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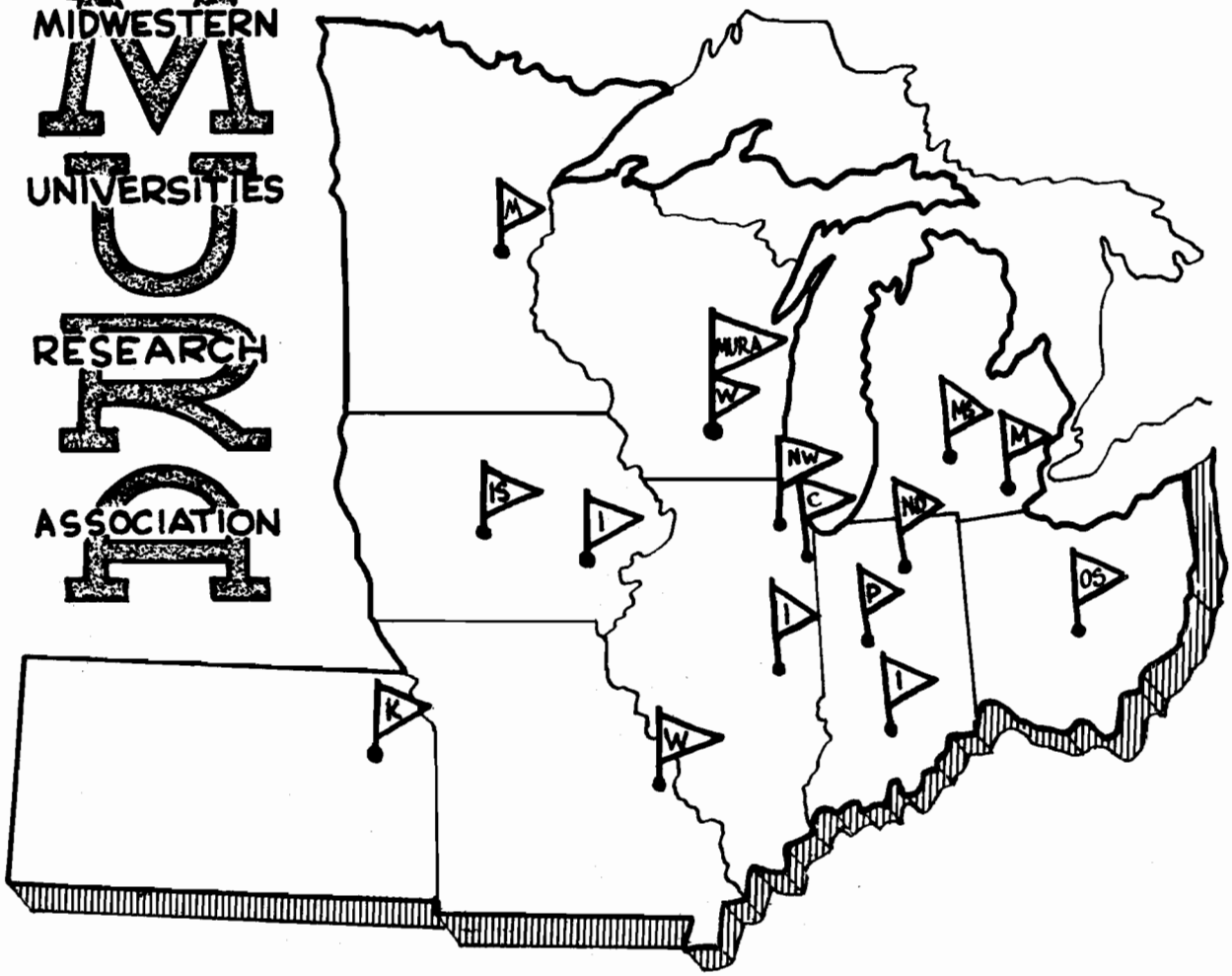

THE EQUILIBRIUM ORBIT OF A FIXED-FIELD ACCELERATOR

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THE EQUILIBRIUM ORBIT OF A FIXED-FIELD ACCELERATOR

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ABSTRACT: This paper presents a method for obtaining an approximate expression for the equilibrium orbit of a fixed field accelerator having an arbitrary magnetic field. Results are given for the general radial sector (Mark I) and the general spiral sector (Mark V) machines. It is believed that the results should be good to 10 or 20% for machines being considered at present. The results of the theory are compared with results obtained numerically.

