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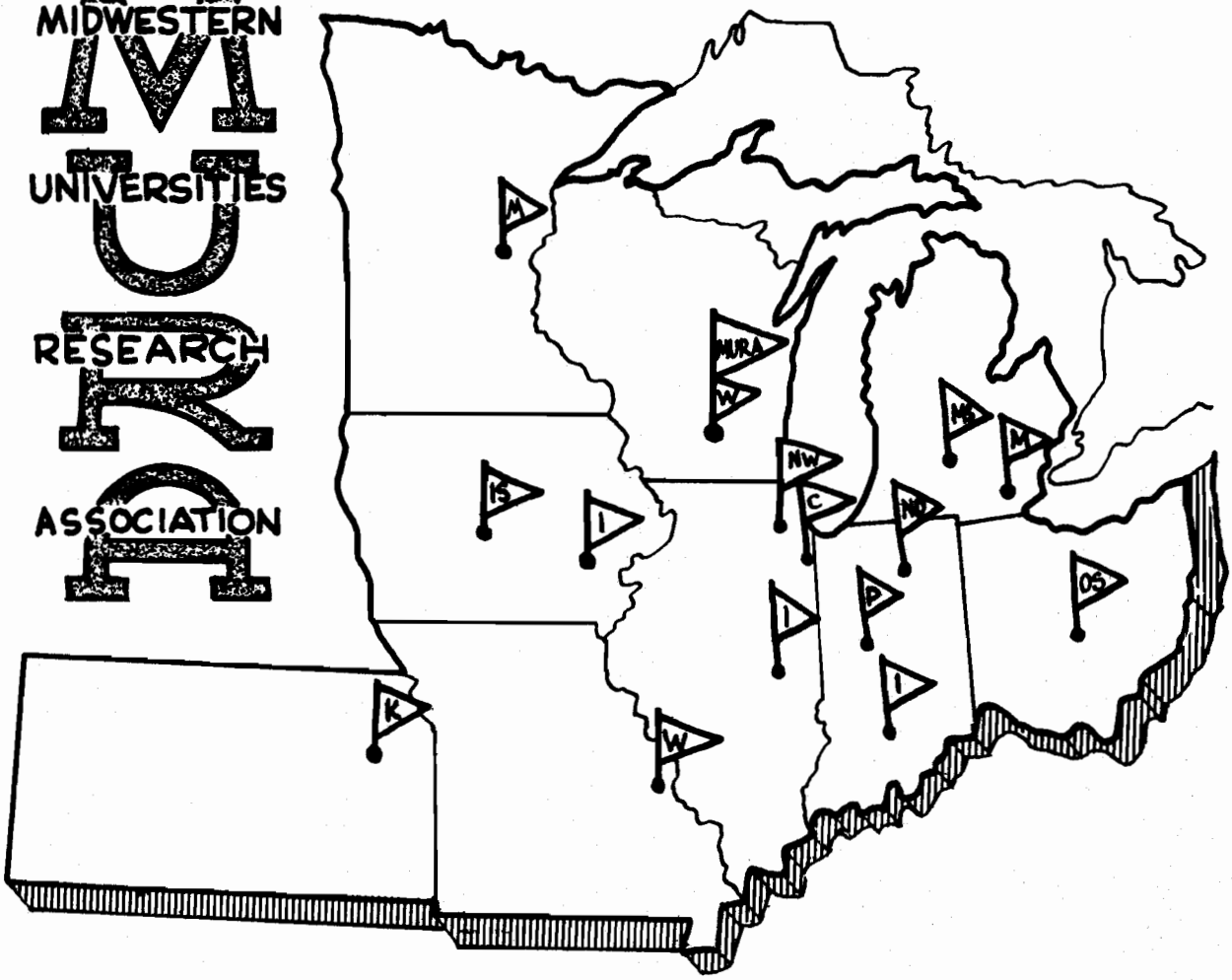

OPTIMUM SPACING OF POLYGONAL COILS FOR UNIFORM CENTRAL FIELD

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OPTIMUM SPACING OF POLYGONAL COILS FOR UNIFORM CENTRAL FIELD

L. Jackson Laslett**

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ABSTRACT: The condition for maximum uniformity of the central magnetic field produced by a pair of polygonal coils is derived. In the limit that the number of sides of the polygons approaches infinity, the condition approaches that given by the Helmholtz relation. A table of optimum spacings for polygons of various number of sides is presented.

