\cos N\theta + \frac{f}{2WN^2} - \frac{f}{2WN^2}\cos 2\nu_x\theta\right].$$

The differential equation for y,

$$y'' + \left[a_y + b_y\cos N\theta - b_5u - \left(\frac{c_{10}}{2}\right)u^2\right]y = 0$$

now becomes (setting $\nu_{y_0}^2 = a_y + f^2/(2W^2N^2)$):

$$y'' + \left\{\nu_{y_0}^2 - \frac{f^2}{2W^2N^2} + \frac{f^2A^2}{8W^4N^2} + \frac{f^2\nu_xA}{W^3N^3}\cos\nu_x\theta - \frac{f^2A^2}{8W^4N^2}\cos 2\nu_x\theta - \frac{f}{W}\left[\left(1 - \frac{1}{2}\left(\frac{A}{W}\right)^2\sin^2\nu_x\theta\right)\cos N\theta + \frac{A}{W}\sin\nu_x\theta\sin N\theta\right]\right\}y = 0.$$

This is conveniently re-written

$$y'' + \left\{\nu_{y_0}^2 - \frac{f^2}{2W^2N^2} + \frac{f^2A^2}{8W^4N^2} + \frac{f^2\nu_xA}{W^3N^3}\cos\nu_x\theta - \frac{f^2A^2}{8W^4N^2}\cos 2\nu_x\theta - \frac{f}{W}\cos\left(N\theta - \frac{A}{W}\sin\nu_x\theta\right)\right\}y = 0,$$

with the "smooth approximation equivalent" (cf. Appendix V)

$$y'' + \left\{ \nu_{y_0}^2 - \frac{f^2}{2W^2N^2} + \frac{f^2A^2}{8W^4N^2} + \frac{f^2\nu_x A}{W^3N^3}\cos\nu_x\theta - \frac{f^2A^2}{8W^4N^2}\cos 2\nu_x\theta + \frac{f^2}{2W^2N^2}\left(1 - \frac{A\nu_x}{WN}\cos\nu_x\theta\right)^{-2} \right\} y = 0$$

or

$$y'' + \left\{ \nu_{y_0}^2 + \frac{f^2A^2}{8W^4N^2} + 2\frac{f^2\nu_x A}{W^3N^3}\cos\nu_x\theta - \frac{f^2A^2}{8W^4N^2}\cos 2\nu_x\theta \right\} y = 0$$

(recalling that ν_x^2 is neglected in comparison to N^2).

The stability boundaries near $\nu_x = \nu_{y_0}$ for this last equation may now be obtained by appeal to the results of Appendix IF, in which the equation

$$y'' + [\alpha + \beta\cos\omega\theta + \gamma\cos 2\omega\theta]\, y = 0$$

is considered

We set

$$\omega = \nu_x \qquad \beta = \frac{2f^2\nu_x A}{W^3N^3}$$

$$\alpha = \nu_{y_0}^2 + \frac{f^2A^2}{8W^4N^2} \qquad \gamma = -\frac{f^2A^2}{8W^4N^2}$$

and obtain

$$\omega^2 - \beta^2/(12\omega^2) + \gamma/2 \leqslant \alpha \leqslant \omega^2 + 5\beta^2/(12\omega^2) - \gamma/2$$

$$-\frac{3}{16}\frac{f^2A^2}{W^4N^2} \leqslant \nu_{y_0}^2 - \nu_x^2 \leqslant -\frac{1}{16}\frac{f^2A^2}{W^4N^2}$$

$$-\frac{3}{4}\left(\frac{f}{WN^2}\right)^2\left(\frac{A}{W}\right)^2 \leqslant \left(\frac{\sigma_{y_0}}{\pi}\right)^2 - \left(\frac{\sigma_x}{\pi}\right)^2 \leqslant -\frac{1}{4}\left(\frac{f}{WN^2}\right)^2\left(\frac{A}{W}\right)^2,$$

the terms which arise from β^2 being neglected since they involve an additional factor $\left(\frac{f}{WN^2}\right)^2$.

This approximate result for estimating the stability boundaries associated with the $\sigma_x \cong \sigma_{y_0}$ resonance suggests a relatively narrow zone of instability whose width is proportional to the <u>square</u> of the radial amplitude and which will be found

exclusively for values of σ_{y_0} below σ_x * Moreover there thus appear to be two "threshold" amplitudes (for specified ν_x, ν_{y_0}), an upper limit

$$|A_2| = 4W\frac{WN^2}{f}\frac{\sqrt{\nu_x^2-\nu_{y_0}^2}}{N} = 2W\frac{WN^2}{f}\sqrt{\left(\frac{\sigma_x}{\pi}\right)^2-\left(\frac{\sigma_{y_0}}{\pi}\right)^2}$$

and the more-pertinent lower limit

$$|A_1| = \left(4/\sqrt{3}\right)W\frac{WN^2}{f}\frac{\sqrt{\nu_x^2-\nu_{y_0}^2}}{N} = \left(2/\sqrt{3}\right)W\frac{WN^2}{f}\sqrt{\left(\frac{\sigma_x}{\pi}\right)^2-\left(\frac{\sigma_{y_0}}{\pi}\right)^2}.$$

The lapse-rate which characterizes exponential growth in the unstable region may be estimated from the result of Appendix IV, noting that the functions c and s are now primarily represented by cosine and sine functions of $\nu_x\theta$ and is conveniently expressed in terms of the threshold amplitude A_1:

$$\mu^2 = -\frac{(a-a_1)(a_2-a)\langle c^2\rangle\langle s^2\rangle}{4\langle sc'\rangle\langle cs'\rangle},$$

$$\mu = \left\{\left[\frac{3}{16}\frac{f^2A^2}{\omega^4N^2}-(\nu_x^2-\nu_{y_0}^2)\right]\left[(\nu_x^2-\nu_{y_0}^2)-\frac{1}{16}\frac{f^2A^2}{\omega^4N^2}\right]\Big/(4\nu_x^2)\right\}^{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{32}\frac{f^2}{\omega^4N^2\nu_x}\sqrt{(A^2-A_1^2)(3A_1^2-A^2)} \qquad \text{nepers/radian}$$

$$= \frac{\sqrt{3}\,\pi}{16}\frac{f^2}{\omega^4N^3\nu_x}\sqrt{(A^2-A_1^2)(3A_1^2-A^2)} \qquad \text{nepers/sector}$$

$$= 0.1477\left(\frac{f}{\omega N^2}\right)^2\frac{N}{\nu_x}\frac{\sqrt{(A^2-A_1^2)(3A_1^2-A^2)}}{\omega^2} \qquad \text{decades/sector.}$$

The maximum lapse-rate, for a given amplitude A, is then estimated to be

$$\mu_{max} = 0.085\left(\frac{f}{\omega N^2}\right)^2\frac{N}{\nu_x}\left(\frac{A}{\omega}\right)^2 \quad \text{decades/sector.}$$

* If the term $(c_{10/2})\,u^2y$ and the perturbation correction to u in b_5uy had been omitted, the stability limits would have appeared (cf. Appendices III B & V B) at $\nu_{y_0}^2 \doteq \nu_x^2 + (1/8)f^2A^2/(\omega^4N^2)$ and $\nu_{y_0}^2 \doteq \nu_x^2 + (3/8)f^2A^2/(\omega^4N^2)$ --i.e. the resonance would have been thought twice as broad.

B. Computational Results:

We have not obtained an example of the $\sigma_x \cong \sigma_{y_0}$ resonance for structures with the small number of sectors (N = 5) employed in the other illustrative cases of this report, but shall present some evidence obtained for a large-N machine. In studying the $\sigma_x \cong \sigma_{y_0}$ resonance, it is desirable to employ low values of σ to avoid interferences from other resonances, but care must then be taken to avoid the $\sigma = 2\pi/4$ resonance, the limitation of stable amplitude attributable to $\sigma_x = 2\pi/5$, and interference from the higher-order resonance $3\sigma_x + 2\sigma_{y_0} = 2\pi$. y-growth attributable to the $\sigma_x \cong \sigma_{y_0}$ resonance was first found with the WELL-TEMPERED FIVE program[20] for a median-plane field expressed by $-h_{y_0} = 1 + 1.1708 \sin 2\pi\xi - .2991 \cos 2(2\pi\xi)$, with N = 40 and with k and 1/w respectively in the neigborhood of 48 and 325.5 ($\frac{\sigma_x}{\pi} \cong .38$, $\frac{\sigma_y}{\pi} \cong .36$). We present here, however, (Table IV) results subsequently obtained by aid of the TEMPERMESH[21] - FORMESH[22] programs for structures having a sinusoidal field variation with f = 1/4. The TEMPERMESH dimensions were a = 70, b = 9.

The results summarized in Table IV are depicted on the Altitude Chart (Fig. 6) attached to this report. The general form of the region of instability appears to be that expected.

TABLE IV.

CHARACTER OF AXIAL MOTION NEAR $\sigma_x = \sigma_{y_0}$

$f = 1/4 \quad 1/\omega = 2500 \quad N = 50$

TEMPER-MESH NO.	k	σ_{x_0}/π	σ_{y_0}/π	$\frac{\sigma_{y_0}}{\pi} - \frac{\sigma_{x_0}}{\pi}$	LAPSE-RATE (Decades/Sector) u_0					
					-.000560	-.000306	-.000262	-.000218	-.000174	-.000130
159	81.7	.3854	.3564	-.0290	.0138					
160	80.4	.3826	$.357_7$	-.0229	.0197	0				
161	79.1	$.379_8$	$.362_9$	-.0169	.0184	.0107	0			
162	77.8	$.376_9$	$.366_1$	$-.010_8$	.0143	.0125	.0063	0		
163	76.5	.3741	$.369_4$	$-.004_7$	0	.0094	.0075	.0048	0	
168	75.85	$.372_7$	$.371_0$	$-.001_7$	--	--	--	--	.0025	0
164	75.2	$.371_3$	$.372_6$	$+.001_3$	--	0	0	0	0	0

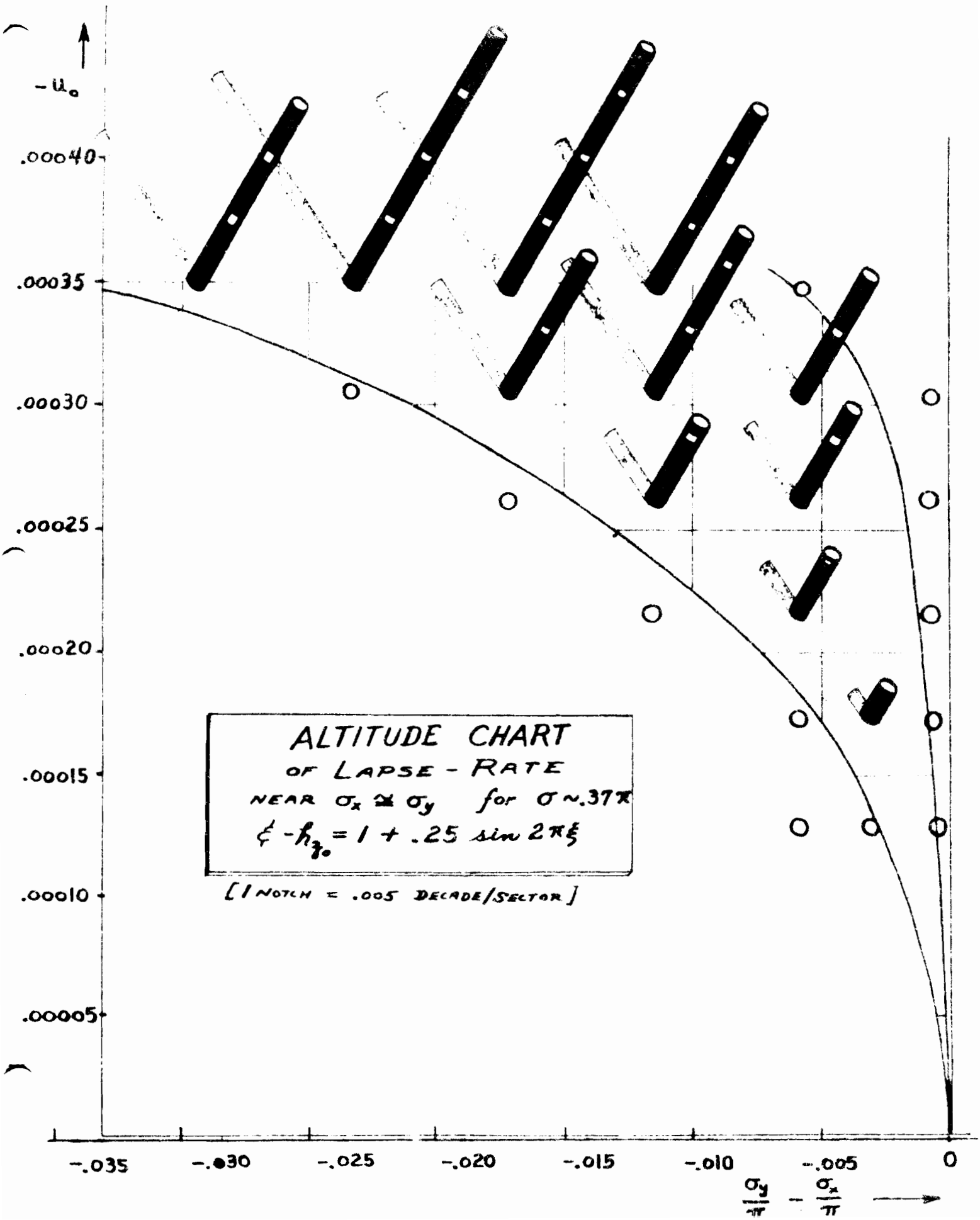
We list below in Table V the threshold amplitudes, $|u_1|_{thr}$, estimated from the data obtained with TEMPERMESH 161, 162, and 163 and, for comparison, the corresponding theoretical estimates, $|A_1|$.

