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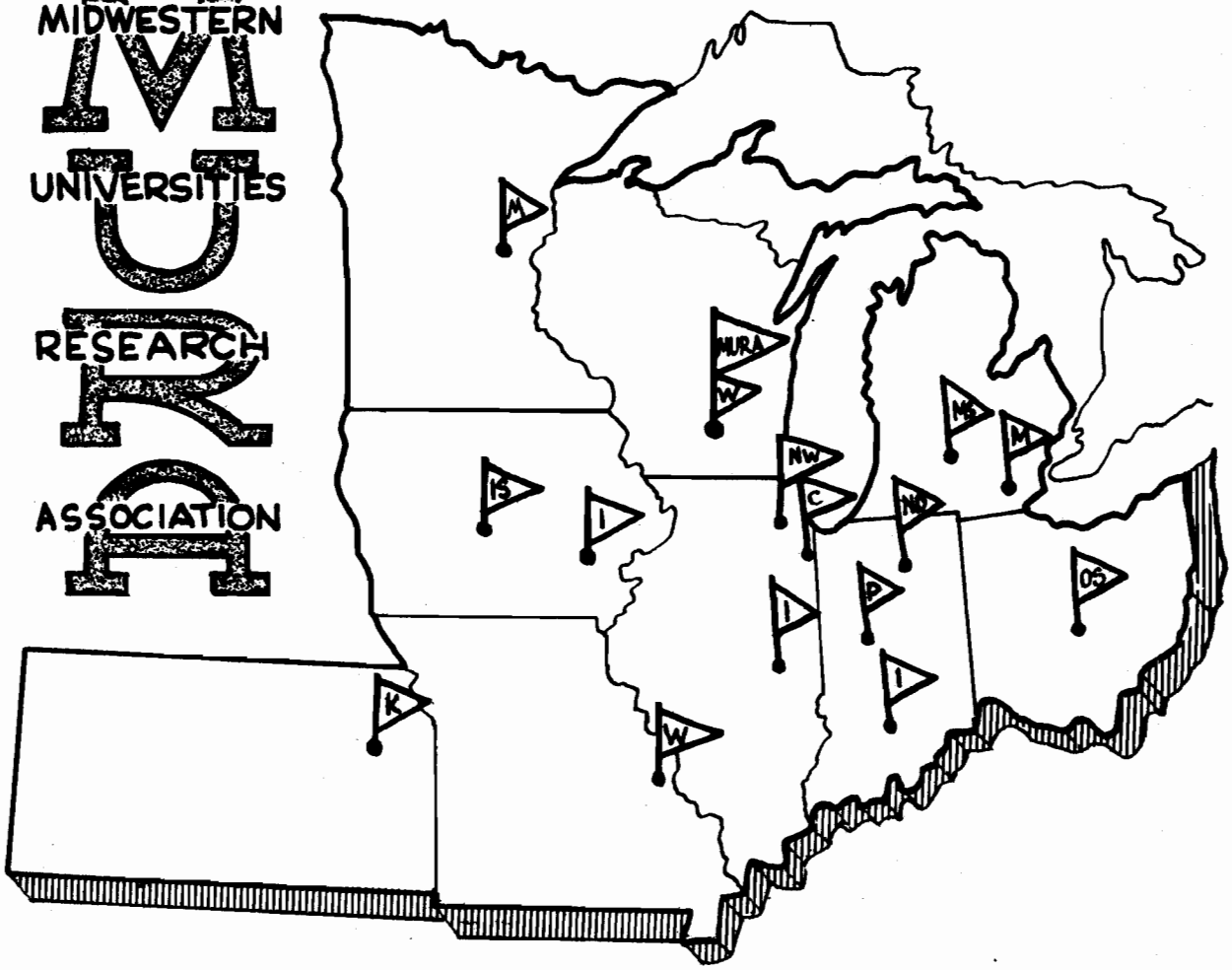

APPROXIMATE SOLUTIONS TO THE MATHIEU EQUATION

L. Jackson Laslett and A. M. Sessler

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APPROXIMATE SOLUTIONS TO THE MATHIEU EQUATION

L. Jackson Laslett* and A. M. Sessler**

April 10, 1957

ABSTRACT: Floquet solutions and the coefficients of a trigonometric series representing such solutions were obtained computationally for the Mathieu equation

$$y'' + (A + B\cos 2t)y = 0$$

for representative cases with B = 1.5 or 1.0 and $0 < \sigma < \pi$. Algebraic formulas, in good agreement with the computational results, are given for the important coefficients in the series expansion.