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** On leave from University of Notre Dame.

I. Introduction

This paper is the first of a series whose final aim is to obtain expressions for the stability limits, due to non linear resonances, of an accelerator having an arbitrary magnetic field. The first step in this program is to determine the equilibrium orbit of the particle in a general magnetic field and this is the problem treated in this paper.

This problem has been treated previously by T. Ohkawa for a Mark I machine and the more general machine is being treated at present by F. T. Cole. Our procedure and results differ from theirs in that we will attempt to get an approximate solution which has an accuracy of 10 to 20%. This will allow us to make approximations at the very beginning which will simplify the equations and will give us a simple general solution. The solutions found by Ohkawa and Cole will provide more accuracy. The solution found in this paper should have the advantage of simplicity.

II. Results

In this section we will state the results for two kinds of magnetic fields. We will give the derivation in section III, where the approximations involved will be explained.

Results for the General Radial Sector Machine (Mark I)

Let us consider first the general radial sector magnetic field (the Mark I). This magnetic field has the form, in the median plane,

$$H_z = -H \left(\frac{r}{r_1}\right)^K h(\theta) . \tag{2.1a}$$

$h(\theta)$ is periodic in θ with the period $2\pi/N$ and we will expand it in a fourier series

$$h(\theta) = \sum_{n=-\infty}^{\infty} h_n e^{i\omega_n \theta} , \tag{2.1b}$$

where $\omega_n = nN$.

To find the equilibrium orbit, we expand the motion around a circle of radius, R, where R will be chosen to make the amplitude of the motion about this circle small. Thus we will write

$$r = R(1+x) , \tag{2.2}$$

where R is given by

$$R = r_1 \left(\frac{pc}{eHr_1}\right)^{\frac{1}{K+1}} b^{\frac{1}{K+1}} , \tag{2.3}$$

and b is given by

$$b = \frac{1}{\beta}\left(\sqrt{h_0^2 + 2\beta} - h_0\right) , \tag{2.4}$$

and $\beta = 4(K+2)\sum_{n\geq 1}\left[|h_n|^2/\omega_n^2\right]$

A graph of b against β is given in Fig. 1. for the case $h_0 = 1$ and for $h_0 = -1$.

The equilibrium orbit is given then by

$$x = b\sum_{n\neq 0} \frac{h_n}{\omega_n^2} e^{i\omega_n\theta} . \tag{2.5}$$

The circumference factor is given by C max, the maximum value of $C(\theta)$, where

$$C(\theta) = \left| 1 + b\sum_{n\neq 0} h_n e^{i\omega_n\theta} \right| . \tag{2.6}$$

If $h_n = 0$ for $n \geq 2$ and the h_n are real, then

$$C_{max} = 1 + 2bh_1 \tag{2.7}$$

(See Appendix B for derivation of circumference factor)

Results for the Spiral Sector Machine (Mark V).

For this machine the magnetic field has the form, in the median plane,

$$H_z = -H\left(\frac{r}{r_1}\right)^K \sum_{n=-\infty}^{\infty} h_n e^{i w_n \varphi}, \tag{2.7a}$$

where

$$\varphi = \theta - \frac{1}{wN} \ln \frac{r}{r_1}. \tag{2.7b}$$

All the results stated above for the radial sector machine, will hold for the spiral sector machine if one replaces in the above formulas h_n by $\bar{h}_n$, where

$$\bar{h}_n = h_n \exp\left[-i\frac{n}{w} \ln \frac{r}{r_1}\right]. \tag{2.8}$$

III. Derivation of the equilibrium orbit.

In this derivation, we will make two assumptions which will allow us to greatly simplify the equations.