TABLE V.

OBSERVED AND THEORETICAL THRESHOLD AMPLITUDES FOR OPERATION NEAR $\sigma_{x_0} = \sigma_{y_0}$

TEMPER-MESH NO.	$\frac{\sigma_{y_0}}{\pi} - \frac{\sigma_{x_0}}{\pi}$	$\lvert u_1 \rvert_{thr}$	$\lvert A_1 \rvert$ (theoretical)
161	-.0169	.00028	.00021
162	-.0108	.00023	.00017
163	-.0047	.00018	.00011

With regard to the lapse-rate, we consider the case described by TEMPERMESH No. 162 with u = -.000306. The lapse-rate calculated from the theoretical estimate of the preceeding sub section, using the observed value of A_1, is 0.0157_6 decades/sector and the observed lapse-rate for this case (Table IV) is 0.0125 decades/sector.


$-u_0$
.00040
.00035
.00030
.00025
.00020
.00015
.00010
.00005
ALTITUDE CHART
OF LAPSE - RATE
NEAR $\sigma_x \cong \sigma_y$ for $\sigma \sim .37\pi$
$\xi - h_{z_0} = 1 + .25 \sin 2\pi\xi$
[1 NOTCH = .005 DECADE/SECTOR]
-.035
-.030
-.025
-.020
-.015
-.010
-.005
0
$\frac{\sigma_y}{\pi} - \frac{\sigma_x}{\pi}$

IV. THE $\sigma_x + 2\sigma_y = 2\pi$ RESONANCE

A. Theory:

An analytic treatment of small-amplitude axial motion in the case that $\sigma_x + 2\sigma_y$ lies in the neighborhood of 2π may be based on the same differential equation as employed for discussion of the $\sigma_x = 2\sigma_y$ resonance in Section II. It is recalled that this equation was obtained by substitution of a solution to the u-equation into

$$y'' + \left[a_y - (f/\omega)\cos N\theta - (f/\omega^2)(\sin N\theta)u\right]y = 0$$

to obtain

$$y'' + \left[a_y - (f/\omega)\cos N\theta - \frac{Af}{\omega^2}\sin \nu_x\theta \sin N\theta + \frac{Af^2\nu_x}{\omega^3 N^3}\cos \nu_x\theta\right]y = 0,$$

terms involving $2N\theta$ being neglected. In the present application it will be seen that the term involving $\cos \nu_x\theta$ in this last equation is of relatively small effect and hence that it would have been sufficient to make the substitution $u = A \sin \nu_x\theta$ in the original y equation.

We now refer to the results of Appendix III C (or Appendix V C), pertaining to

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-\nu_x)\theta \pm (c/2)\cos(N+\nu_x)\theta + d\cos \nu_x\theta\right]y = 0$$

with stability limits

$$\left|(N-\nu_x)^2 - (2\nu_{y_0})^2\right| \doteq \left|c + \frac{2bd}{N^2}\right|.$$

We identify

$b = -f/w$,

$c = \pm Af/w^2$, taking the lower sign in the equation of the appendix,

and

$d = Af^2\nu_x/(\omega^3 N^3)$.

We note that the magnitude of $2bd/N^2$ is less than c by the factor $2\left[f/(\omega N^2)\right]^2(\nu_x/N)$ and hence may safely be neglected.

Accordingly we find

$$\left|(N-\nu_x)^2-(2\nu_{y_0})^2\right| \doteq |c|$$
$$= (f/\omega^2)|A| \ ,$$

with the threshold amplitude then being explicitly

$$|A|_{thr} = (\omega^2/f)\left|(N-\nu_x)^2-(2\nu_{y_0})^2\right|$$

$$= \frac{\omega}{4}\left(\frac{\omega N^2}{f}\right)\left|\left(2-\frac{\sigma_x}{\pi}\right)^2-\left(2\,\frac{\sigma_y}{\pi}\right)^2\right|.$$

An estimate of the lapse-rate for y-growth in the unstable region is again given by the formula (Appendix IV)

$$\mu^2 = -\frac{(a-a_1)(a_2-a)\langle c^2\rangle\langle s^2\rangle}{4\langle sc'\rangle\langle cs'\rangle} \ ,$$

with c and s now represented by circular functions of argument $(N-\nu_x)/2$.

Accordingly

$$\mu = \left\{\left(\frac{f}{4\omega^2}A+\left[\nu_{y_0}^2-\left(\frac{N-\nu_x}{2}\right)^2\right]\right)\left(\frac{f}{4\omega^2}A-\left[\nu_{y_0}^2-\left(\frac{N-\nu_x}{2}\right)^2\right]\right)\Big/(N-\nu_x)^2\right\}^{1/2}$$

$$= \frac{f}{4\omega^2(N-\nu_x)}\sqrt{A^2-A_{thr}^2} \qquad \text{nepers/radian}$$

$$= \frac{\pi}{2}\,\frac{f}{\omega^2 N(N-\nu_x)}\sqrt{A^2-A_{thr}^2} \qquad \text{nepers/sector}$$

$$= 0.682\,\frac{f}{\omega N^2}\,\frac{1}{1-\nu_x/N}\,\frac{\sqrt{A^2-A_{thr}^2}}{\omega} \qquad \text{decades/sector}$$

The maximum μ, for a given amplitude A, is

$$\mu_{max.} = 0.682\,\frac{f}{\omega N^2}\,\frac{1}{1-\nu_x/N}\,\frac{A}{\omega} \qquad \text{decades/sector.}$$

It is noted that for a <u>sum</u> resonance, such as the one considered here, ν_x and ν_y cannot both be arbitrarily small in comparison to N

B. Computational Results:

FECKLESS FIVE data pertaining to the resonance of present interest were obtained for seven operating points, for which the characteristics are given in Table VI. The lapse-rates were readily measured by semi-logarithmic plots of K_y vs. the number of sectors traversed (c.f. Fig. 2) and a threshold value for the onset of y-growth could be estimated. A comparison of the threshold value of u_0, as observed from the computations, and the theoretical threshold amplitude, $A_{thr.}$, is included in Table VI.

TABLE VI.

CHARACTERISTICS OF AXIAL MOTION NEAR $\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$

k = 0.2 f = 1/4 N = 5

Point	$1/\omega$	σ_{x_0}/π	σ_{y_0}/π	ν_{x_0} ν_{y_0} $\nu_{x_0}+2\nu_{y_0}$	$\lvert u_0\rvert_{thr}$	$\lvert A\rvert_{thr.}$ (Theoretical)
1	31.5243	.4826	.6824	1.206$_5$ 1.706 4.618$_5$	0.0075	0.011
2	32.2167	.4845	.7053	1.211+ 1.763+ 4.738	0.0075	0.007
3	32.9091	.4865	.7295	1.216 1.824 4.864	0.003	0.004
4	33.2223	.4875	.7410	1.219 1.852$_5$ 4.924	0.001	0.002
5	33.5555	.4885	.7537	1.221+ 1.884+ 4.990	0.0002	0.0003
6	34.1615	.4904	.7781	1.226 1.945 5.116	0.001	0.003
7	34.7676	.4924	.8051	1.231 2.013 5.257	0.004	0.0066

We also list below, in Table VII, observed lapse-rates and those calculated theoretically from the observed threshold values, for runs made using operating points 5 and 7. The first of these operating points is very close to the resonance in question and the calculated lapse-rate corresponds in effect to μ_{max}.

TABLE VII.

OBSERVED AND THEORETICAL LAPSE-RATES FOR OPERATION NEAR $\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$

Point	u_0	$\sqrt{\lvert u_0\rvert^2 - \lvert u\rvert^2_{thr\ obs.}}$	μ (decades/sector) observed	μ (decades/sector) theor. calc.
5	.01500	.015	.179	.15
	-.00750	.0075	.086	.076
	-.00375	.00375	.042	.038
7	-.0150	.0145	.22	.16
	-.0075	.0063	.089	.069

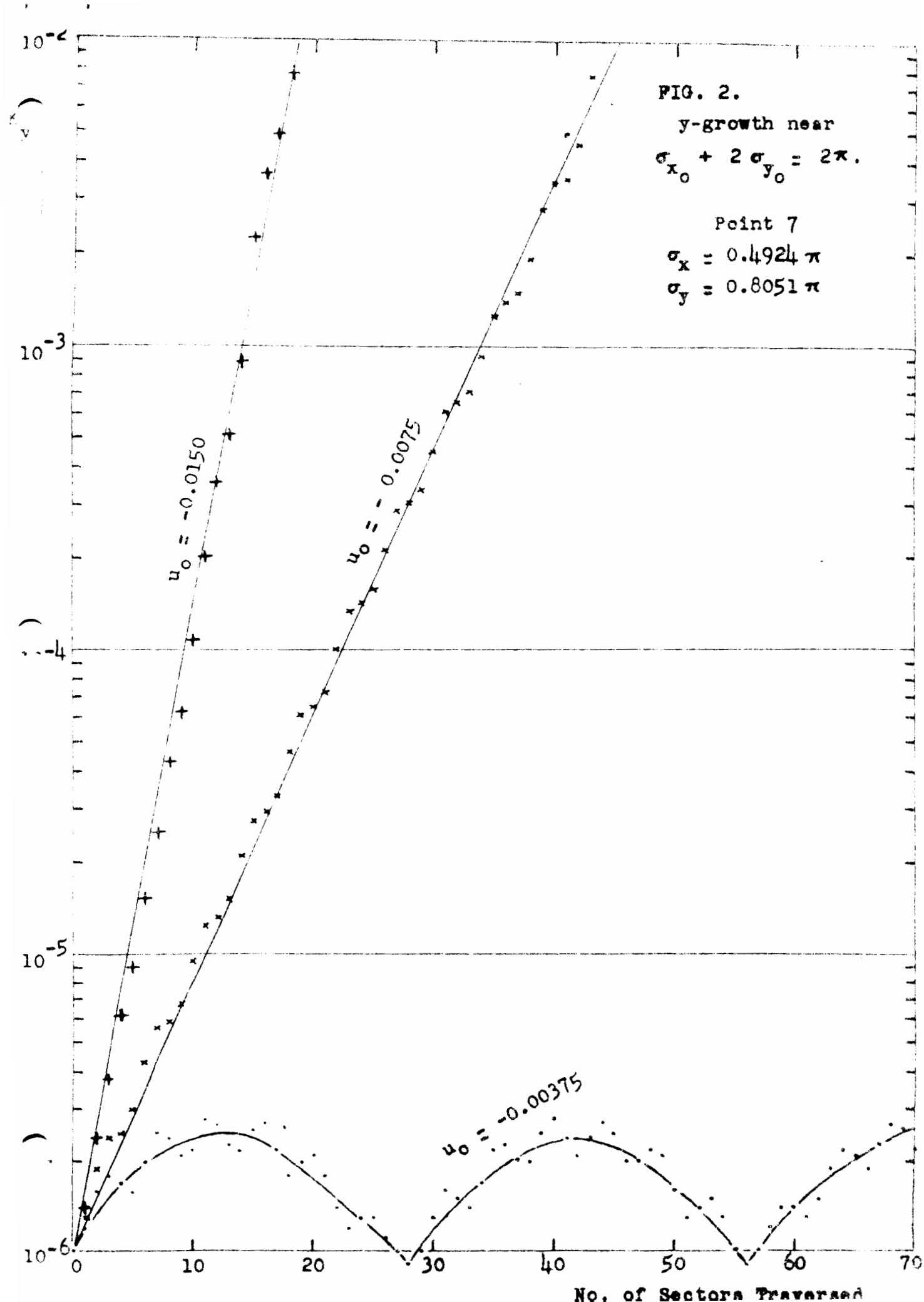


FIG. 2.
y-growth near
$\sigma_{x_o} + 2\sigma_{y_o} = 2\pi$.
Point 7
$\sigma_x = 0.4924\pi$
$\sigma_y = 0.8051\pi$
$u_o = -0.0150$
$u_o = -0.0075$
$u_o = -0.00375$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
10
20
30
40
50
60
70
No. of Sectors Traversed

V. THE $2\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$ RESONANCE

A. Theory

In an analysis of the resonance to be expected when $2\sigma_x + 2\sigma_{y_0}$ is close to 2π, the obvious term to invoke in the y-equation is $(1/2)\, c_{10}\, u^2 y$. It is necessary, however, also to consider the double-frequency $(2\nu_x)$ terms which can enter the term $b_5\, u\, y$ by virtue of supplementary terms in u obtainable by a perturbation solution of the non-linear u-equation (cf. Section III). It will appear that the direct contribution from $(1/2)\, c_{10}\, u^2 y$ nonetheless definitely dominates.