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** On leave from Iowa State College, Ames, Iowa.

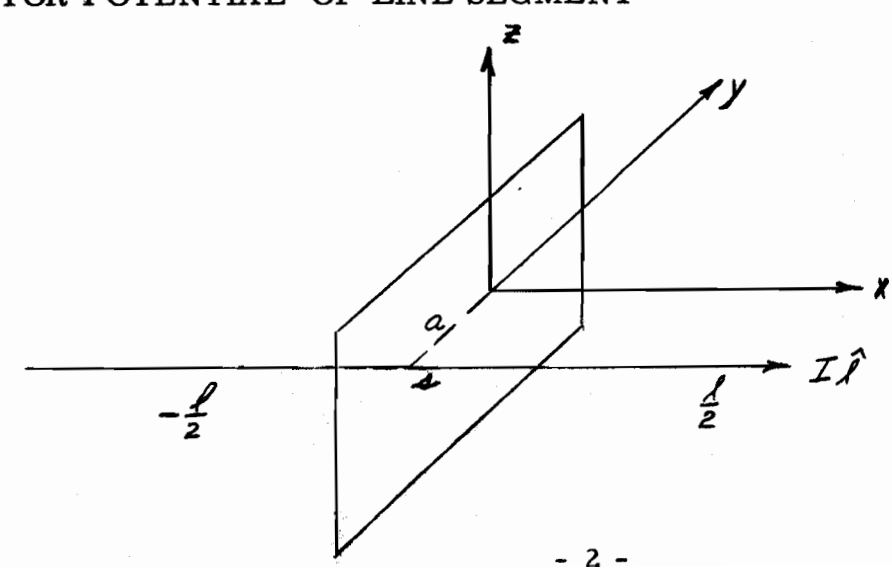
I. MOTIVATION

For the purpose of annulling the earth's magnetic field, interest has been expressed in the uniformity of field midway between two thin coaxial parallel polygonal coils of identical shape and size. With circular coils, maximum uniformity of the central field leads, of course, to the HELMHOLTZ spacing and the same type of optimization would be of interest for polygonal coils, which may in some cases be more conveniently constructed or installed.

II. METHOD

We first derive the contribution to the vector potential, at points near the axis of a single complete coil, from the current in a line segment representing one of the sides of the polygon. This result need only be developed through first order terms in the distance of the field point from the coil axis. The resultant expression is summed over the n sides of the polygon and its curl taken to give the axial field of one complete coil. The sum of two such expressions gives the axial field of the coil-pair and may be expanded in even powers of the distance z of the field point from the point midway between the coils. Suppression of the coefficient of z^2 in this expansion then constitues the criterion for maximum uniformity of the field.

III. VECTOR POTENTIAL OF LINE SEGMENT



In unrationalized (absolute) units we take, for a line-segment of length ℓ,

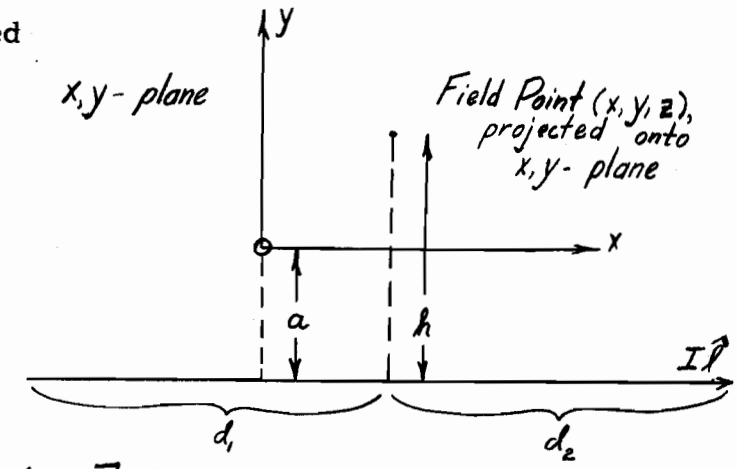
$$\frac{A_x}{\mu I} = \int \frac{ds}{r}$$