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I. INTRODUCTION

For analytic work concerning the motion of particles in a cyclic accelerator, as in the study of imperfections when 2ν has values near an integer, solutions to the Mathieu equation may be required which remain accurate when σ is near π. Results which appear to describe such solutions adequately are presented here, for convenient reference in future work, although it is recognized that essentially the same problem has received considerable attention previously by other workers,[1-6] motivated either by accelerator problems or by an interest in solid-state physics.

The present report consists of two parts: Firstly, the Floquet solution of the Mathieu equation is obtained in representative cases and analysized in a series of cosine functions by means of digital computation (MURA I.B.M. "DUCK-ANSWER" and "FORANAL" programs). The adequacy of retaining only a limited number of terms in this expansion is examined. Secondly, algebraic expressions for the coefficients in such an expansion are obtained by harmonic balance and compared with the coefficients given by digital computation.

The differential equation with which we shall be concerned throughout the report is written

$$y'' + [A + B\cos Nt]y = 0 , \tag{1}$$

with $N = 2$ and with representative values of the constants considered to be

$B = 1.0$ or 1.5

A such that $0 < \sigma < \pi$.

Expansion of the <u>even</u> Floquet solution ($y_0 = 1$, $y_0' = 0$) in the form

$$y = g_0 \cos \nu t + \sum_{m=1} \left[f_m \cos (mN - \nu) t + g_m \cos (mN + \nu) t \right] \tag{2a}$$

$$= g_0 \cos \nu t + \sum_{m=1} \left[(f_m + g_m) \cos mNt \cos \nu t + (f_m - g_m) \sin mNt \sin \nu t \right] \tag{2b}$$

is specifically treated, from which the general solution may be written by replacing ν t with $\nu t + \epsilon$:

$$y = g_0 \cos (\nu t + \epsilon) + \sum_{m=1} \left\{ f_m \cos \left[(mN - \nu) t - \epsilon \right] + g_m \cos \left[(mN + \nu) t + \epsilon \right] \right\}. \tag{2c}$$

Note that, with N=2, ν represents σ/π. In terms of this notation it may also be noted that the matrix which carries the solution (y y') between $t = 0$ and $t = 2\pi/N$ is $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

$$= \begin{pmatrix} \cos \sigma & \dfrac{g_0 + \sum_{m=1} (f_m + g_m)}{\nu g_0 - \sum_{m=1} \left[(mN - \nu) f_m - (mN + \nu) g_m \right]} \sin \sigma \\ \dfrac{\nu g_0 - \sum_{m=1} \left[(mN - \nu) f_m - (mN + \nu) g_m \right]}{g_0 + \sum_{m=1} (f_m + g_m)} \sin \sigma & \cos \sigma \end{pmatrix}.$$

II. COMPUTATIONAL RESULTS:

A series of DUCK-ANSWER solutions to the equation $y'' + (A + B\cos 2t)y = 0$ were obtained with B = 1.5 and with B = 0. Values of A were chosen which were intended to lead to values of σ/π close to simple rational fractions in the range between 0 and 1. From the computational solutions, $M_{11} = M_{22} = \cos\sigma$ (and, hence, σ/π) could be determined, as could also the remaining matrix-elements M_{12} and M_{21}. These results are summarized in the accompanying table.

Taking as a guide the nearest simple rational fraction which would serve to represent σ/π in each case, an interval of t was found within which the solution was substantially periodic. The solution within such an interval was then subjected to Fourier analysis, by the FORANAL program, to yield the coefficients of a cosine series representing the solution. The coefficients so obtained are given in tabular form below and serve as the basis for the curves denoted "Digital" on the graphs appended to this report.

From the results of these computations - - considered representative of particle motion in a spirally-ridged FFAG accelerator with sinusoidally varying median-plane field -- it appears that the coefficients g_2 and certainly f_3 are quite small. One measure of the adequacy of the previous coefficients to describe the solution is provided by a comparison, given in tabular form below (for the case B = 1.5 and $\sigma = 0.889\pi$), of the actual solution with that computed from a limited number of coefficients. A more sensitive test, which may be relevant only in certain applications, is a comparison of the actual computed matrix-element M_{12} with that given by retaining only the first few terms of the series. The result of such a comparison, for a few representative cases, is

also tabulated below. The conclusion may be drawn that the retention of 4 or 5 terms (through f_2 or, possibly, through g_2) affords a quite good representation of the Floquet solution in the cases considered. In the following section, therefore, we direct our attention to formulating algebraic expressions suitable for calculating the relative coefficients $f_1/g_0 \cdots g_2/g_0$.