Let us expand the motion about a circle of radius R, where R is chosen to make the motion about the circle as small as possible. The method of chosing R is given below. Let us then write

$$r = R(1+x) \tag{3.1}$$

The two assumptions we will make can then be written as

$$\left(\frac{dx}{d\theta}\right)^2 << 1, \tag{3.2a}$$

and

$$Kx << 1. \tag{3.2b}$$

The first condition just says that the slope of the equilibrium orbit $dr/d\theta$ obeys the condition $(dr/d\theta) << r$. The second condition says that the magnetic field does not vary much in the radial direction over the amplitude of oscillation in the equilibrium orbit; that is, $(Rx/H_z)(\partial H_z/\partial r) << 1$ on the orbit. It turns out, that the above two conditions are fulfilled by a broad class of machines.

The equation of motion in the median plane is

$$\frac{d}{d\theta}\left\{\frac{r'}{\sqrt{r^2+r'^2}}\right\} - \frac{r}{\sqrt{r^2+r'^2}} = \frac{e}{pc}\, r H_z \,. \tag{3.3}$$

Using condition (3.2a), that $r'^2 << r^2$, Eq. (3.3) can be simplified to

$$r'' - r = \frac{e}{pc}\, r^2 H_z \,. \tag{3.4}$$

Eq. (3.4) can be rewritten in terms of x as

$$x'' - (1+x) = \frac{e}{pc}\, R\,(1+x)^2\, H_z \,. \tag{3.5}$$

We will do all our calculations for the case of a general radial sector machine for the sake of simplicity. It will be clear, however, that any other magnetic field could be treated in the same way with very little change.

Thus we will assume that H_z is given by

$$H_z = -H\left(\frac{r}{r_1}\right)^K \sum_{n=-\infty}^{\infty} h_n\, e^{i\omega_n\theta} \,, \tag{3.6}$$

where the quantities involved are defined in Sect. II.

For this magnetic field Eq. (3.5) becomes

$$x'' - (1+x) = -\frac{eHR}{pc}\left(\frac{R}{r_1}\right)^K (1+x)^{K+2} \sum_n h_n e^{i\omega_n\theta} \,. \tag{3.7}$$

Making use of the condition $Kx << 1$, we write Eq. (3.7) as

$$x'' + (E_s - g(\theta))\,x = f(\theta), \tag{3.8a}$$

where

$$E_s = b(K+2)\,h_0 - 1, \tag{3.8b}$$

$$g(\theta) = -b \sum_{n\neq 0} (K+2)\, h_n\, e^{i\omega_n\theta} \,, \tag{3.8c}$$

$$f(\theta) = 1 - b \sum_n h_n\, e^{i\omega_n\theta} \,. \tag{3.8d}$$

and b is given by

$$b = \frac{eHR}{pc} \left(\frac{R}{r_1}\right)^K . \tag{3.9}$$

So far, the value of R and thus b has not been specified. As mentioned previously, we will chose R and b to make χ small.

One could solve Eqs. (3.8) for χ by expanding χ in a fourier series. However, we think it is more convenient to expand χ in the flocquet solutions, $u_\nu(\theta)$ of the equation

$$u'' + (E_\nu - g(\theta))\, u = 0 \tag{3.10}$$

which is just the left hand side of Eq. (3.8a) except that E_s is replaced by E_ν, and E_ν may take on all values for which Eq. (3.10) has stable solutions. These u_ν can be made to form an orthonormal complete set (see the report MURA-200 for details)

We expand χ in these flocquet solutions as

$$\chi = \sum_i a_i\, u_i(\theta) \tag{3.11}$$

where $u_i(\theta)$ is the solution of Eq. (3.10) having the tune ν_i and the E -value E_i.