The solution of the u-equation is taken to be that employed previously in Section III, namely

$$u = A\left[\sin \nu_x\theta + \frac{f}{\omega N^2}\sin \nu_x\theta \cos N\theta - 2\frac{f\nu_x}{\omega N^3}\cos \nu_x\theta \sin N\theta\right]$$

$$+ \frac{fA^2}{4\omega^2 N^2}\left[\sin N\theta - \sin N\theta \cos 2\nu_x\theta + 4\frac{\nu_x}{N}\cos N\theta \sin 2\nu_x\theta\right].$$

In forming $(1/2)\, c_{10}\, u^2$, or $-\frac{f}{2\omega^3}(\cos N\theta)\, u^2$, the term of major importance in exciting the resonance of present interest is $\frac{fA^2}{4\omega^3}\cos 2\nu_x\theta \cos N\theta$, although the following terms might all be kept in mind:

$$\frac{1}{2}c_{10}u^2 = -\frac{fA^2}{4\omega^3}\cos N\theta + \frac{fA^2}{4\omega^3}\cos 2\nu_x\theta \cos N\theta$$

$$- \frac{f^2A^2}{4\omega^4N^2} + \frac{f^2A^2}{4\omega^4N^2}\cos 2\nu_x\theta + \cdots\cdots$$

Likewise, the following terms might be noted to arise from $b_5 u \doteq (f/w^2)(\sin N\theta)\, u$:

$$b_5 u \doteq \frac{f^2A^2}{8\omega^4N^2} - \frac{f^2A^2}{8\omega^4N^2}\cos 2\nu_x\theta.$$

With the foregoing expressions for the u-dependent terms, the differential equation for axial motion,

$$y'' + \left[a_y + b_y\cos N\theta - b_5 u - (c_{10}/2)u^2\right]y = 0$$

becomes

$$\ddot{y} + \left[a + b\cos N\theta + c\cos 2\nu_x\theta\cos N\theta + d\cos 2\nu_x\theta\right] y = 0 .$$

where

$$a = a_y + \text{terms of order } \frac{f^2}{w^2 N^2}\left(\frac{A}{w}\right)^2$$

$$b = -\frac{f}{w}\left(1 + \text{terms of order}\left(\frac{A}{w}\right)^2\right)$$

$$c = -\frac{f}{4w}\left(\frac{A}{w}\right)^2 \qquad \text{and}$$

$$d \text{ is of order } \frac{f^2}{w^2 N^2}\left(\frac{A}{w}\right)^2 .$$

This equation is of the form considered in Appendix III (see Item 4 of summary), for which the stability boundaries are represented by

$$\left|(N - 2\nu_x)^2 - (2\nu_{y_0})^2\right| = \left|c + \frac{2bd}{N^2}\right| .$$

In this last relation, $\nu_{y_0}^2$ refers to the square of the y-frequency when the coefficients c and d vanish; it differs, however, only by terms of order $\frac{f^2}{w^2 N^2}\left(\frac{A}{w}\right)^2$ from the square of the y-frequency for $A = 0$. The factor $2bd/N^2$, moreover, is less than c by a factor of order $\left(\frac{f}{wN^2}\right)^2$. Regarding $f/(wN^2)$ as small in comparison to unity, we are thus led to the result which would have been obtained if only the term $\frac{fA^2}{4\omega^3}\cos 2\nu_x\theta\cos N\theta$ in $\frac{c_{10}}{2}u^2$ had been retained:

$$\left|(N - 2\nu_x)^2 - (2\nu_{y_0})^2\right| \doteq \frac{f}{4\omega}\left(\frac{A}{\omega}\right)^2 ,$$

this result involving A <u>squared</u>, as was also the case for the $2\sigma_x = 2\sigma_{y_0}$ resonance treated in Section III.

The threshold amplitude is correspondingly

$$|A|_{thr} = 2\omega\sqrt{\frac{\omega}{f}\cdot\left|(N - 2\nu_x)^2 - (2\nu_{y_0})^2\right|}$$

$$= 2\omega\sqrt{\frac{\omega N^2}{f}\cdot\left|\left(1 - \frac{\sigma_x}{\pi}\right)^2 - \left(\frac{\sigma_{y_0}}{\pi}\right)^2\right|} .$$

An estimate of the lapse-rate for the axial amplitude in the unstable region is given by the formula (Appendix IV).

$$\mu^2 = -\frac{(a-a_1)(a_2-a)\langle c^2\rangle\langle s^2\rangle}{4\langle s c'\rangle\langle c s'\rangle}$$

with c and s represented by circular functions of argument $(N - 2\nu_x)/2$.

Accordingly,

$$\mu = \left\{\left(\frac{fA^2}{16\omega^3} + \left[\nu_{y_0}^2 - \left(\frac{N}{2} - \nu_x\right)^2\right]\right)\left(\frac{fA^2}{16\omega^2} - \left[\nu_{y_0}^2 - \left(\frac{N}{2} - \nu_x\right)^2\right]\right)\Big/(N-2\nu_x)^2\right\}^{1/2}$$

$$= \frac{f}{16\omega^3(N-2\nu_x)}\sqrt{A^4 - A^4_{thr.}} \qquad \text{nepers/radian}$$

$$= \frac{\pi}{8}\frac{f}{\omega^3 N(N-2\nu_x)}\sqrt{A^4 - A^4_{thr.}} \qquad \text{nepers/sector}$$

$$= 0.17\,\frac{f}{\omega N^2}\,\frac{1}{1-2\nu_x/N}\,\frac{\sqrt{A^4 - A^4_{thr}}}{\omega^2} \qquad \text{decades/sector.}$$

The maximum μ, for a given amplitude A, is then

$$\mu_{max.} = 0.17\frac{f}{\omega N^2}\,\frac{1}{1-2\nu_x/N}\left(\frac{A}{\omega}\right)^2 \qquad \text{decades/sector,}$$

with a quadratic dependence on A.

B. Computaticnal Results:

Data pertaining to the resonance treated here were obtained for three operating points, with results listed in Table VIII.

TABLE VIII.

CHARACTERISTICS OF AXIAL MOTION NEAR $2\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$

$k = 0.25$ $f = 1/4$ $N = 5$

Point	$1/\omega$	σ_{x_0}/π	σ_{y_0}/π	ν_{x_0} ν_{y_0} $2\nu_{x_0}+2\nu_{y_0}$	$\lvert u_0\rvert_{thr}$	$\lvert A\rvert_{thr.}$ (Theoretical)
12	26	.4785	.5169	1.196 1.292 4.977	0.010	0.010
13	25	.4762	$.491\dot{2}$	$1.19\dot{1}$ 1.228 4.837	0.017	0.029
14	24.38	$.474\dot{9}$	.4754	1.187 $1.18\dot{9}$ 4.751	0.024	0.037

A comparison between the observed lapse-rates and those calculated theoretically is given in table IX for the structure represented by Point 12. This operating point is close to the resonance in question and the calculated lapse-rate is close to μ_{max}.

TABLE IX.

OBSERVED AND THEORETICAL LAPSE-RATES FOR OPERATION NEAR $2\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$ [OPERATING POINT 12]

u_0	$\sqrt{\lvert u_0\rvert^4 - \lvert u\rvert^4_{thr,obs.}}$	μ (decades/sector). observed	theor. calc.
-.0300	.00089	.12	.051
-.0225	.00050	.05	.029
-.0150	.00020	.02	.011

VI. CONCLUSION

A. General:

In addition to the resonances reported here, for which positive evidence of y-growth was obtained, operating points near $2\sigma_{x_0} \cong 3\sigma_{y_0}$ and others near $3\sigma_{x_0} + \sigma_{y_0} \cong 2\pi$ were also studied. These latter resonances (for which the coefficient of σ_{y_0} is odd) showed no evidence of y-growth for the structures studied (N = 5). Recently some positive evidence of y-growth near $3\sigma_{x_0} + 2\sigma_{y_0} = 2\pi$ has been found with aid of the MURA I. B. M.-704 computer and the results of further studies will be reported at a later date.

For the resonances treated in the present report, the computational results and the theoretical estimates are in fair agreement - generally within a factor of two. This agreement may be considered satisfactory at this stage in view of (i) the data inaccuracies associated with determining the small-amplitude oscillation frequencies and extrapolated thresholds, (ii) the presence of additional (possibly non-scaling) terms which may be prominent in small-scale machines,[15] and (iii) the approximations inherent in the analytic work. We would like to infer, therefore, that the equations presented in this report afford a semi-quantitative account of the resonances considered, when the median-plane field has a sinusoidal variation characterized by a modest flutter-factor ($f \cong 1/4$, or smaller). These formulas are summarized at the end of this Section for convenience.

As was pointed out in the Introduction, the viewpoint taken in the analysis has been that a prescribed u-oscillation is assumed for the radial motion and is introduced into a linear differential equation for y which is taken to characterize the axial oscillations. If large axial amplitudes are built up, the

radial motion will certainly be affected, however, and the amplitude of radial oscillations has then been seen to decrease noticeably in certain cases

It would be of interest to extend this investigation, possibly with a more refined theoretical approach,[16-17] to cases in which the flutter-factor f is large (so that additional terms, which here could be considered negligible, become important) and to cases in which a significant harmonic content is present in the magnetic field (as for separated-sector structures). In further computational examples it may prove somewhat more convenient to employ machine parameters characteristic of large scale accelerators, so that the differential equations on which the theoretical analysis is based are more clearly defined. As indicated in the Introduction, separate computational checks can be made of formulas believed to describe the instabilities of the simplified linear equations assumed to represent the small-amplitude axial motion and the results of such work may be reported separately at a later date.

It is important, of course, also to obtain an understanding of the "leveling off" which the y growth may exhibit (cf. Figs. 3 and 4), the danger that y-growth arising from a difference-resonance which might in itself be innocuous would aggravate the effects of other resonances, and the manner in which these phenomena may be correlated with observational experience acquired with a FFAG model.[23] The graphs of Figs. 3 and 4 are schematic in that high-frequency oscillations of K_y, with relatively small amplitudes, have been omitted in the interest of simplifying the drawing. It appears from Fig. 4, for example, that y-growth which "levels off" at relatively large values of y (near the y-stability limit in the cases investigated) is followed by oscillations of K_y whose wave-length becomes longer as $|u_o|$ is decreased -- in contrast, the small-amplitude runs which

level off early, and should not be regarded as showing y-growth in the sense treated here, exhibit more rapid oscillation of K_y as $|u_o|$ takes on smaller values (e.g. the three runs of Fig. 4 with $u_o = -.0150$, $-.0125$, and $-.0075$). For analysis of such mutual coupling, additional terms involving y should of course be introduced into the u-equation so as to result in a Hamiltonian system of equations

The phenomena discussed here of course have their analogues in "machine resonances", which may be engendered when misalignments are present. It would be desirable ultimately also to obtain a semi-quantitative understanding of the corresponding effects produced by such imperfections, both in regard to their ability to excite machine resonances and with respect to their effect on the true stability or instability of orbits strongly affected by some inherent sector resonance It may be noted that one can expect to encounter certain imperfection resonances whose analogous sector resonances are absent by virtue of median-plane symmetry, since in the presence of misalignments symmetry about the "median-plane" need no longer obtain.

Computations directed to a study of "turn-over", which will be summarized in a later report, suggest questions concerning the ultimate stability of particles whose axial motion is subject to growth and exhibits turn-over. The repeated rise and fall of y-amplitude in such cases may conceal an ultimate instability which would only appear if undesirably-protracted runs were made. For this reason there has been an awakening of interest in the possible utility of ALGYTEE runs[18] which can readily execute for tens of thousands of iterations algebraic transformations which also exhibit[19] the phenomena of interest here.

B. Summary:

We list here for convenient reference the equations developed in this report, which serve as "handy formulas" to describe approximately the characteristics of y-growth at the several coupling resonances in spirally-ridged accelerators with modest flutter-factor and σ_x, y not close to π.

1. The $\sigma_x = 2\sigma_y$ Resonance:

$$|A|_{thr} = \frac{1}{16}\omega\left(\frac{\omega N^2}{f}\right)^2 \frac{N}{\nu_x}\left|\left(\frac{\sigma_x}{\pi}\right)^2 - \left(2\frac{\sigma_{y_0}}{\pi}\right)^2\right|,$$

$$\mu = 2.79\left(\frac{f}{\omega N^2}\right)^2 \frac{\sqrt{A^2 - A_{thr}^2}}{\omega} \quad \text{decades/sector.}$$

2. The $\sigma_x = \sigma_y$ Resonance:

$$|A_1| = \frac{|A_2|}{\sqrt{3}} = \frac{2\omega}{\sqrt{3}}\frac{\omega N^2}{f}\sqrt{\left(\frac{\sigma_x}{\pi}\right)^2 - \left(\frac{\sigma_{y_0}}{\pi}\right)^2};$$

$$\mu = 0.1477\left(\frac{f}{\omega N^2}\right)^2 \frac{N}{\nu_x}\frac{\sqrt{(A^2 - A_1^2)(3A_1^2 - A^2)}}{\omega^2} \quad \text{decades/sector,}$$

$$\mu_{max} = 0.085\left(\frac{f}{\omega N^2}\right)^2 \frac{N}{\nu_x}\left(\frac{A}{\omega}\right)^2 \quad \text{decades/sector.}$$

3. The $\sigma_x + 2\sigma_y = 2\pi$ Resonance:

$$|A|_{thr} = \frac{\omega}{4}\left(\frac{\omega N^2}{f}\right)\left|\left(2 - \frac{\sigma_x}{\pi}\right)^2 - \left(2\frac{\sigma_{y_0}}{\pi}\right)^2\right|,$$

$$\mu = 0.682\frac{f}{\omega N^2}\frac{1}{1 - \nu_x/N}\frac{\sqrt{A^2 - A_{thr}^2}}{\omega} \quad \text{decades/sector.}$$

4. The $2\sigma_x + 2\sigma_y = 2\pi$ Resonance:

$$|A|_{thr.} = 2\omega\sqrt{\frac{\omega N^2}{f}\left|\left(1-\frac{\sigma_x}{\pi}\right)^2-\left(\frac{\sigma_{y0}}{\pi}\right)^2\right|},$$

$$\mu = 0.17\,\frac{f}{\omega N^2}\,\frac{1}{1-2\nu_x/N}\,\frac{\sqrt{A^4-A^4_{thr}}}{\omega^2}\ \text{decades/sector}.$$

It is felt that these formulas may be of assistance in the event that it is desired to select an operating point such that the threshold for y-growth exceeds the limit imposed by inherent radial resonances or by mechanical limitations of the vacuum chamber. In selecting a suitable operating point it is of course well to keep in mind also the acceptance in phase-space, which involves the oscillation frequencies as well as the permissible coordinate amplitudes.