III. ALGEBRAIC EVALUATION OF COEFFICIENTS:

We seek the even solution of

$$y'' + [A + B\cos Nt]y = 0 \tag{1}$$

in the approximate form

$$\begin{aligned} y = g_0 \cos \nu t &+ f_1 \cos(N-\nu)t + g_1 \cos(N+\nu)t \\ &+ f_2 \cos(2N-\nu)t + g_2 \cos(2N-\nu)t \,. \end{aligned} \tag{2}$$

By application of harmonic balance the following set of five algebraic equations is obtained:

$$\begin{aligned} A - \nu^2 + (B/2)(F_1 + G_1) &= 0 \\ [A - (N-\nu)^2]F_1 + (B/2)(1 + F_2) &= 0 \\ [A - (N+\nu)^2]G_1 + (B/2)(1 + G_2) &= 0 \\ [A - (2N-\nu)^2]F_2 + (B/2)F_1 &= 0 \\ [A - (2N+\nu)^2]G_2 + (B/2)G_1 &= 0\,, \end{aligned} \tag{3}$$

where $F_{1,2}$ and $G_{1,2}$ denote respectively $f_{1,2}/g_0$ and $g_{1,2}/g_0$.

From these simultaneous equations it follows that

$$F_1 \equiv f_1/g_0 = \frac{B}{2\left[-A + (N-\nu)^2 - \dfrac{B^2}{4[-A+(2N-\nu)^2]}\right]}$$

$$G_1 \equiv g_1/g_0 = \frac{B}{2\left[-A + (N+\nu)^2 - \dfrac{B^2}{4[-A+(2N+\nu)^2]}\right]}$$

$$F_2 \equiv f_2/g_0 = \frac{B}{2[-A+(2N-\nu)^2]}\left(\frac{f_1}{g_0}\right)$$

$$G_2 \equiv g_2/g_0 = \frac{B}{2[-A+(2N+\nu)^2]}\left(\frac{g_1}{g_0}\right), \tag{4}$$

provided the frequency satisfies

$$\nu^2 = A + \frac{B^2}{4\left[-A + (N-\nu)^2 - \dfrac{B^2}{4[-A+(2N-\nu)^2]}\right]} + \frac{B^2}{4\left[-A + (N+\nu)^2 - \dfrac{B^2}{4[-A+(2N+\nu)^2]}\right]}. \tag{5}$$

It is noted that in some applications, when $\nu \ll N$, equation (5) and the first two of

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$$G_1 \equiv g_1/g_0 = \frac{B}{2\left[-A + (N+\nu)^2 - \dfrac{B^2}{4[-A+(2N+\nu)^2]}\right]}$$

$$F_2 \equiv f_2/g_0 = \frac{B}{2[-A+(2N-\nu)^2]}\left(\frac{f_1}{g_0}\right)$$

$$G_2 \equiv g_2/g_0 = \frac{B}{2[-A+(2N+\nu)^2]}\left(\frac{g_1}{g_0}\right), \tag{4}$$

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It is noted that in some applications, when $\nu \ll N$, equation (5) and the first two of

difficulties seem to be encountered when σ is near π .

IV. ACKNOWLEDGEMENTS:

The writers wish to thank Mr. Mills, Mr. Morton, and Mr. Westlund, who assisted in the numerical work.

V. REFERENCES:

1. A Spiral-Ridged Bevatron, W. Walkinshaw, A.E.R.E. Harwell (1956).

2. Expansions of the Characteristic Exponent and Solutions..., Nils Vogt-Nilsen, MURA Report-118 (May, 1956).

3. Approximations for Linear Betatron Oscillations, Adler and Baroncini, Nuovo Cimento (10) 4, 959-74 (1956).

4. Non-Linear Resonances in the Radial Equation, G. Parzen, MURA Report-200 (October 12, 1956).

5. Concerning Resonances in the Spirally-Ridged FFAG Accelerator, Laslett and Sessler, MURA Report in preparation -- Appendix II.