$$= \int_{-\ell/2}^{\ell/2} \frac{ds}{\sqrt{(s-x)^2 + (y+a)^2 + z^2}}$$

$$= \operatorname{Sinh}^{-1} \frac{\ell/2 + x}{\sqrt{z^2+(y+a)^2}} + \operatorname{Sinh}^{-1} \frac{\ell/2 - x}{\sqrt{z^2+(y+a)^2}}$$

$$= 2 \operatorname{Sinh}^{-1} \frac{\ell/2}{\sqrt{z^2 + (y+a)^2}} + O(x^2).$$

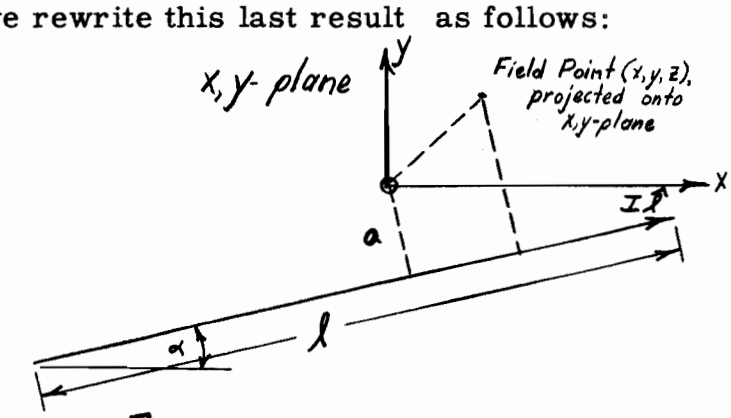
This result may be conveniently expressed



$$\frac{\vec{A}}{\mu I} = \left[\operatorname{Sinh}^{-1} \frac{d_1}{\sqrt{z^2+h^2}} + \operatorname{Sinh}^{-1} \frac{d_2}{\sqrt{z^2+h^2}}\right] \hat{\ell}$$

$$\cong 2\left[\operatorname{Sinh}^{-1} \frac{\ell/2}{\sqrt{z^2+h^2}}\right] \hat{\ell},$$

where $\hat{\ell}$ denotes a unit-vector in the direction of the current. In preparation for summing over the n sides of a regular polygon we rewrite this last result as follows:



$$\frac{\vec{A}}{\mu I} = 2\left[\operatorname{Sinh}^{-1} \frac{\ell/2}{\sqrt{z^2+(a-x\sin\alpha+y\cos\alpha)^2}}\right]\left[\hat{i}\cos\alpha + \hat{j}\sin\alpha\right]$$

$$\cong 2\left[\operatorname{Sinh}^{-1} \frac{\ell/2}{\sqrt{z^2+a^2}} + \frac{a(\ell/2)(x\sin\alpha - y\cos\alpha)}{[z^2+a^2]\sqrt{z^2+a^2+(\ell/2)^2}}\right]\left[\hat{i}\cos\alpha + \hat{j}\sin\alpha\right].$$

IV. VECTOR POTENTIAL AND AXIAL FIELD OF A SINGLE COMPLETE POLYGONAL COIL

The vector potential for a complete polygonal coil (regular polygon) is found by summing, over alpha, the expression just found. We take $\alpha = \alpha_0 + \frac{2\pi}{n} m$, with $m = 0, 1, \ldots n-1$ and obtain $(n > 2)$

$$\frac{\vec{A}}{\mu I} \doteq \frac{nal}{2[z^2+a^2]\sqrt{z^2+a^2+(l/2)^2}}\left[x\hat{j} - y\hat{i}\right]$$

$$= \frac{\text{Area}}{[z^2+a^2]\sqrt{z^2+a^2+(l/2)^2}}\left[x\hat{j} - y\hat{i}\right]$$