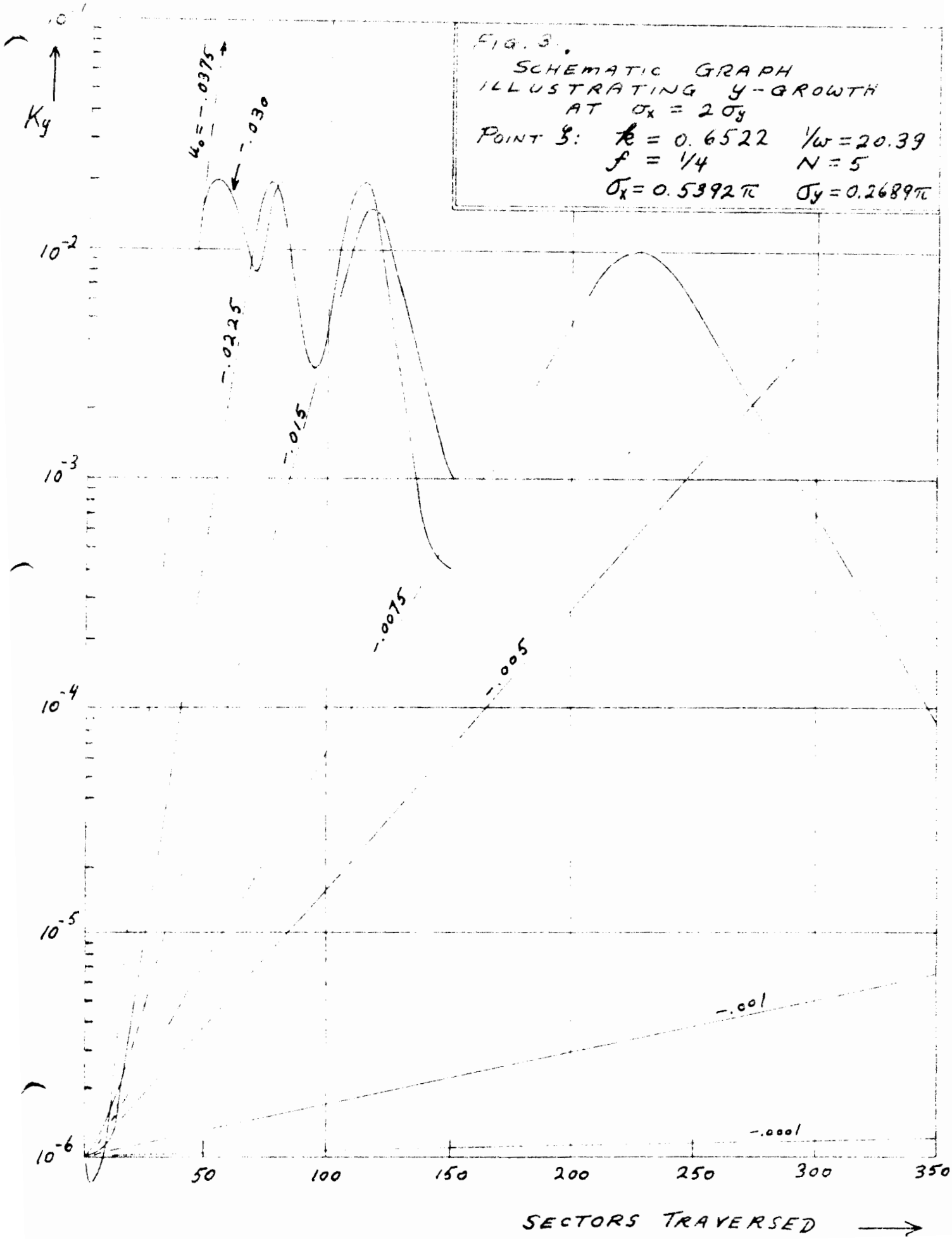


Fig. 3.
SCHEMATIC GRAPH
ILLUSTRATING y-GROWTH
AT $\sigma_x = 2\sigma_y$
POINT 3: $k = 0.6522$ $1/w = 20.39$
$f = 1/4$ $N = 5$
$\sigma_x = 0.5392\pi$ $\sigma_y = 0.2689\pi$
K_y
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$u_0 = -.0375$
-.030
-.0225
-.015
-.0075
-.005
-.001
-.0001
50
100
150
200
250
300
350
SECTORS TRAVERSED

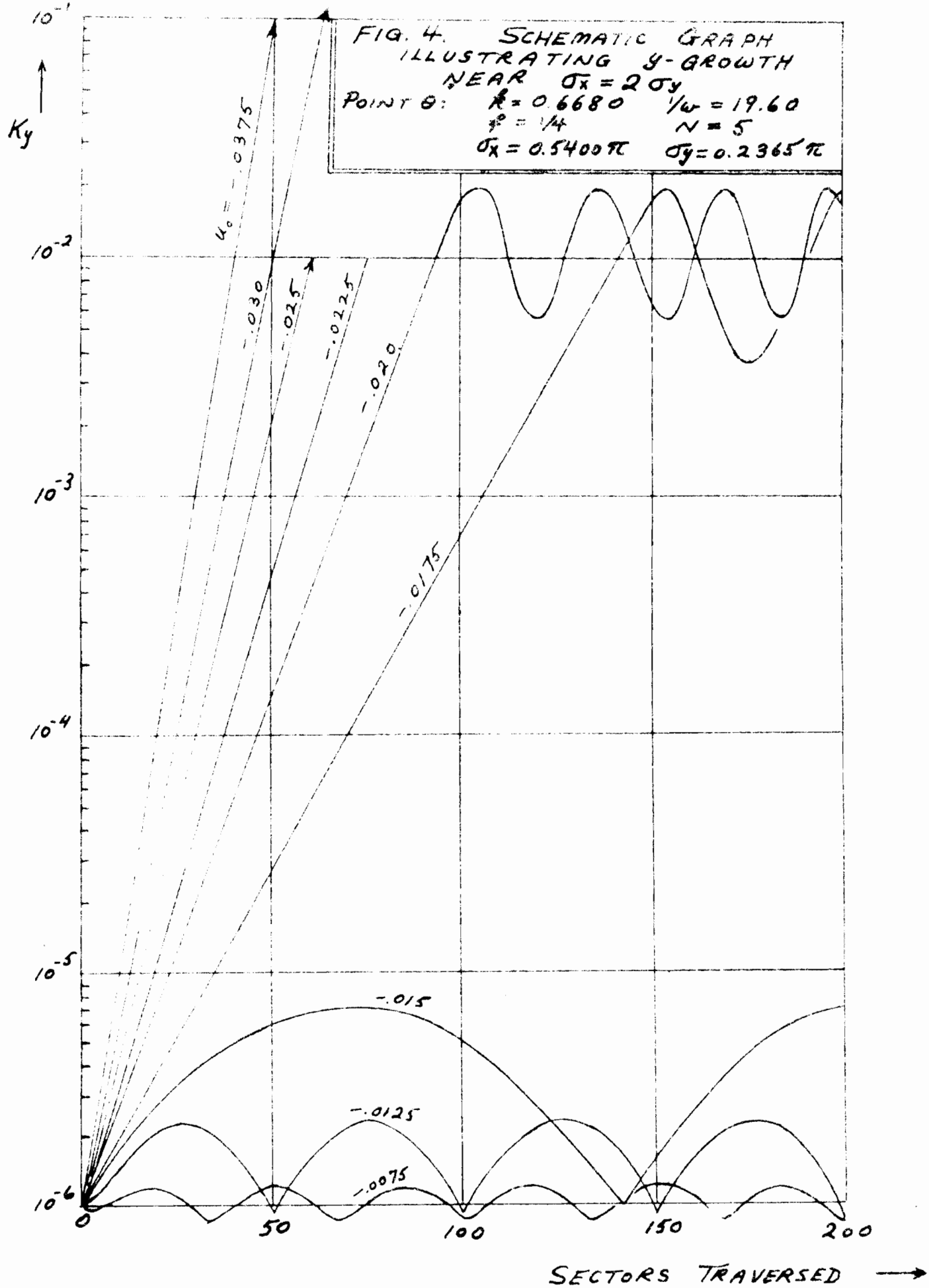


FIG. 4. SCHEMATIC GRAPH ILLUSTRATING y-GROWTH NEAR $\sigma_x = 2\sigma_y$
POINT θ: $k = 0.6680$ $1/\omega = 19.60$
$\varphi^\circ = 1/4$ $N = 5$
$\sigma_x = 0.5400\pi$ $\sigma_y = 0.2365\pi$
K_y
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$u_0 = -.0375$
-.030
-.025
-.0225
-.020
-.0175
-.015
-.0125
-.0075
0
50
100
150
200
SECTORS TRAVERSED

VII. REFERENCES AND NOTES

1. The Feckless Five equations represent a Hamiltonian system and are designed to represent, with fair accuracy, the trajectory which a particle of constant energy follows in a spirally-ridged FFAG accelerator. The forces which affect the radial motion (x-motion) are even about the median plane ($y = 0$) and those affecting the axial motion are odd. Small-amplitude motion about the stable equilibrium orbit is normally represented with reasonable accuracy by decoupled linear equations of the Hill form, but for larger amplitudes of oscillation, non-linear terms and coupling become significant [cf. LJL (MURA)-5, esp. Appendix II].

2. Since $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$ and $\cos\sigma = (A + D)/2$, it is immediately found that $\xi - 1 = \frac{\eta^2}{4\zeta}$. The equation obtained by setting K_y = const. is that of an "invariant ellipse" and the ellipse will be aligned with the coordinate axes if $\eta = 0$ ($A = D$, $\xi = 1$). [Cf. Internal MURA Report 206 (21 November, 1956)]. In representative cases the small-amplitude y-motion, observed at $N\theta = 0$, mod. 2π, is such that ξ exceeds unity by about two-tenths of one percent.

3. F.T. Cole, MURA/FTC-3.

4. W. Walkinshaw, "A Spiral Ridged Bevatron", A.E.R.E., Harwell (1956).

5. (a) The application of Walkinshaw's Mathieu equation to our data was described in a paper prepared for the Geneva conference of 11-15 June 1956, and (b) the methods employed in the present report were illustrated in MURA lectures given by one of us (L.J.L.) in Madison on 20 June 1956.

6. This observation of ours has been pointed out previously in a similar context by one of us (A.M.S.) in informal MURA Notes (20 July 1956).

7. Some discussion of this method was given earlier by one of us (L. J. L.) in informal MURA Notes (1 February 1955); cf. also LJL (MURA)-5, Appendix III

8. Notation of Whittaker and Watson "Modern Analysis" (Cambridge University Press, 1927), Sect. 19.3. These authors use 16q in place of our coefficient b and take N = 2.

9. Previously a variational solution was undertaken by one of us (L. J. L.) -- for example in reference 7--through use of the phase-amplitude representation of Floquet solutions, since the separate functions w and Ψ are periodic and so implement formulation of a variational equation equivalent to the differential equation.

10. K. R. Symon, KRS (MURA)-1 and -4.

11. Laslett, Snyder, and Hutchinson, MURA Notes (20 April 1955). The tables were prepared by a variational method[9] for the case N = 2; the results for other values of N are readily found by a transformation of the independent variable ($N\theta = 2t$).

12. Relations of the sort obtained here may alternatively be sought by substitution of the supposed solution into the differential equation and use of harmonic balance.

13. N. W. McLachlan, "Theory and Application of Mathieu Functions" (Clarendon Press, Oxford, 1947), Sects. 4.90-4.91.

14. L. Jackson Laslett and A. M. Sessler, MURA Internal Report 252 (10 April 1957). It should be mentioned that Parzen has developed, by a systematic perturbation procedure, algebraic expressions for expansion of a Floquet solution which appear to be of the same accuracy as our own (informal communication) and which possess the feature of not involving both the coefficient a and and the oscillation frequency ν

15. Minutes of MURA General Sesquimonthly Technical Meeting, # 13 (4-5 January 1957).

16. G. Parzen, "Coupled Non-Linear Resonance in A-G Accelerators", MURA Report 217 (20 February 1957); cf. also G. Parzen, "The $2\nu_y - \nu_x$ Difference Resonance", MURA Internal Report 250 (1 April 1957).

17. J. Moser, Nachr. Akad. Wiss. Göttingen, Math.-physik Kl. IIa, No. 6, 87 (1955); also Commun. Pure and Appl. Math. 8, 409 (1955).

18. ALGYTEE, MURA Internal Report 233 (1957).

19. L. Jackson Laslett, MURA Internal Report 246 (11 March 1957) and correction dtd. 25 March 1957; also MURA Internal Report 247 (18 March 1957).

20. WELL-TEMPERED FIVE, MURA Internal Report 226 (1957).

21. TEMPERMESH, MURA Internal Report 241 (1957).

22. FORMESH, MURA Internal Report 222 (1957).

23. F. T. Cole, et. al., MURA Report 219 (5 March 1957) --to be published in the Review of Scientific Instruments.

VIII. ACKNOWLEDGEMENTS

We were happy to have had an opportunity to discuss the phenomenon of y-growth with W. Walkinshaw shortly after the FECKLESS FIVE computations had revealed this effect for the $\sigma_x = 2\sigma_y$ resonance. We are also greatly indebted to Dr. J. N. Snyder for his invaluable assistance with the computational phases of the work. It is a pleasure, finally, to acknowledge the aid of Mr. E. C. Weinberg, of Purdue University, and Mr. R. E. Mills, of the Ohio State University, in checking some of the algebraic work and performing certain arithmetic calculations.