6. Soluble Problem in Energy Bands, J. C. Slater, Physical Review 87, 807-835 (September 1, 1952), Sect. 2.
For the case N = 2, our notation may be related to that of Slater as follows: His variables u and w are identified with our y and t, the parameters a and s with our A and 2B, his g_0 with our σ/π, and $v(g_0 \mp 2m)$ with our F_m and G_m, respectively.

7. MURA Tables No. 5 and accompanying graphs, Laslett, Snyder, and Hutchinson (April 20, 1955).

8. MURA Tables No. 6, Belford, Laslett, and Snyder, (April 3, 1956).

LINEAR EQUATIONS FOR WHICH FLOQUET SOLUTIONS OBTAINED

$$y'' + (A + B \cos 2t)y = 0$$

Run Nos.		Param^{s}.		Matrix Elements			σ/π *	
DUCK -ANS.	FOR-ANAL	B	A	M_{21}	M_{12}	$M_{11}=M_{22}$ $= \cos\sigma$	From $\cos\sigma$	Nominal
779	30	1.5	.1782	-.010 328	4.1584	-.978 291	.933 557	14/15
780	34		.1648	-.027 410	4.2329	-.940 199	.889 362	8/9
781	37		.1514	-.043 702	4.3085	-.900 950	.857 129	6/7
782	38		.1216	-.076 968	4.4800	-.809 434	.800 226	4/5
783	39		.0904	-.107 163	4.6646	-.707 195	.750 040	3/4
784	40		.0327	-.149 384	5.0202	-.500 067	.666 691	2/3
785	41		-.0352	-.173 818	5.4628	-.224 636	.572 119	4/7
786	42		-.0850	-.172 270	5.8048	+.0004918	.499 843	1/2
787	43		-.1470	-.144 598	6.2520	.309 801	.399 737	2/5
788	44		-.1823	-.114 921	6.5175	.501 004	.332 964	1/3
789	45		-.2170	-.075 130	6.7864	$.700\ 100_5$	.253 139	1/4
790	46		-.2330	-.053 061	6.9131	$.795\ 728_4$	.207 088	1/5 †
779	50	1.0	.450	-.028 369	2.3618	-.965 918	.916 658	11/12
780	54		.430	-.054 616	2.4415	-.930 942	.881 012	8/9 †
781	57		.411	-.078 378	2.5188	-.895 869	.853 446	6/7 †
782	58		.368	-.127 621	2.6993	-.809 637	.800 336	4/5
783	59		.323	-.171 786	2.8967	-.708 788	.750 758	3/4
784	60		.244	-.228 507	3.2654	-.503 814	.668 070	2/3
785	61		.180	-.252 125	3.5856	-.309 811	.600 266	3/5
786	62		.092	-.246 364	4.0590	$+.001\ 299_4$	.499 586	1/2
787	63		.017	-.201 068	4.4945	.310 321	.399 563	2/5
788	64		-.025	-.157 353	4.7518	$.502\ 277_5$	.332 496	1/3
789	65		-.065	-.102 323	5.0062	.698 389	.253 901	1/4
790	66		-.086	-.067 892	5.1435	.806 719	.201 241	1/5

* With N = 2, $\nu = \sigma/\pi$

† Approx.

Unnormalized FORANAL Coefficients For

$$y = g_0 \cos \nu\tau + \sum_{m=1} \left[f_m \cos (mN-\nu)\tau + g_m \cos (mN+\nu)\tau \right] \; *$$

$$= g_0 \cos \nu\tau + \sum_{m=1} \left[(f_m+g_m) \cos mN\tau \cos \nu\tau + (f_m-g_m) \sin mN\tau \sin \nu\tau \right]$$