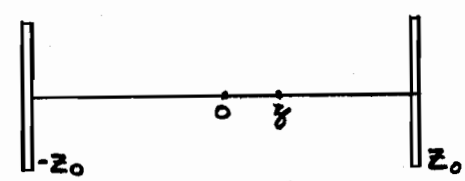
$$= \frac{\text{Area}}{[z^2+a^2]\sqrt{z^2+a^2+\sec^2 \pi/n}}\left[x\hat{j} - y\hat{i}\right].$$

The axial flux-density, $\vec{B} = \text{curl}\,\vec{A}$, is then given by

$$\frac{\vec{B}_0}{\mu I} = \frac{2\cdot\text{Area}}{[z^2+a^2]\sqrt{z^2+a^2\sec^2 \pi/n}}\,\hat{k}\,.$$

(It may be noted that this reduces to the well-known result for a circular coil when $n \to \infty$ ($\sec \pi/n \to 1$).)

V. AXIAL FIELD OF A COIL-PAIR



For a parallel coaxial pair of such coils, carrying current in the same sense, we now write

$$\frac{B_0}{\mu I} = 2\cdot\text{Area}\left\{\frac{1}{\sqrt{(z_0+z)^2+a^2\sec^2\pi/n}\,\left[(z_0+z)^2+a^2\right]} + \frac{1}{\sqrt{(z_0-z)^2+a^2\sec^2\pi/n}\,\left[(z_0-z)^2+a^2\right]}\right\}$$

$$\doteq 2\cdot\text{Area}\left\{\frac{2}{\sqrt{z_0^2+a^2\sec^2\pi/n}\,\left[z_0^2+a^2\right]} + z^2\cdot\frac{d^2}{dz_0^2}\left[\frac{1}{\sqrt{z_0^2+a^2\sec^2\pi/n}\,\left[z_0^2+a^2\right]}\right]\right\}.$$

It may be noted that, to the order considered, the field is, in fact, that derivable from the scalar potential

$$\frac{V}{\mu I} = -2 \cdot \text{Area}\left\{ \frac{2z}{\sqrt{z_0^2 + a^2 \sec^2 \pi/n}\,\left[z_0^2 + a^2\right]} + \frac{2z^3 - 3(x^2+y^2)z}{6} \frac{d^2}{dz_0^2}\left[\frac{1}{\sqrt{z_0^2 + a^2 \sec^2 \pi/n}\,\left[z_0^2 + a^2\right]}\right]\right\}.$$

The second derivative which appears in these formulas is

$$\frac{d^2}{dz_0^2}\left[\ \right] = \frac{\left[\begin{array}{l} 12 z_0^6 + 3 z_0^4 a^2 (2 + 5 \sec^2 \pi/n) \\ + 2 z_0^2 a^4 (1 - \sec^2 \pi/n + 3 \sec^4 \pi/n) - a^6 \sec^2 \pi/n (1 + 2 \sec^2 \pi/n) \end{array}\right]}{(z_0^2 + a^2 \sec^2 \pi/n)^{5/2} (z_0^2 + a^2)^3}.$$

VI. CONDITION FOR UNIFORMITY OF CENTRAL FIELD

Maximum uniformity of the central field is attained by so proportioning "z_0" and "a" as to cause the coefficient of z^2 in B_0 to vanish, or the coefficient of $2z^3 - 3(x^2 + y^2)z$ in V to vanish. In terms of the dimensionless ratio $\lambda \equiv z_0/a$, this condition leads to the following cubic requation in λ^2 :

$$12\lambda^6 + 3(2 + 5\sec^2 \pi/n)\lambda^4 + 2(1 - \sec^2 \pi/n + 3\sec^4 \pi/n)\lambda^2 - \sec^2 \pi/n (1 + 2\sec^2 \pi/n) = 0.$$