Putting (3.11) in Eq. (3.8a) we can find the a_i using the orthonormality of the u_i. The a_i are given by

$$a_i = \frac{1}{E_s - E_i} \int d\theta\; u_i^*(\theta)\, f(\theta), \tag{3.12}$$

and thus our equilibrium orbit is given by

$$\chi = \sum_i u_i(\theta)\, \frac{1}{E_s - E_i} \int d\theta'\, u_i^*(\theta')\, f(\theta') . \tag{3.13}$$

Now we will chose R and b so that the term in the expansion Eq. (3.11) which correspond to $\nu_i = 0$ vanishes. This means that b is determined by

the condition

$$\int d\theta \, u_0^*(\theta) f(\theta) = 0, \tag{3.14}$$

where $u_0(\theta)$ is the flocquet solution of Eq. (3.10) corresponding to $\nu = 0$.

The flocquet solution $u_0(\theta)$ is given by[1]

$$u_0(\theta) = 1 - \sum_{n \neq 0} \frac{g_n}{\omega_n^2} e^{i\omega_n \theta}, \tag{3.15}$$

where $g(\theta) = \sum_{n \neq 0} g_n e^{i\omega_n \theta}$, and for the radial-sector machine (Mark I), g_n is given by Eq. (3.8c) as

$$g_n = -b(K+2) h_n. \tag{3.16}$$

Substituting Eq. (3.15) for u_0 into Eq. (3.14) and using Eq. (3.8d) for $f(\theta)$, we get an equation for b

$$1 - b h_0 - 2b^2 (K+2) \sum_{n \geq 1} \frac{|h_n|^2}{\omega_n^2} = 0 \tag{3.17}$$

By solving Eq. (3.17) for b, we find

$$b = \frac{1}{\beta} \left\{ \sqrt{h_0^2 + 2\beta} - h_0 \right\}, \tag{3.18}$$

where $\beta = 4(K+2) \sum_{n \geq 1} \left[|h_n|^2 / \omega_n^2 \right]$. This determines b, and now we can find R by using Eq. (3.9)

$$R = r_1 \left(\frac{pc}{eHr_1} \right)^{\frac{1}{K+1}} b^{\frac{1}{K+1}}. \tag{3.19}$$

In Eq. (3.18) for b, we chose the plus sign of the radical to get a positive b. We could also choose the minus sign, which would mean that b, and thus p, is negative and this choice of sign will lead to the equilibrium orbit for the particle going in the opposite direction.

Now let us return to Eq. (3.13) and find the equilibrium orbit. Let us notice that since $f(\theta)$ is periodic, only those $u_i(\theta)$ appear for which $\nu_i = 0, \pm N, \pm 2N \ldots$

[1] G. Parzen - MURA-200

We will assume that $E_s << N^2$ or according to Eq. (3.8b) that $bKh_0/N^2 << 1$. This will turn out to be guaranteed by the condition $Kx << 1$. Then in Eq. (3.13) we will drop E_s compared with E_i, as $E_i \simeq \nu_i^2$ so $E_i \gtrsim N^2$.

Further, the flocquet solutions $u_i(\theta)$ in Eq. (3.13) for $\nu_i = \omega_n$, $n \neq 0$, can be taken as just $e^{i\omega_n\theta}$ (see Appendix A).

Thus Eq. (3.13) gives for the equilibrium orbit

$$x = b \sum_{n \neq 0} \frac{h_n}{\omega_n^2} e^{i\omega_n\theta} . \tag{3.14}$$

Now that we have found the equilibrium orbit, let us find what restrictions conditions (3.2) put on the magnetic field. The condition $(dx/d\theta)^2 << 1$ gives

$$\frac{1}{n^2} (bh_n/N)^2 << 1 , \tag{3.15a}$$

and the condition $Kx << 1$ gives

$$b\left(\frac{K}{N^2}\right) h_n \frac{1}{n^2} << 1 . \tag{3.15b}$$

In Section IV, we will show how well these conditions are fulfilled.