with $y(0) = 1$

FOR-ANAL NO.	NOM-INAL ν	g_0	f_1	g_1	f_2	g_2	f_3
30	14/15	.5000	.4173	.0446	.0340	~ .002 ‡	~ .002 ‡
34	8/9	.5260	.3903	$.0483_5$	$.0308_3$	$.0015_2$	.0009
37	6/7	.5449	.3726	$.0516_6$	.02880	.00164	.00082
38	4/5	$.5744_6$	.34116	.05599	.02533	.00183	.00070
39	3/4	.59808	.31626	.06023	.02269	.00201	.00062
40	2/3	.63147	.27953	.06715	.01896	.00232	.00050
41	4/7	.6616	.2445	.0749	.01555	.00267	.00038
42	1/2	.6796	$.2226_6$	.08082	.01356	.00299	.00034
43	2/5	.6988	.1968	.08918	.01128	.00344	.00027
44	1/3	$.740_4$	$.159_6$	$.083_4$	.010	.006	in noise
45	1/4	.721	.167	.103	.0087	.0042	~ .0002
46	1/5 †	.73	.16	.11	.008	.004	- - -
50	11/12	.5488	.3942	~ .035 ‡	~ .022 ‡		
54	8/9 †	.5850	.3794	.0373	.0208	.0009	.0005
57	6/7 †	.60	.35	$.04_9$	$.02_{75}$	- - -	- -
58	4/5	.6363	.3038	$.0426_2$	$.0153_8$	.00093	.00028
59	3/4	.665	$.273_3$	$.045_9$	.0133	.0010	.0002
60	2/3	.7034	$.232_3$	.0512	.0106	$.0011_4$	.00015
61	3/5	$.727_2$	$.206_7$	$.0553_4$	$.0090_8$	$.0013_1$	— —
62	1/2	$.753_4$	.1763	$.0613_3$	.00728	.00153	.00013
63	2/5	$.771_6$	.1530	$.0673_7$	.00593	.00175	.00010
64	1/3	.780	.1402	$.0715_5$	.00524	.00192	~ .0001
65	1/4	.793	.127	.077	.00441	.00206	- -
66	1/5	.795	.120	$.080_9$	.00410	.00228	- -

* In general $\nu\tau$ may be replaced by $\nu\tau + \epsilon$

† Approx.

‡ Est. (not fm. Foranal)

COMPARISON OF CALCULATED AND TRUE EVEN FLOQUET SOLUTION

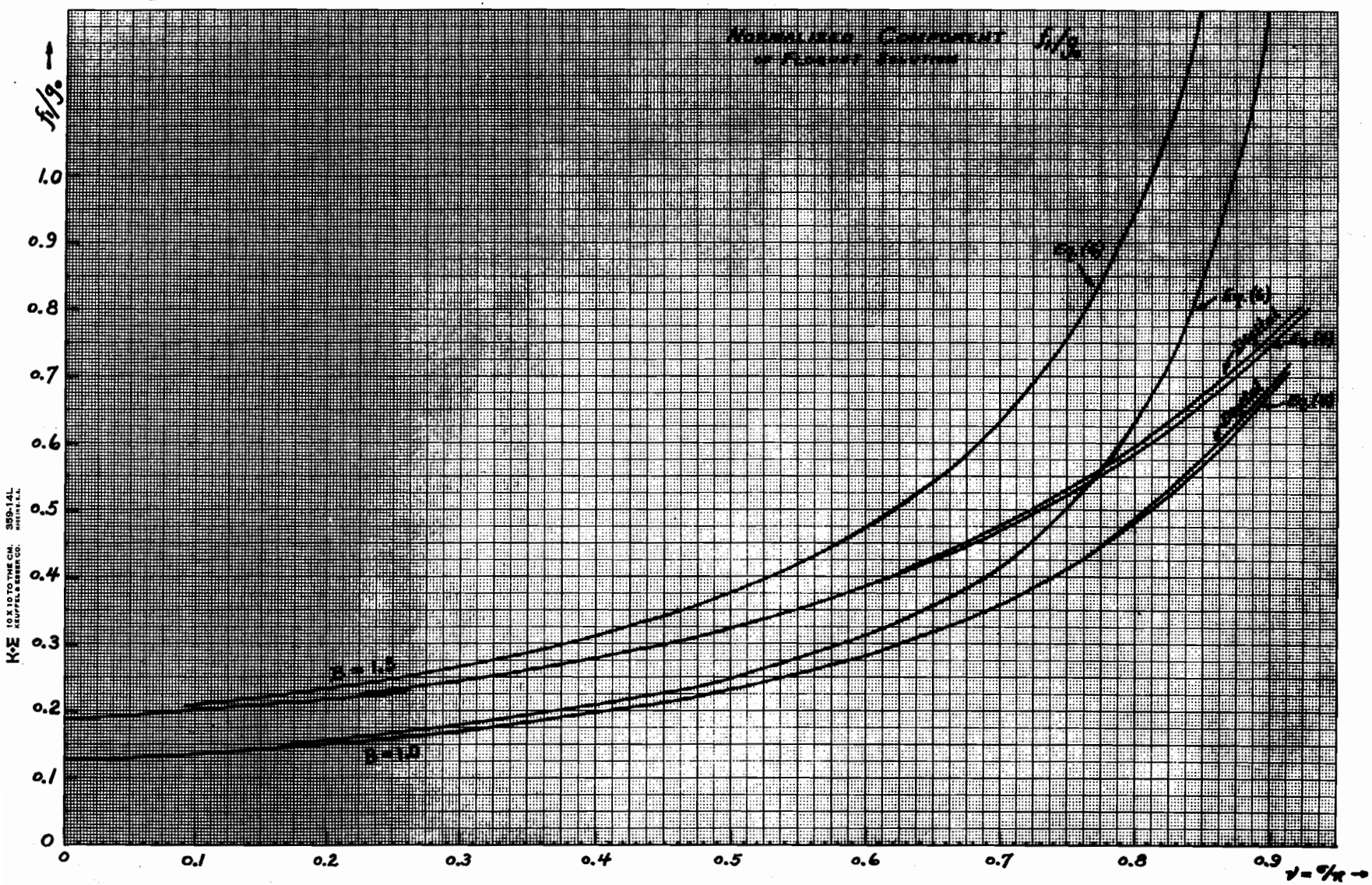
FOR MATHIEU EQUATION $y'' + (A + B\cos 2t)\,y = 0$