APPENDICES

I. ESTIMATION OF STABILITY BOUNDARIES FOR A MATHIEU EQUATION

Summary

(1) For the Mathieu Equation

$$y'' + \left[a + b \cos N\theta\right] y = 0$$

regions of instability are

$$a < -b^2/(2N^2)$$

$$N^2/4 - b/2 < a < N^2/4 + b/2$$

$$N^2 - b^2/(12N^2) < a < N^2 + 5b^2/(12N^2).$$

(2) For the Hill Equation

$$y'' + \left[\alpha + \beta \cos \omega\theta + \gamma \cos 2\omega\theta\right] y = 0$$

a region of instability exists for α between

$$\omega^2 - \beta^2/(12\omega^2) + \gamma/2 \text{ and } \omega^2 + 5\beta^2/(12\omega^2) - \gamma/2.$$

II. APPROXIMATE SOLUTION OF A MATHIEU EQUATION

Summary:

For the Mathieu Equation

$$y + \left[a + b \cos N\theta\right] y = 0$$

we take the solution to be, when ν/N is small,

$$y = A_y \left[\sin(\nu\theta + \epsilon) + (b/N^2) \sin(\nu\theta + \epsilon) \cos N\theta - (2b\nu/N^3) \cos(\nu\theta + \epsilon) \sin N\theta\right.$$

with

$$\nu^2 \cong a + b^2/(2N^2)$$

III. STABILITY LIMITS FOR A HILL EQUATION

Summary:

Stability limits for the equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-\nu_x)\theta \pm (c/2)\cos(N+\nu_x)\theta + d\cos\nu_x\theta\right] y = 0$$

are found associated with zones of instability as follows:

(1) $$\left|\nu_x^2 - (2\nu_{y_0})^2\right| \doteq 2\left|\frac{bc}{N^2} + d\right|,$$

when the upper sign is taken;

$$\left|\nu_x^2 - (2\nu_{y_0})^2\right| \doteq 2\left|\frac{\nu_x bc}{N^3} + d\right|,$$

when the lower sign is taken.

(2) $$\frac{c^2}{8N^2} - \frac{5}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2 \leqslant \nu_x^2 - \nu_{y_0}^2 \leqslant \frac{3}{8}\frac{c^2}{N^2} + \frac{1}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2$$

when the lower sign applies.

(3) $$\left|(N-\nu_x)^2 - (2\nu_{y_0})^2\right| \doteq \left|c + \frac{2bd}{N^2}\right|$$

for <u>either</u> sign of the term which involves $\cos(N+\nu_x)\theta$.

(4) If ν_x is replaced by $2\nu_x$ in this last result, the equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-2\nu_x)\theta \pm (c/2)\cos(N+2\nu_x)\theta + d\cos 2\nu_x\theta\right] y = 0$$

has a zone of instability defined by

$$\left|(N-2\nu_x)^2 - (2\nu_{y_0})^2\right| \doteq \left|c + \frac{2bd}{N^2}\right|.$$

IV. ESTIMATE OF CHARACTERISTIC EXPONENT IN THE UNSTABLE REGION OF A HILL EQUATION

Summary

The lapse-rate, μ nepers/radian, characterizing unstable solutions of the differential equation

$$y'' + f(a, \theta)\, y = 0$$

is given in terms of the eigenvalues a_1, a_2 and eigen functions $c(\theta)$, $s(\theta)$ associated with the boundaries of the unstable region:

$$\mu^2 \cong - \frac{(a - a_1)(a_2 - a) \langle c^2 \rangle \langle s^2 \rangle}{4 \langle s c' \rangle \langle c s' \rangle},$$

$$\mu^2_{max} \cong - \frac{\langle c^2 \rangle \langle s^2 \rangle}{\langle s c' \rangle \langle c s' \rangle} \left[\frac{a_2 - a_1}{4} \right]^2 .$$

V. SMOOTH APPROXIMATION METHOD

Summary

Application of "smooth-approximation" methods is shown to lead to results for stability boundaries of a Hill equation in agreement with those derived by a variational method in Appendix III.

APPENDIX I.

ESTIMATION OF STABILITY BOUNDARIES FOR A MATHIEU EQUATION

In part for orientation purposes, we outline here a variational method[7] for determining the first few stability boundaries of the Mathieu equation

$$y'' + \left[a + b \cos N\theta\right] y = 0.$$

At stability boundaries the differential equation admits a periodic solution such that, formally

$$\delta \int_0^{4\pi/N} \frac{1}{2}\left[y'^2 - (a + b\cos N\theta) y^2\right] d\theta = 0,$$

the constant $-a$ playing the rôle of a Lagrange multiplier.

A. At the first stability boundary, corresponding[8] to ce_0 and for which $a = 0$ when $b = 0$, a suitable trial function is

$$y = A_0 + A_1 \cos N\theta.$$

Insertion of this trial function into the integral and setting the partial derivatives of the result (taken with respect to A_0, A_1) equal to zero leads to the simultaneous linear homogeneous algebraic equations:

$$-2a A_0 - b A_1 = 0$$

$$-b A_0 + (N^2 - a) A_1 = 0.$$

Accordingly, for a non-trivial solution,

$$2a(N^2 - a) + b^2 = 0,$$

with a root

$$a \cong -\frac{b^2}{2N^2} \quad \text{for } b \text{ small.}$$

Also

$$A_1/A_0 \cong b/N^2.$$

[This result, and the others obtained in this Appendix, are, of course, approximations representing the initial term or terms of well-known series expansions.[8]]

B. At the second stability boundary, near $a = N^2/4$ and corresponding to ce_1, we take

$$y = B_1 \cos N\theta/2 + B_2 \cos 3N\theta/2$$

as the trial function. One then obtains in a similar manner

$$a \cong N^2/4 - b/2$$

and

$$B_2/B_1 \cong b/(4N^2).$$

C. At the third stability boundary, again near $a = N^2/4$ but corresponding to se_1, we take

$$y = C_1 \sin N\theta/2 + C_2 \sin 3N\theta/2.$$

In this case one obtains

$$a \cong N^2/4 + b/2$$

and

$$C_2/C_1 \cong b/(4N^2).$$

D. At the fourth stability boundary, near $a=N^2$ and corresponding to se_2, a suitable trial function may be taken of the form

$$y = D_1 \sin N\theta + D_2 \sin 2N\theta.$$

One obtains in this case

$$a \cong N^2 - b^2/(12N^2)$$

and

$$D_2/D_1 \cong b/(6N^2).$$

E. At the other stability boundary near $a = N^2$, one may employ the trial function

$$y = E_o + E_1 \cos N\theta + E_2 \cos 2N\theta$$

to obtain

$$a \cong N^2 + 5b^2/(12N^2),$$

$$E_o/E_1 \cong -b/(2a) \approx -b/(2N^2)$$

$$E_2/E_1 \cong b/[2(4N^2-a)] \approx b/(6N^2).$$

It is of interest to note from the results of this and the preceding subsection that the stability boundaries are not symmetrically located about $a = N^2$.

Series expansions for all these various stability boundaries are, of course, given in published texts.[8,13]

F. A case involving a special Hill equation may also be considered here because of certain similarities to D and E above. The equation

$$y'' + [\alpha + \beta \cos \omega\theta + \cos 2\omega\theta] y = 0,$$

with β and γ considered small, will exhibit a narrow zone of instability for α near ω^2. When $\beta = 0$, the equation is of the form considered in subsections B and C (with 2ω corresponding to N) and the width of the unstable region will be proportional to γ; when $\gamma = 0$, the resonance in question is that considered in subsections D and E (with $\omega = N$) and the width will be proportional to β^2.

The corresponding result for the general case (β and γ both different from zero) may be obtained for circumstances in which β^2 and γ are of the same order of magnitude. The variational statement

$$\delta \int \{y'^2 - [\alpha + \beta \cos \omega\theta + \gamma \cos 2\omega\theta] y^2\} d\theta = 0$$

is used, with the trial functions

$$y = D_1 \sin \omega\theta + D_2 \sin 2\omega\theta$$

and

$$y = E_o + E_1 \cos \omega\theta + E_2 \cos 2\omega\theta.$$

One then finds that instability will occur when

$$\alpha - \omega^2$$

lies between

$$-\beta^2/(12\,\omega^2) + \gamma/2 \quad \text{and} \quad 5\beta^2/(12\,\omega^2) - \gamma/2 .$$

[When $\beta = 0$, this result is consistent with subsections B and C ($2\,\omega = N$); when $\gamma = 0$, the result is consistent with D and E ($\omega = N$).]

Summary of Zones of Instability found in Appendix 1

(1) For the Mathieu Equation

$$y'' + \left[a + b \cos N\theta\right] y = 0$$

regions of instability are

$$a < -b^2/(2N^2)$$

$$N^2/4 - b/2 < a < N^2/4 + b/2$$

$$N^2 - b^2/(12\,N^2) < a < N^2 + 5\,b^2/(12\,N^2).$$

(2) For the Hill Equation

$$y'' + \left[\alpha + \beta \cos \omega\theta + \gamma \cos 2\omega\theta\right] y = 0$$

a region of instability exists for α between

$$\omega^2 - \beta^2/(12\,\omega^2) + \gamma/2 \quad \text{and} \quad \omega^2 + 5\beta^2/(12\,\omega^2) - \gamma/2 .$$

APPENDIX II.

APPROXIMATE SOLUTION OF A MATHIEU EQUATION

We are again concerned with the Mathieu equation

$$y'' + \left[a + b \cos N\theta\right] y = 0,$$

seeking an approximate representation of the Floquet solutions and an estimate of the characteristic oscillation frequency. In contrast to the variational procedure previously employed[9] for these purposes by one of us, a simplification results if one imagines that the characteristic period of the solution and the period of the coefficient $\cos N\theta$ are commensurate in some (possibly large) interval and that the Floquet solution is accordingly periodic in this interval.

By the foregoing ruse we then again write

$$\delta \int 1/2 \left[y'^2 - (a + b \cos N\theta)\, y^2\right] d\theta = 0,$$

with the integral now covering a sufficient number of periods of the cosine coefficient that the periodicity of the solution in this interval may be exploited. Seeking a solution whose variation with θ is roughly that of $\cos \nu\theta$ or $\sin \nu\theta$, effective trial functions are

$$y = A_1 \cos \nu\theta + B_1 \cos (N-\nu)\theta + C_1 \cos (N+\nu)\theta$$

or

$$y = A_2 \sin \nu\theta + B_2 \sin (N-\nu)\theta + C_2 \sin (N+\nu)\theta.$$

We proceed to a solution of the problem by use of these trial functions under the supposition that ν is small in comparison to N, results containing this limitation being suitable for the purposes of the present report.*

* Subsequent to the preparation of this Appendix, a more generally-valid representation of the Floquet solutions, for $0 < \nu < N/2$, has been undertaken and reported elsewhere.[14]

The first of the trial functions, when adjusted to make the integral stationary, leads to the simultaneous equations

$$
\begin{aligned}
(\nu^2 - a)\, A_1 \quad & - (b/2)\, B_1 \quad - (b/2)\, C_1 = 0 \\
-(b/2)\, A_1 \quad + & \left[(N-\nu)^2 - a\right] B_1 = 0 \\
-(b/2)\, A_1 \quad & + \left[(N+\nu)^2 - a\right] C_1 = 0.
\end{aligned}
$$

Approximate solution of these equations gives

$$B_1 \cong \frac{b}{2N^2}\left(1 + \frac{2\nu}{N}\right) A_1$$

$$C_1 \cong \frac{b}{2N^2}\left(1 - \frac{2\nu}{N}\right) A_1 \qquad \text{and}$$

$$\nu^2 \cong a + b^2 / (2N^2),$$

this last relation being in agreement with the "smooth approximation" result.[10]

The second trial function, involving sine terms, leads similarly to

$$
\begin{aligned}
(\nu^2 - a)\, A_2 \quad & + (b/2)\, B_2 \quad - (b/2)\, C_2 = 0 \\
(b/2)\, A_2 \quad + & \left[(N-\nu)^2 - a\right] B_2 = 0 \\
-(b/2)\, A_2 \quad & + \left[(N+\nu)^2 - a\right] C_2 = 0,
\end{aligned}
$$

with approximate solutions identical in form to those for the cosine series, save for a change in sign for B_2. Thus, although the procedure employed is formally similar to that which can be used to find stability boundaries (cf. Appendix I B, C), the relations connecting ν, a, and b are <u>identical</u> for the two cases considered here and we may write the general approximate solution as an arbitrary linear combination of the two solutions

$$A_1 \left[\cos \nu\theta + (b/N^2) \cos N\theta \cos \nu\theta + 2b\nu/N^3 \sin N\theta \sin \nu\theta\right]$$

and

$$A_2 \left[\sin \nu\theta + (b/N^2) \cos N\theta \sin \nu\theta - 2b\nu/N^3 \sin N\theta \cos \nu\theta\right],$$

viz.:

$$y = A_y \left[\sin(\nu\theta + \epsilon) + (b/N^2) \sin(\nu\theta + \epsilon) \cos N\theta - (2b\nu/N^3) \cos(\nu\theta + \epsilon) \sin N\theta\right],$$

with

$$\nu^2 \cong a + b^2 \ (2N^2)$$

[It may be noted that this solution is identical to that obtained by expansion of Walkinshaw's form [4] The result may also be compared with the expansion of the form $y = A_y (1 + 2P \cos N\theta + \ldots) \sin(\nu\theta + \epsilon + 2R \cos N\theta + \ldots)$ employed in digital computations for the preparation of tables [11] pertaining to a Hill equation. By inspection of the tables, particularly for ν/N small, one notes that $2P \doteq b/N^2$ and $2R \doteq -2b\nu/N^3$.]

Illustration

The application of this result to the differential equation for u cited in the Introduction leads to the following expression for the radial oscillations.

$$u = A_x \left[\sin(\nu_x\theta + \epsilon) + f/(wN^2) \sin(\nu_x\theta + \epsilon) \cos N\theta - 2\nu_x f/(wN^3) \cos(\nu_x\theta + \epsilon) \sin N\theta\right],$$

where $\nu_x^2 \cong k + 1$ Similarly, for the axial oscillations, one expects solutions of the form

$$y = A_y \left[\sin(\nu_y\theta + \epsilon') - f/(wN^2) \sin(\nu_y\theta + \epsilon') \cos N\theta + 2\nu_y f/(wN^3) \cos(\nu_y\theta + \epsilon') \sin N\theta\right],$$

with $\nu_y^2 \cong -k + \left(\frac{f}{wN^2}\right)^2$.