The above derivation was carried through for the general radial-sector magnetic field. The general spiral-sector field goes through in much the same way. The results are given in Sect. II.

IV. Comparison with numerical results

We will compare our theoretical results with numerical calculations for two radial sector machines. Machine I is a "small" machine with $N = 12$. Machine II is a "large" machine with $N = 54$.

We will find x at $\theta = 0$ on the equilibrium orbit which is the "stable fixed point".

Let us introduce the variable $\bar{x}$ defined by

$$r = r_1 (1 + \bar{x}). \tag{4.1}$$

$\bar{x}$ is the quantity computed on the IBM 704 and it is related to x in our paper by

$r_1(1+\bar{x}) = R(1+x)$ or

$$\bar{x} = x + (1+x)\left(\frac{R}{r_1} - 1\right). \tag{4.2}$$

If the momentum p, and H and r_1 in Eq. (2.7a) for the magnetic field, are so chosen that $pc = eHr_1$ then by Eq. (2.3), $R = r_1 b^{\frac{1}{k+1}}$, and Eq. (4.2) can be written as

$$\bar{x} = x + (1+x)\left(b^{\frac{1}{k+1}} - 1\right). \tag{4.3}$$

Machine I is defined by $N = 12$, $K = 8$, $h_0 = 1$ $h_1 = 3.1$

Machine II is defined by $N = 54$, $K = 168$, $h_0 = 1$, $h_1 = 3.1$

Comparison of theory and experiment is given in the following table

	$\bar{x}$ at $\theta = 0$.	
Machine	Theory	Experiment
I	-.030	-.0376
II	-1.74×10^{-3}	-1.83×10^{-3}

Let us see how well conditions (3.2) are obeyed.

For Machine I, b = .61

and $(dx/d\theta)^2 \simeq (b h_1/N)^2 = .025$

and $Kx \simeq b(K/N^2) h_1 = .1$

For Machine II, b = .6

and $(dx/d\theta)^2 \simeq 1.23 \times 10^{-3}$

and $Kx \simeq .1$

Thus we can expect an accuracy of 10 - 20% in our results.

Appendix A

In Section III, we assumed that $u_i(\theta) \simeq e^{i\omega_n\theta}$ and that $E_i \simeq \omega_n^2$ when the tune $\nu_i = \omega_n$. We would like to discuss here the error introduced by these assumptions.

The error in the above approximations for $u_i(\theta)$ and E_i is of order g_{2n}/N^2, the g_n being defined by Eq. (3.16). The error is then of order $bK\,h_{2n}/N^2$. This error can at most be as large as $K\chi \ll 1$ if there is a large second harmonic

Appendix B

We would like to indicate here the derivation of the circumference factor results given in Section II.

Let $\rho(\theta)$ be the radius of curvature of the orbit and let ρ min be the minimum value. We define the circumference factor as $C_{max} = R/\rho_{min}$. Now since $1/\rho = eH_z/pc$ we can use Eq. (3.5) to eliminate H_z and we find for $C(\theta) = R/\rho(\theta)$

$$C(\theta) = |\chi'' - 1| . \tag{B.1}$$

In Eq.(B.1) we have assumed $\chi \ll 1$ and have dropped terms of order χ. Now using Eq. (3.14) for χ, the equilibrium orbit, we find that

$$C(\theta) = \left| 1 + b \sum_{n \neq 0} h_n e^{i\omega_n\theta} \right| . \tag{B.2}$$