VII. SPECIAL CASES

A. For the case of circular coils, $n = \infty$ and $\sec^2 \pi/n = 1$. The equation for λ then becomes

$$12\lambda^6 + 21\lambda^4 + 6\lambda^2 - 3 = 0 \qquad \text{or}$$

$$4\lambda^6 + 7\lambda^4 + 2\lambda^2 - 1 = 0 \qquad \text{or}$$

$$4(\lambda^2 - 1/4)(\lambda^2 + 1)^2 = 0.$$

The root $\lambda = 1/2$ corresponds to the well-known Helmholtz condition $z_0 = a/2$.

B. For the case of square coils, n=4 and $\sec^2 \pi/n = 2$. The equation for λ now assumes the form

$$12\lambda^6 + 36\lambda^4 + 22\lambda^2 - 10 = 0 \qquad \text{or}$$

$$6\lambda^6 + 18\lambda^4 + 11\lambda^2 - 5 = 0.$$

This equation has previously been obtained by R. O. Haxby (private communication), and has the approximate root

$$\lambda \cong 0.5445$$

$$z_0 \cong 0.5445\,a.$$

VIII. TABLE OF SOLUTIONS

The equation for λ may be conveniently rewritten in terms of $\lambda^2 = (1/4) + \psi$ and $\sec^2 \pi/n = 1 + \epsilon$ $\quad (\epsilon = \tan^2 \pi/n)$:

$$[300 + 8\epsilon(12\epsilon + 35)]\psi$$
$$= \epsilon(8\epsilon + 25) - 240(\epsilon + 2)\psi^2 - 192\psi^3.$$

This last-named equation may conveniently be solved by iteration. Solutions have been obtained by aid of the MURA I. B. M. -704 ALGYTEE computational program for a number of values of n, with the results tabulated below.

n	$\Psi = \lambda^2 - 0.25$	$\lambda \equiv z_0/a$
3	.070 355 89	.565 9999
4	.046 486 40	.544 5056
5	.031 245 47	.530 3258
6	.022 142 51	.521 6728
7	.016 435 01	.516 1734
8	.012 657 07	.512 5008
9	.010 037 52	.509 9387
10	.008 150 54	.508 0852
11	.006 747 72	.506 7028
12	.005 677 20	.505 6453
15	.003 641 80	.503 6286
16	.003 202 29	.503 1921
20	.002 051 95	.502 0478
25	.001 314 25	.501 3125
30	.000 913 03	.500 9122
32	.000 802 57	.500 8019
40	.000 513 80	.500 5135
50	.000 328 88	.500 3288
64	.000 200 76	.500 2007
100	.000 082 24	.500 0822

It appears that a limiting algebraic form for the solution λ, valid for large n, is

$$\lambda \doteq \frac{1}{2} + \frac{1}{12}\frac{\pi^2}{n^2} - \frac{49}{3600}\frac{\pi^4}{n^4}$$