As a check of the accuracy of this analysis, we have evaluated this last formula for a representative axial oscillation in a structure with parameters identical to those of Point 6 introduced in Section II B of this report. The parameters of this structure are $k = 0.668$, $1/w = 19.6$, $f = 1/4$, and $N = 5$ For convenience we consider $\nu_y = 53\ 5/90$ and evaluate

$$y = 1.243_{78} \times 10^{4} \left\{ \cos(53.5\theta/90) \left[1 - 0.196 \cos 5\theta \right] - 0.046 \sin(53.5\theta/90) \sin 5\theta \right\},$$

for which $y(0) = 0.0001$ and $y'(0) = 0$. The results of this theoretical evaluation are compared with computational results from the FECKLESS FIVE program in Table X. The results are in sufficiently good agreement as to be virtually indistinguishable graphically. The nature of the oscillation represented by this example is illustrated in Fig. 5.

Summary of Floquet Solution Obtained in Appendix II for the Mathieu Equation

For the Mathieu equation

$$y'' + \left[a + b \cos N\theta \right] y = 0$$

we take the solution to be, when ν/N is small,

$$y = A_y \left[\sin(\nu\theta + \epsilon) + (b/N^2) \sin(\nu\theta + \epsilon) \cos N\theta - (2b\nu/N^3) \cos(\nu\theta + \epsilon) \sin N\theta \right],$$

with

$$\nu^2 \cong a + b^2/(2N^2).$$

If a more accurate representation of the Floquet solution is required, valid for $0 < \nu < N/2$, the reader is referred to reference 14.

TABLE X

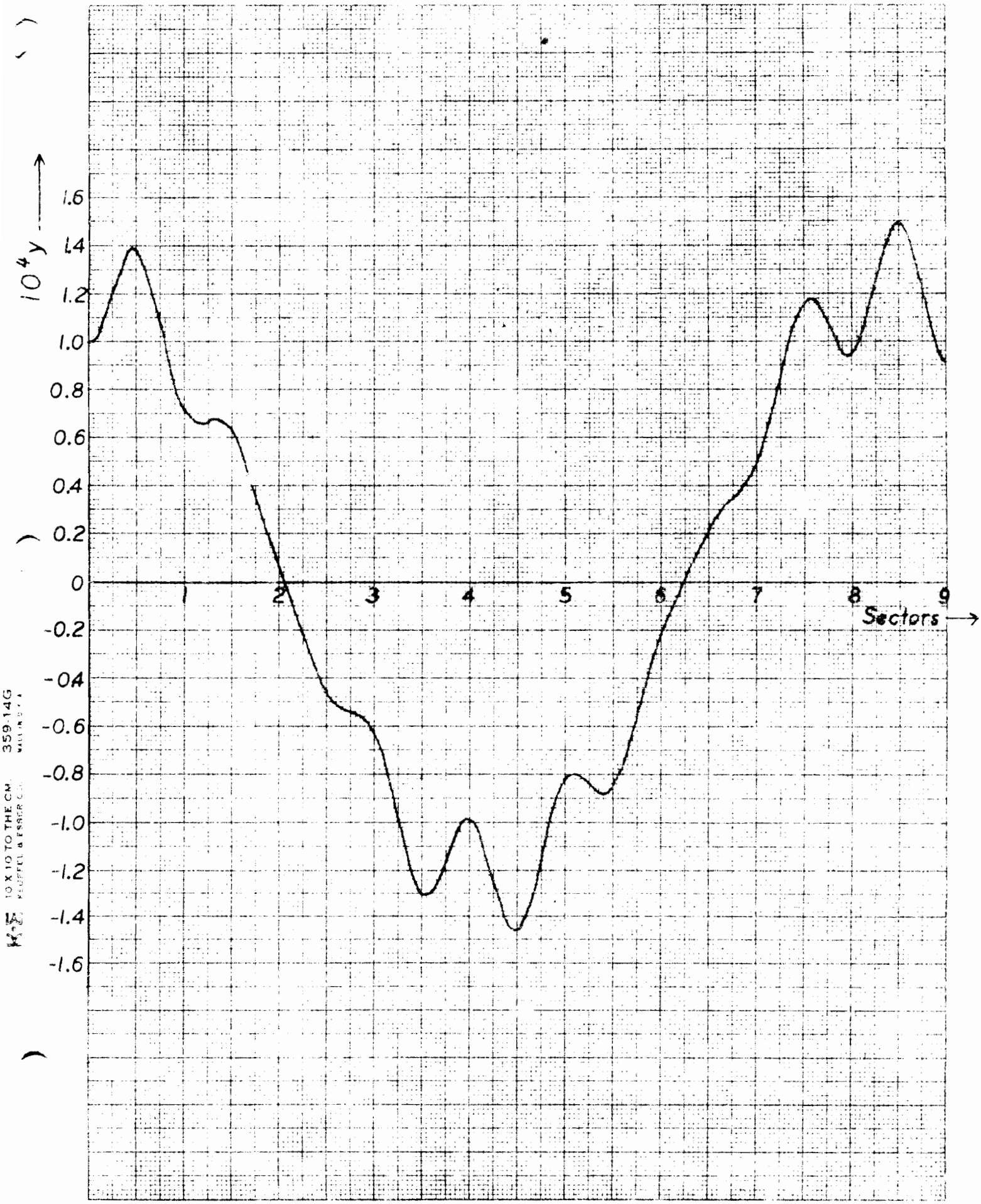
EXAMPLE OF AXIAL OSCILLATION AT OPERATING - POINT θ

$f_0 = 0.668$ $\nu_{\omega} = 19.6$ $f = \nu_f$ $N = 5$

$\frac{N\theta}{2\pi}$	$10^4 y$ Analytic Formula	$10^4 y$ Illiac Run FF 2903*	$\frac{N\theta}{2\pi}$	$10^4 y$ Analytic Formula	$10^4 y$ Illiac Run FF 2903*
0	1.000	1.000	5	-0.829	-0.823
1/4	1.212	1.210	5 - 1/4	-0.844	-0.841
1/2	1.335	1.379	5 - 1/2	-0.845	-0.843
3/4	1.084	1.069	5 - 3/4	-0.556	-0.551
1	0.734	0.715	6	-0.228	-0.221
1 - 1/4	0.694	0.673	6 - 1/4	0.003	0.009
1 - 1/2	0.647	0.623	6 - 1/2	0.212	0.216
1 - 3/4	0.379	0.355	6 - 3/4	0.349	0.350
2	0.077	0.054	7	0 494	0.497
2 - 1/4	-0.193	-0.220	7 - 1/4	0.848	0.854
2 - 1/2	-0.435	-0.462	7 - 1/2	1.156	1.161
2 - 3/4	-0.528	-0.547	7 - 3/4	1.068	1.067
3	-0.621	-0.636	8	0.953	0.954
3 - 1/4	-0.978	-0.996	8 - 1/4	1.242	1.249
3 - 1/2	-1.286	-1.303	8 - 1/2	1.484	1.494
3 - 3/4	-1.153	-1.160	8 - 3/4	1.218	1.220
4	-0.988	-0.991	9	0.905	0.907
4 - 1/4	-1.241	-1.247			
4 - 1/2	-1.452	-1.457			
4 - 3/4	-1.165	-1.161			

*$x_0 = -0.0021530$ $y_0 = 0.0001000$ $p_{x_0} = -0.0553885$ $p_{y_0} = 0$
$x_f = -0.0021530$ $p_{x_f} = -0.0553885$

FIG 5. AXIAL OSCILLATION AT OPERATING POINT θ



APPENDIX III.

STABILITY LIMITS FOR A HILL EQUATION

We are concerned here with the differential equation

$$y'' + \left[a + b \cos N\theta + (c/2) \cos (N - \nu_x)\theta \pm (c/s) \cos (N + \nu_x)\theta + d \cos \nu_x \theta\right] y = 0,$$

where we presume that $p \equiv N/\nu_x$ may be regarded as a rational number and the coefficients c and d are regarded as small. For the work to follow, the differential equation is replaced by the variational statement

$$\delta \int \tfrac{1}{2}\left\{y'^2 - \left[a + b \cos p\nu_x\theta + (c/2)\cos(p-1)\nu_x\theta \pm (c/2)\cos(p+1)\nu_x\theta + d\cos\nu_x\theta\right] y^2\right\} d\theta = 0.$$

We then proceed to determine in turn stability boundaries near $\nu_x = 2\nu_{y_0}$, $\nu_x = \nu_{y_0}$, $\nu_x + 2\nu_{y_0} = N$, and $2\nu_x + 2\nu_{y_0} = N$.

A (1) The location of the first stability limit of interest here is determined by aid of the trial function

$$y = B_1 \cos \nu_x\theta/2 + B_2 \cos 3\nu_x\theta/2 + R_1 \cos (2p-3)\nu_x\theta/2 + P_1 \cos (2p-1)\nu_x\theta/2 + P_2 \cos (2p+1)\nu_x\theta/2 + R_2 \cos (2p+3)\nu_x\theta/2,$$

although, as we shall see, inclusion of the terms with coefficients B_2, R_1, and R_2 is unnecessary for the accuracy desired here. Insertion of this trial solution into the integral and formation of the appropriate derivatives leads to the following simultaneous equations:

$$\left[\frac{\nu_x^2}{4} - a - \frac{d}{2}\right]B_1 - \frac{d}{2}B_2 - \frac{b + c/2}{2}P_1 - \frac{b \pm c/2}{2}P_2 - \frac{c}{4}R_1 \mp \frac{c}{4}R_2 = 0$$

$$-\frac{d}{2}B_1 + \left[\frac{9\nu_x^2}{4} - a\right]B_2 \mp \frac{c}{4}P_1 - \frac{c}{4}P_2 - \frac{b}{2}R_1 - \frac{b}{2}R_2 = 0$$

$$-\frac{b + c/2}{2}B_1 \mp \frac{c}{4}B_2 + \left[\frac{(2p-1)^2}{4}\nu_x^2 - a\right]P_1 - \frac{d}{2}P_2 - \frac{d}{2}R_1 = 0$$

$$-\frac{b \pm c/2}{2}B_1 - \frac{c}{4}B_2 - \frac{d}{2}P_1 + \left[\frac{(2p+1)^2}{4}\nu_x^2 - a\right]P_2 - \frac{d}{2}R_2 = 0$$

$$-\frac{c}{4}B_1 - \frac{b}{2}B_2 - \frac{d}{2}P_1 + \left[\frac{(2p-3)^2}{4}\nu_x^2 - a\right]R_1 = 0$$

$$\mp\frac{c}{4}B_1 - \frac{b}{2}B_2 - \frac{d}{2}P_2 + \left[\frac{(2p+3)^2}{4} - a\right]R_2 = 0$$

It can be seen from these equations that B_2, R_1, and R_2 are of the same order as c and d and hence that these coefficients will play no significant rôle in the first equation if terms of second and higher order in c and d are ignored.

If $d/(2p^2\nu_x^2)$, or $d/(2N^2)$, is neglected in comparison to unity, the coefficients P_1 and P_2 are then found to be given by

$$P_1 \doteq \frac{b + c/2}{2p^2\nu_x^2}\left[1 + 1/p\right] B_1$$

and

$$P_2 \doteq \frac{b \pm c/2}{2p^2\nu_x^2}\left[1 - 1/p\right] B_1 \quad .$$

By use of the first equation one then obtains, if the upper sign is used,

$$\frac{\nu_x^2}{4} - \left(a + \frac{b^2}{2p^2\nu_x^2}\right) \doteq \frac{bc}{2p^2\nu_x^2} + \frac{d}{2} \quad ,$$

or

$$\frac{\nu_x^2}{4} - \left(a + \frac{b^2}{2N^2}\right) \doteq \frac{bc}{2N^2} + \frac{d}{2} \quad .$$

If the lower sign is used, one obtains instead

$$\frac{\nu_x^2}{4} - \left(a + \frac{b^2}{2N^2}\right) \doteq \frac{\nu_x bc}{2N^3} + \frac{d}{2} \quad .$$

Since, by the results of Appendix II, $\nu_{y_0}^2 \equiv a + b^2/(2N^2)$ represents the square of the frequency of the y-oscillations for the case $c = d = 0$, we may conveniently write

$$(\nu_x/2)^2 - \nu_{y_0}^2 \doteq bc/(2N^2) + d/2 \quad ,$$

for the upper sign,

and

$$(\nu_x/2)^2 - \nu_{y_0}^2 \doteq \nu_x\, bc/(2N^3) + d/2 \quad ,$$

for the lower sign

A (2). A second stability limit to the differential equation is similarly obtained in the same neighborhood by use of the trial function

$$y = C_1 \sin \nu_x \theta/2 + \cdots + Q_1 \sin (2p-1)\nu_x \theta/2 + Q_2 \sin (2p+1)\nu_x \theta/2 + \cdots$$

In this case one finds

$$Q_1 \doteq -\frac{b - c/2}{2p^2 \nu_x^2}\left[1 + 1/p\right] C_1$$

$$Q_2 \doteq \frac{b \mp c/2}{2p^2 \nu_x^2}\left[1 - 1/p\right] C_1$$

with the relations

$$\nu_{y_0}^2 - (\nu_x/2)^2 \doteq bc/(2N^2) + d/2,$$

for the upper sign

and

$$\nu_{y_0}^2 - (\nu_x/2)^2 \doteq \nu_x\, bc/(2N^3) + d/2$$

for the lower sign.