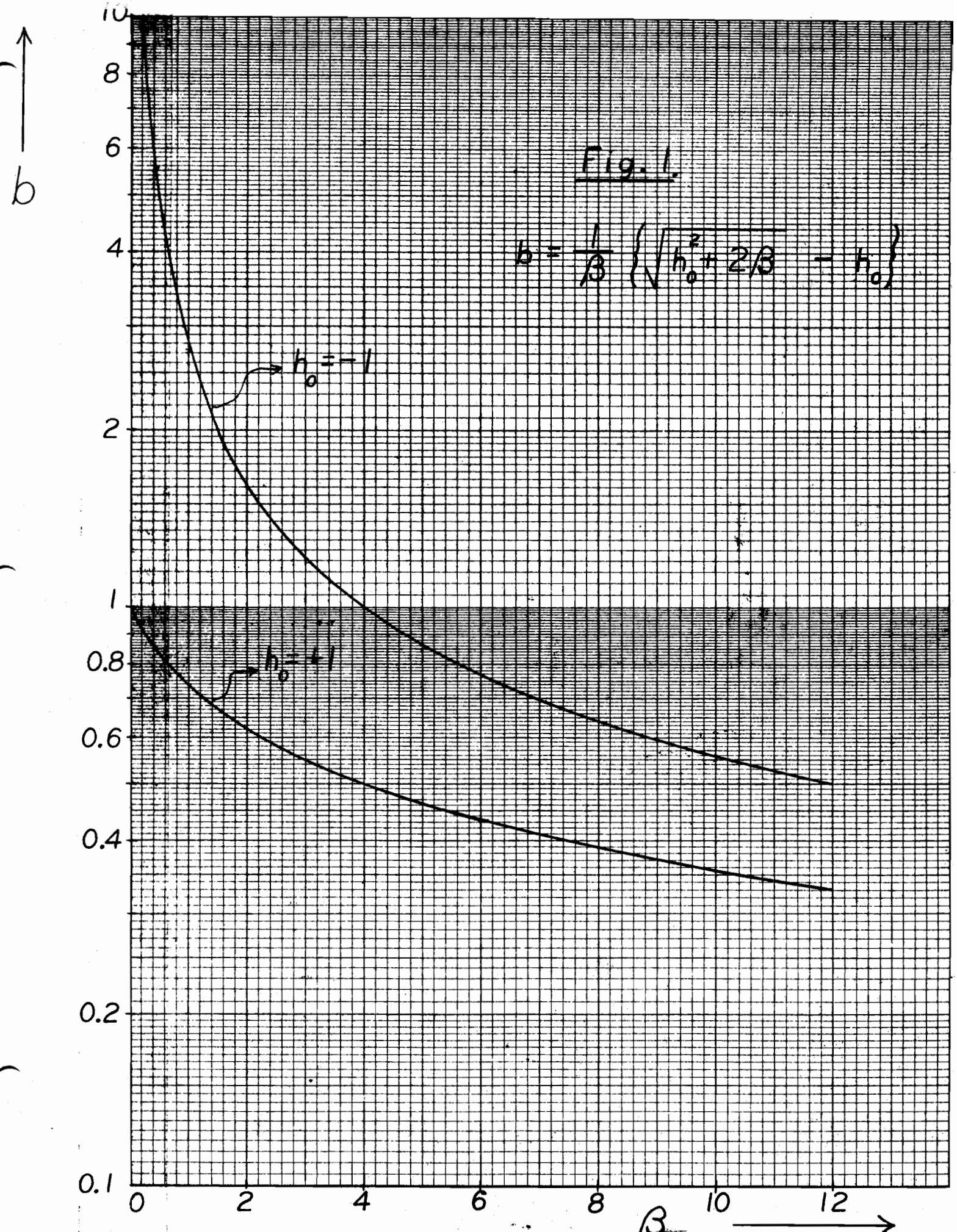
Fig. 1.
$b = \frac{1}{\beta}\left\{\sqrt{h_0^2 + 2\beta} - h_0\right\}$
$h_0 = -1$
$h_0 = +1$
b
β
10
8
6
4
2
1
0.8
0.6
0.4
0.2
0.1
0
2
4
6
8
10
12