$$\doteq 0.5 + 0.82246703/n^2 - 1.325846/n^4$$

$$\doteq 0.5 + 0.82247/n^2 - 1.326/n^4.$$

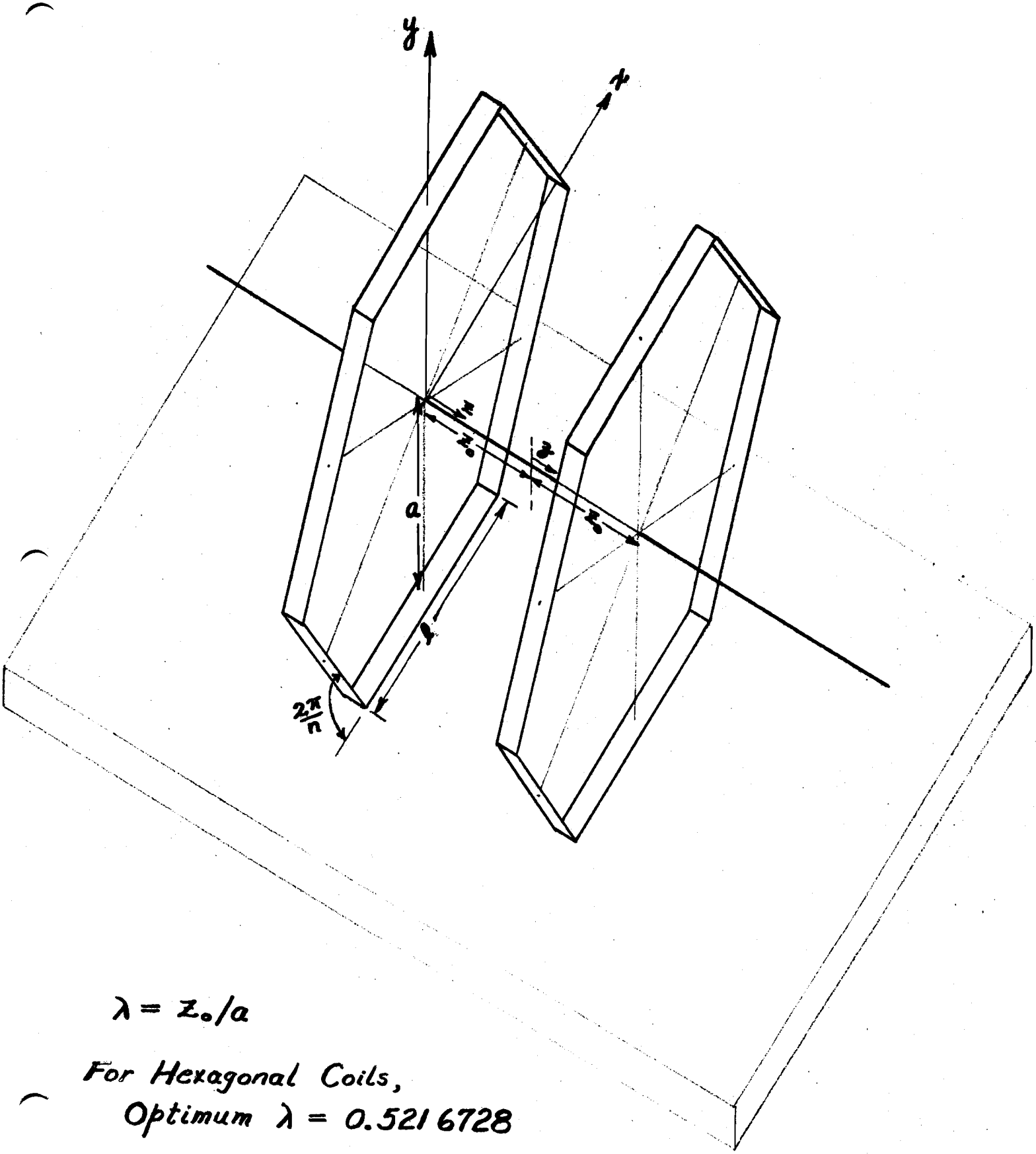
IX. SUMMARY

The central field from a pair of polygonal coils is given by

$$\frac{B_0}{\mu I} = \frac{4.\ \text{Area}}{\sqrt{z_0^2 + a^2 \sec^2 \pi/n}\ \left[z_0^2 + a^2\right]}$$

where I denotes the number of ab-ampere turns on a single coil. The optimum spacing is given by the values of $\lambda \equiv z_0/a$ listed in the Table.

It is recognized that it would be of interest to generalize the work reported here to treat coils of appreciable thickness and to give the appropriate spacing for the production of fields uniform over an extended area.



$\lambda = z_0/a$

For Hexagonal Coils,
Optimum $\lambda = 0.5216728$