The associated stabilify limits discussed here [subsections A (1) and A (2)] may thus be summarized as follows:

$$\left|\nu_x^2 - (2\nu_{y_0})^2\right| \doteq 2\left|\frac{bc}{N^2} + d\right|,$$

when the upper sign is taken,

and

$$\left| \nu_x^2 - (2\nu_{y_0})^2 \right| \doteq 2 \left| \frac{\nu_x bc}{N^3} + d \right|,$$

when the lower sign applies.

B. Higher-order resonances, as $\nu_x \approx \nu_{y_0}$, may involve the perturbation coefficients c and d to a higher power, but, with some additional algebraic complexity, may be investigated in a similar manner.

We consider here the equation

$$y'' + \Big[a + b\cos p\nu_x\theta + (c/2)\cos(p-1)\nu_x\theta - (c/2)\cos(p+1)\nu_x\theta + d\cos\nu_x\theta\Big]y = 0$$

or

$$\delta\int \tfrac{1}{2}\Big\{y'^2 - \Big[a + b\cos p\nu_x\theta + (c/2)\cos(p-1)\nu_x\theta - (c/2)\cos(p+1)\nu_x\theta + d\cos\nu_x\theta\Big]y^2\Big\}\,d\theta = 0,$$

in which p represents N/ν_x, and seek the stability boundaries in the neighborhood of $\nu_x \sim \nu_{y_0}$.

B.(I). We first employ the odd trial function

$$y = F_1 \sin\nu_x\theta + F_2 \sin 2\nu_x\theta + W_1 \sin(p-2)\nu_x\theta + V_1 \sin(p-1)\nu_x\theta + U \sin p\nu_x\theta + V_2 \sin(p+1)\nu_x\theta + W_2 \sin(p+2)\nu_x\theta$$

and, taking $a \ll p^2\nu_x^2 \equiv N^2$, obtain the simultaneous algebraic equations:

$$
\begin{aligned}
&(\nu_x^2-a)F_1 - \frac{d}{2}F_2 + \frac{c}{4}W_1 + \frac{b}{2}V_1 - \frac{c}{2}U - \frac{b}{2}V_2 + \frac{c}{4}W_2 = 0\\
&-\frac{d}{2}F_1 + (4\nu_x^2-a)F_2 + \frac{b}{2}W_1 - \frac{c}{4}V_1 - \frac{c}{2}V_2 - \frac{b}{2}W_2 = 0\\
&\frac{c}{4}F_1 + \frac{b}{2}F_2 + \rho^2\nu_x^2\left(1-\frac{2}{\rho}\right)^2 W_1 - \frac{d}{2}V_1 = 0\\
&\frac{b}{2}F_1 - \frac{c}{4}F_2 - \frac{d}{2}W_1 + \rho^2\nu_x^2\left(1-\frac{1}{\rho}\right)^2 V_1 - \frac{d}{2}U = 0\\
&-\frac{c}{2}F_1 - \frac{d}{2}V_1 + \rho^2\nu_x^2 U - \frac{d}{2}V_2 = 0\\
&-\frac{b}{2}F_1 - \frac{c}{4}F_2 - \frac{d}{2}U + \rho^2\nu_x^2\left(1+\frac{1}{\rho}\right)^2 V_2 - \frac{d}{2}W_2 = 0\\
&\frac{c}{4}F_1 - \frac{b}{2}F_2 - \frac{d}{2}V_2 + \rho^2\nu_x^2\left(1+\frac{2}{\rho}\right)^2 W_2 = 0.
\end{aligned}
$$

It appears appropriate to solve approximately the last six of these equations for F_2, W_1, ... W_2 in terms of the main coefficient F_1 and then to substitute the result into the first equation to obtain the location of the stability boundary in question. A guide to this solution is obtained by first solving certain special cases, in which one or more of the coefficients in the original differential equation are taken to be zero, and it may be helpful to keep in mind the result suggested for this case by the "smooth approximation"

method outlined in Appendix V. In simplifying certain correction terms the approximation may be made that ν^2 is close to $\nu_{y_0}^2 \cong a + b^2/(2p^2\nu_x^2)$.

A solution which appears adequate for the present purpose is

$$F_2 = \left[\frac{bc}{6p^3\nu_x^4} + \frac{d}{6\nu_x^2}\right]F_1$$

$$U = \left[\frac{c}{2p^2\nu_x^2} - \frac{bd}{p^5\nu_x^4}\right]F_1 \qquad \text{(in which the second term is not needed)}$$

$$V_1 = \frac{1+2/p}{p^2\nu_x^2}\left[-\frac{b}{2} + \frac{bc^2}{24p^3\nu_x^4} + \frac{cd}{24\nu_x^2}\right]F_1$$

$$V_2 = \frac{1-2/p}{p^2\nu_x^2}\left[\frac{b}{2} + \frac{bc^2}{24p^3\nu_x^4} + \frac{cd}{24\nu_x^2}\right]F_1$$

$$W_1 = \frac{1+4/p}{p^2\nu_x^2}\left[-\frac{c}{4} - \frac{b^2c}{12p^3\nu_x^4} - \frac{bd}{12\nu_x^2}\right]F_1$$

$$W_2 = \frac{1-4/p}{p^2\nu_x^2}\left[-\frac{c}{4} + \frac{b^2c}{12p^3\nu_x^4} + \frac{bd}{12\nu_x^2}\right]F_1$$

with the equation

$$(\nu_x^2 - a)F_1 - \frac{d}{2}F_2 + \frac{c}{4}(W_1 + W_2) + \frac{b}{2}(V_1 - V_2) - \frac{c}{2}U = 0$$

yielding the result [a term $bcd/(2p^5\nu_x^4)$ from U being ignored)

$$\nu_x^2 - a - \frac{b^2}{2p^2\nu_x^2} = \frac{3}{8}\frac{c^2}{p^2\nu_x^2} + \frac{b^2c^2}{12p^6\nu_x^6} + \frac{bcd}{6p^3\nu_x^4} + \frac{d^2}{12\nu_x^2}$$

$$\nu_x^2 - \nu_{y_0}^2 = \frac{3}{8}\frac{c^2}{p^2\nu_x^2} + \frac{1}{12}\left[\frac{bc}{p^3\nu_x^3} + \frac{d}{\nu_x}\right]^2$$

$$= \frac{3}{8}\frac{c^2}{N^2} + \frac{1}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2.$$

B(2). For the second stability boundary in the neighborhood of $\nu_x \sim \nu_{y_0}$ we employ the even trial function

$$y = G_0 + G_1 \cos \nu_x \theta + G_2 \cos 2\nu_x \theta$$
$$+ H_1 \cos (p-2)\nu_x \theta + J_1 \cos (p-1)\nu_x \theta + L \cos p\nu_x \theta$$
$$+ J_2 \cos (p+1)\nu_x \theta + H_2 \cos (p+2)\nu_x \theta$$

in which G_1 represents the major coefficient. The simultaneous algebraic equations obtained in this case, again taking $a \ll p^2\nu_x^2$, are:

$$-2aG_0 - d\cdot G_1 - \frac{c}{2}J_1 - b\cdot L + \frac{c}{2}J_2 = 0$$

$$-d\cdot G_0 + (\nu_x^2 - a)G_1 - \frac{d}{2}G_2 - \frac{c}{4}H_1 - \frac{b}{2}J_1 - \frac{b}{2}J_2 + \frac{c}{4}H_2 = 0$$

$$-\frac{d}{2}G_1 + (4\nu_x^2 - a)G_2 - \frac{b}{2}H_1 + \frac{c}{4}J_1 - \frac{c}{4}J_2 - \frac{b}{2}H_2 = 0$$

$$-\frac{c}{4}G_1 - \frac{b}{2}G_2 + p^2\nu_x^2\left(1-\frac{2}{p}\right)^2 H_1 - \frac{d}{2}J_1 = 0$$

$$-\frac{c}{2}G_0 - \frac{b}{2}G_1 + \frac{c}{4}G_2 - \frac{d}{2}H_1 + p^2\nu_x^2\left(1-\frac{1}{p}\right)^2 J_1 - \frac{d}{2}L = 0$$

$$-b\cdot G_0 - \frac{d}{2}J_1 + p^2\nu_x^2 L - \frac{d}{2}J_2 = 0$$

$$\frac{c}{2}G_0 - \frac{b}{2}G_1 - \frac{c}{4}G_2 - \frac{d}{2}L + p^2\nu_x^2\left(1+\frac{1}{p}\right)^2 J_2 - \frac{d}{2}H_2 = 0$$

$$\frac{c}{4}G_1 - \frac{b}{2}G_2 - \frac{d}{2}J_2 + p^2\nu_x^2\left(1+\frac{2}{p}\right)^2 H_2 = 0$$

An adequate solution of the last six of these equations taken together with the first is apparently,

$$G_0 = \left[-\frac{bc}{2p^3\nu_x^4} - \frac{d}{2\nu_x^2}\right] G_1 \qquad G_2 = \left[\frac{bc}{6p^3\nu_x^4} + \frac{d}{6\nu_x^2}\right] G_1$$

$$J_1 = \frac{1+2/p}{p^2\nu_x^2}\left[\frac{b}{2} - \frac{7}{24}\frac{bc^2}{p^3\nu_x^4} - \frac{7}{24}\frac{cd}{\nu_x^2}\right] G_1$$

$$J_2 = \frac{1-2/p}{p^2\nu_x^2}\left[\frac{b}{2} + \frac{7}{24}\frac{bc^2}{p^3\nu_x^4} + \frac{7}{24}\frac{cd}{\nu_x^2}\right] G_1$$

$$L = \left[-\frac{b^2c}{2p^5\nu_x^6} - \frac{bd}{2p^2\nu_x^4}\right] G_1$$

$$H_1 = \frac{1+4/p}{p^2\nu_x^2}\left[\frac{c}{4} + \frac{b^2c}{12p^3\nu_x^4} + \frac{bd}{12\nu_x^2}\right] G_1$$

$$H_2 = \frac{1-4/p}{p^2\nu_x^2}\left[-\frac{c}{4} + \frac{b^2c}{12p^3\nu_x^4} + \frac{bd}{12\nu_x^2}\right] G_1 \;;$$

with the second equation,

$$-d\cdot G_0 + (\nu_x^2 - a)G_1 - (d/2)G_2 - (c/4)(H_1 - H_2) - (b/2)(J_1 + J_2) = 0$$

this gives

$$\nu_x^2 - a - \frac{b^2}{2p^2\nu_x^2} = \frac{c^2}{8p^2\nu_x^2} - \frac{5}{12}\frac{b^2c^2}{p^6\nu_x^6} - \frac{5}{6}\frac{bcd}{p^3\nu_x^4} - \frac{5}{12}\frac{d^2}{\nu_x^2}$$

$$\nu_x^2 - \nu_{y_0}^2 = \frac{c^2}{8p^2\nu_x^2} - \frac{5}{12}\left[\frac{bc}{p^3\nu_x^3} - \frac{d}{\nu_x}\right]^2$$

$$= \frac{c^2}{8N^2} - \frac{5}{12}\left[\frac{bc}{N^3} - \frac{d}{\nu_x}\right]^2.$$

The results of subsections (1) and (2) can thus be summarized as

$$\frac{c^2}{8N^2} - \frac{5}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2 \leqslant \nu_x^2 - \nu_{y_0}^2 \leqslant \frac{3}{8}\frac{c^2}{N^2} + \frac{1}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2 ,$$

for the zone of instability associated with the $\nu_x \approx \nu_{y_0}$ resonance. It is noted that the width of this resonance is proportional to the <u>square</u> of the perturbation

coefficients c and d that the values of $\nu_{y_0}^2$ at the limits of stability are in general not situated symmetrically about ν_x^2, and moreover that when the term in c^2 dominates [as it will in our application with $\left(\frac{f}{\omega N^2}\right)^2$ small] the unstable region occurs exclusively for values of ν_{y_0} below ν_x.

C (1) An additional zone of instability occurs near $\nu_x + 2\nu_{y_0} = N$. We write for convenience $N = q\nu_0$ and $\nu_x = (q-1)\nu_0$, where $\nu_0 \equiv N - \nu_x$. The equations

$$y'' + \Big[a + b\cos q\nu_0\theta + (c/2)\cos\nu_0\theta \pm (c/2)\cos(2q-1)\nu_0\theta + d\cos(q-1)\nu_0\theta\Big]y = 0$$

or

$$\delta\int\frac{1}{2}\Big\{y'^2 - \Big[a + b\cos q\nu_0\theta + (c/2)\cos\nu_0\theta \pm (c/2)\cos(2q-1)\nu_0\theta + d\cos(q-1)\nu_0\theta\Big]y^2\Big\}d\theta = 0$$

are then solved approximately by the trial function

$$y = E_1\cos\nu_0\theta/2 + T_1\cos(2q-1)\nu_0\theta/2 + T_2\cos(2q+1)\nu_0\theta/2 .$$

The following conditions are found to apply (for either sign of the coefficient of $\cos(2q-1)\nu_0\theta$):

$$T_1 \doteq \left[(b+d)/(2q^2\nu_0^2)\right]E_1$$

$$T_2 \doteq \left[b/(2q^2\nu_0^2)\right]E_1$$

and, again noting $\nu_{y_0}^2 \doteq a + b^2/(2N^2)$,

$$(\nu_0/2)^2 - \nu_{y_0}^2 \doteq c/4 + bd/(2N^2).$$

C (2). A similar result, with a reversal in sign of the entire right-hand side of the equation, can be obtained for the companion stability boundary if sine functions are used in place of cosine functions in the trial solution We accordingly write

$$\left| \nu_0^2 - (2\nu_{y_0})^2 \right| \doteq \left| c + 2\, bd/N^2 \right|,$$

or

$$\left| (N-\nu_x)^2 - (2\nu_{y_0})^2 \right| \doteq \left| c + 2\, bd/N^2 \right|.$$

Summary of Stability Limits Found in Appendix III.

Stability limits for the equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-\nu_x)\theta \pm (c/2)\cos(N+\nu_x)\theta + d\cos\nu_x\theta\right] y = 0$$

are found associated with zones of instability as follows:

(1) $$\left| \nu_x^2 - (2\nu_{y_0})^2 \right| \doteq 2\left| \frac{bc}{N^2} + d \right|,$$

when the upper sign is taken;

$$\left| \nu_x^2 - (2\nu_{y_0})^2 \right| \doteq 2\left| \frac{\nu_x bc}{N^3} + d \right|,$$

when the lower sign is taken.

(2) $$\frac{c^2}{8N^2} - \frac{5}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2 \leq \nu_x^2 - \nu_{y_0}^2 \leq \frac{3}{8}\frac{c^2}{N^2} + \frac{1}{12}\left[\frac{bc}{N^3} + \frac{d}{\nu_x}\right]^2$$

when the lower sign applies.

(3)

$$\left| (N-\nu_x)^2 - (2\nu_{y_0})^2 \right| \doteq \left| c + \frac{2bd}{N^2} \right|$$

for <u>either</u> sign of the term which involves $\cos(N + \nu_x)\theta$.

(4) If we replace ν_x by $2\nu_x$ in this last result, the equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-2\nu_x)\theta \pm (c/2)\cos(N+2\nu_x)\theta + d\cos 2\nu_x\theta\right]y = 0$$

has a zone of instability defined by

$$\left| (N-2\nu_x)^2 - (2\nu_{y_0})^2 \right| \doteq \left| c + \frac{2bd}{N^2} \right| .$$

APPENDIX IV.

ESTIMATE OF CHARACTERISTIC EXPONENT IN THE UNSTABLE REGION OF A HILL EQUATION

An approximate expression may be derived for the characteristic exponent, μ, which characterizes the lapse-rate of an exponentially-growing solution in the unstable region of a Hill equation. For this purpose we follow a procedure analogous to that described by McLachlan.[13]

We denote the even and odd characteristic solutions at the associated stability boundaries by $c(\theta)$ and $s(\theta)$ respectively. The solutions near the vertex of the zone of instability may then be written approximately as

$$y = e^{\pm\mu\theta}\left[C_c(\theta) \pm S_s(\theta)\right]$$

in place of representing the Floquet factor within the bracket by expansion in a complete orthogonal set of functions. Substitution of this solution into the differential equation

$$y'' + f(a,\theta)\,y = 0$$

yields

$$C_c''(\theta) \pm S_s''(\theta) \pm 2\mu\left[C_c'(\theta) \pm S_s'(\theta)\right] + \left[\mu^2 + f(a,\theta)\right]\left[C_c(\theta) \pm S_s(\theta)\right] = 0.$$

The eigensolutions satisfy

$$c'' + f(a_1,\theta)\,c = 0 \quad , \quad s'' + f(a_2,\theta)\,s = 0.$$

and a_1 and a_2 are eigenvalues corresponding to the stability boundaries of the problem. Thus

$$\pm 2\mu\left[C_c'(\theta) \pm S_s'(\theta)\right] + C\left[\mu^2 + a - a_1\right]c(\theta) \pm S\left[\mu^2 + a - a_2\right]s(\theta) = 0.$$

The coefficients of the even and odd functions in this approximate identity may be related by multiplying through by $c(\theta)$ and by $s(\theta)$ in turn and inte-

grating, making use of the orthogonality of the (periodic) eigenfunctions which correspond to the two distinct eigenvalues a_1 and a_2. In this way one obtains

$$C\left[\mu^2 + a - a_1\right]\langle c^2\rangle + 2\mu S\langle c s'\rangle = 0$$

$$2\mu C\langle s c'\rangle + S\left[\mu^2 + a - a_2\right]\langle s^2\rangle = 0,$$

where $\langle\ \rangle$ denotes that the average value is taken. An approximate solution of the resulting determinantal equation yields

$$\mu^2 \cong -\frac{(a-a_1)(a_2-a)\langle c^2\rangle\langle s^2\rangle}{4\langle s c'\rangle\langle c s'\rangle}.$$

If the parameter "a" lies midway between the two eigenvalues a_1 and a_2, the lapse-rate will thereby be maximized.

$$\mu^2_{max.} \cong -\frac{\langle c^2\rangle\langle s^2\rangle}{\langle s c'\rangle\langle c s'\rangle}\left[\frac{a_2-a_1}{4}\right]^2$$

for $a = (a_1+a_2)/2$.

The foregoing expressions for μ^2 may readily be applied to estimate the lapse-rates associated with the resonances which form the subject of this report, employing the estimates for their respective stability boundaries derived in Appendix III. For the eigenfunctions $c(\theta)$ and $s(\theta)$ it is convenient merely to take the cosine and sine functions which constitute the dominant terms of the trial functions employed in estimating the stability boundaries.

Summary of Expressions Derived in Appendix IV for the Lapse-Rate

The lapse-rate, μ nepers/radian, characterizing unstable solutions of the differential equation

$$y'' + f(a,\theta)\,y = 0$$

is given in terms of the eigenvalues a_1, a_2 and eigenfunctions $c(\theta)$, $s(\theta)$ associated with the boundaries of the unstable region:

$$\mu^2 \cong -\frac{(a-a_1)(a_2-a)\langle c^2\rangle\langle s^2\rangle}{4\langle sc'\rangle\langle cs'\rangle},$$

$$\mu^2_{max} \cong -\frac{\langle c^2\rangle\langle s^2\rangle}{\langle sc'\rangle\langle cs'\rangle}\left[\frac{a_2-a_1}{4}\right]^2.$$

APPENDIX V.

SMOOTH APPROXIMATION METHOD

It may be recalled[5a] that Walkinshaw[4] applied a type of "smooth approximation" to an equation of the form

$$y'' + \left[a + g(\theta) + b\cos(N\theta + h(\theta))\right] y = 0,$$

in which the coefficient of y is slightly modulated in amplitude and frequency by the relatively slowly-varying (periodic) functions $g(\theta)$ and $h(\theta)$. The procedure followed appeared essentially to replace this equation by

$$y'' + \left[a + g(\theta) + \frac{b^2}{2[N + h'(\theta)]^2}\right] y = 0$$

and perform an expansion of the last term to obtain

$$y'' + \left[a + \frac{b^2}{2N^2} + g(\theta) - \frac{b^2 h'(\theta)}{N^3}\right] y = 0.$$

With the elimination of the high-frequency component, a simple Hill (Mathieu) equation is thus obtained

It may be of interest to carry through a similar procedure for the equations considered in Appendix III.

A (1). The equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N - \nu_x)\theta + (c/2)\cos(N + \nu_x)\theta + d\cos\nu_x\theta\right] y = 0$$

may be rewritten for the present purpose as

$$y'' + \left[a + d\cos\nu_x\theta + (b + c\cos\nu_x\theta)\cos N\theta\right] y = 0.$$

In this case it appears reasonable to replace this equation by

$$y'' + \left[a + d\cos\nu_x\theta + \frac{(b + c\cos\nu_x\theta)^2}{2N^2}\right] y = 0$$

or

$$y'' + \left[\nu_{y_0}^2 + \left(\frac{bc}{N^2} + d\right)\cos \nu_x \theta\right] y = 0.$$

This simple Mathieu equation will exhibit instability for ν_{y_0} near $\nu_x/2$, with the stability boundaries given by (see, for example, Appendix I)

$$\left|\nu_x^2 - (2\nu_{y_0})^2\right| \doteq 2\left|\frac{bc}{N^2} + d\right|,$$

in agreement with the result of Appendix III A.

A (2) Likewise the equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-\nu_x)\theta - (c/2)\cos(N+\nu_x)\theta + d\cos\nu_x\theta\right] y = 0$$

may be written

$$y'' + \left[a + b\cos N\theta + c\sin\nu_x\theta\sin N\theta + d\cos\nu_x\theta\right] y = 0$$

or, approximately

$$y'' + \left[a + d\cos\nu_x\theta + b\cos\left(N\theta - \frac{c}{b}\sin\nu_x\theta\right)\right] y = 0.$$

If the suggested smooth approximation is now made, one obtains

$$y'' + \left[a + d\cos\nu_x\theta + \frac{1}{2}\frac{b^2}{[N-\nu_x(c/b)\cos\nu_x\theta]^2}\right] y = 0$$

$$y'' + \left[a + \frac{b^2}{2N^2} + \left(\frac{\nu_x bc}{N^3} + d\right)\cos\nu_x\theta\right] y = 0$$

$$y'' + \left[\nu_{y_0}^2 + \left(\frac{\nu_x bc}{N^3} + d\right)\cos\nu_x\theta\right] y = 0.$$

Thus a simple Mathieu equation is again obtained, with the stability boundaries of interest now given by

$$\left|\nu_x^2 - (2\nu_{y_0})^2\right| = 2\left|\frac{\nu_x bc}{N^3} + d\right|,$$

in agreement with the result found in Appendix III A for this case.

B. On the basis of the differential equation which resulted in Section A (2) from application of the smooth approximation, it appears that a narrow zone of instability would be expected near $\nu_x = \nu_{y_0}$ for the equation

$$y'' + \left[a + b\cos N\theta + c\sin \nu_x\theta \sin N\theta + d\cos \nu_x \theta\right] y = 0.$$

To obtain a quantitative description of this resonance, the smooth approximation must be applied with somewhat more care, however, than sufficed in A (2), since the perturbation coefficients c and d now enter in a more subtle way.

We commence by rewriting our differential equation in the form

$$y'' + \left[a + b\left(1 + \frac{c^2}{2b^2}\sin^2 \nu_x\theta\right)\cos\left(N\theta - \frac{c}{b}\sin \nu_x\theta\right) + d\cos \nu_x\theta\right] y = 0,$$

a form which is equivalent through terms in c^2. Application of the "smooth approximation" in the manner employed previously, to remove the rapid variation arising from the argument $N\theta$, yields the "equivalent" Hill equation

$$y'' + \left[a + \frac{b^2}{2N^2}\frac{\left(1 + \frac{c^2}{2b^2}\sin^2 \nu_x\theta\right)^2}{\left(1 - \frac{\nu_x c}{Nb}\cos \nu_x\theta\right)^2} + d\cos \nu_x\theta\right] y = 0,$$

or

$$y'' + \left\{a + \frac{b^2}{2N^2}\left[1 + \frac{c^2}{2b^2}\left(1 + 3\frac{\nu_x^2}{N^2}\right)\right] + \left[\frac{\nu_x bc}{N^3} + d\right]\cos \nu_x\theta - \frac{c^2}{4N^2}\left(1 - 3\frac{\nu_x^2}{N^2}\right)\cos 2\nu_x\theta\right\} y = 0.$$

At the $\nu_x \approx \nu_{y_0}$ resonance, the terms involving $\cos \nu_x \theta$ can contribute to the width of the unstable zone to second order in the coefficient $\left[\frac{\nu_x bc}{N^3} + d\right]$; the terms involving c^2, on the other hand, can contribute directly and their effect will actually dominate in our application $\left(\frac{f}{\omega N^2} \text{ small}\right)$. We proceed by

application of the result of Appendix I F, with

$$\alpha = a + \frac{b^2}{2N^2}\left[1 + \frac{c^2}{8b^2}\left(1 + 3\frac{\nu_x^2}{N^2}\right)\right]$$

$$\beta = \frac{\nu_x bc}{N^3} + d$$

$$\gamma = -\frac{c^2}{4N^2}\left(1 - 3\frac{\nu_x^2}{N^2}\right)$$

$$\omega = \nu_x$$

$$\nu_{y_0}^2 \doteq a + \frac{b^2}{2N^2} ,$$

to obtain, in agreement with Appendix III B,

$$-\frac{3}{8}\frac{c^2}{N^2}\left(1+\frac{\nu_x^2}{N^2}\right) - \frac{1}{12}\left[\frac{bc}{N^3}+\frac{d}{\nu_x}\right]^2 \leqslant \nu_{y_0}^2 - \nu_x^2 \leqslant -\frac{c^2}{8N^2}\left(1+9\frac{\nu_x^2}{N^2}\right) + \frac{5}{12}\left[\frac{bc}{N^3}+\frac{d}{\nu_x}\right]^2$$

as the condition for instability.

This result gives a width of the unstable zone which is proportional to the square of the perturbation coefficients c and d. It is noted that if the term in c^2 dominates, as it normally will in our application, the instability associated with this resonance will appear only for values of ν_{y_0} below ν_x. Since ν_x^2/N^2 is presumably quite small in this approximation, the correction terms which involve this quantity might be dropped.

C. With respect to the resonance for which $\nu_x + 2\nu_{y_0} \cong N$, considered in Appendix III C, it will suffice for our application to consider here the simplified equation

$$y'' + \left[a + b\cos N\theta + (c/2)\cos(N-\nu_x)\theta\right] y = 0 .$$

Regarding the term in cos N θ as definitely more rapidly varying than that involving cos $(N-\nu_x)\,\theta$, it appears natural to replace this differential equation by

$$y'' + \left[a + \frac{b^2}{2N^2} + \frac{c}{2}\cos(N-\nu_x)\theta\right] y = 0 ,$$

or

$$y'' + \left[\nu_{y_0}^2 + \frac{c}{2}\cos(N-\nu_x)\theta\right] y = 0 .$$

Again the stability boundaries of interest are expected to be given by the condition

$$\left| (N - \nu_x)^2 - (2\nu_{y_0})^2 \right| = |c| ,$$

which is consistent with the result of Appendix III C when the coefficient d is sufficiently small that its effect can be neglected.

Summary of Results of Appendix V

Application of "smooth-approximation" methods is shown to lead to results for stability boundaries of a Hill equation in agreement with those derived by a variational method in Appendix III.