WITH B = 1.5 and $\sigma = 0.889\pi$

t/π (Sectors)	Calculated y		True y
	$y(g_0, f_1, g_1)$	$y(g_0, f_1, g_1, f_2)$	Duck-Ans. Run 780
0	.9647	.9955	1.0000
1/8	.8683	.8788	.8800
1/4	.6226	.5991	.5989
3/8	.3173	.2907	.2938
1/2	.0152	.0206	.0213
5/8	-.2755	-.2452	-.2470
3/4	-.5592	-.5438	-.5420
7/8	-.7961	-.8158	-.8146
1	-.9065	-.9355	-.9379
1-1/8	-.8367	-.8361	-.8357
1-1/4	-.6112	-.5822	-.5808
1-3/8	-.3208	-.3406	-.3030
1-1/2	-.0437	-.0591	-.0597
1-5/8	.2006	.1703	.1718
1-3/4	.4281	.4228	.4210
1-7/8	.6276	.6543	.6523
2	.7390	.7625	.7636
2-1/8	.7030	.6925	.6915
2-1/4	.5259	.4951	.4932
2-3/8	.2855	.2750	.2761
2-1/2	.0670	.0905	.0910
2-5/8	-.1015	-.0748	-.0781
2-3/4	-.2455	-.2508	-.2496
2-7/8	-.3834	-.4138	-.4119
3	-.4824	-.4977	-.4981
3-1/8	-.4851	-.4653	-.4646
3-1/4	-.3773	-.3483	-.3466
3-3/8	-.2158	-.2158	-.2161
3-1/2	-.0821	-.1111	-.1114
3-5/8	-.0099	-.0297	-.0288
3-3/4	.0332	.0485	.0484
3-7/8	.0930	.1233	.1222
4	.1675	.1729	.1729
4-1/8	.2086	.1820	.1820
4-1/4	.1831	.1595	.1586
4-3/8	.1201	.1307	.1302
4-1/2	.0874	.0874	.1185

CALCULATED AND COMPUTED MATRIX-ELEMENT M_{12}

Duck-Ans. Run No.	Foranal Run No.	B	ν	M_{12} Calc. fm. 3 Terms	M_{12} Calc. fm. 4 Terms	M_{12} Calc. fm. 5 Terms	M_{12} From I. B. M. Computations
780	34	1.5	.889362	1.8878	4.3391	3.9684	4.2329
781	37		.857129	2.2272	4.4057	4.0819	4.3085
789	45		.253139	5.7592	7.9081	6.6296	6.7864
780	54	1.0	.881012	1.8442	2.7978	2.7111	2.4415
788	64		.332496	4.4561	4.9766	4.7577	4.7518
789	65		.253901	4.6595	5.2460	4.9400	5.0062
790	66		.201241	4.8153	5.5416	5.0963	5.1435



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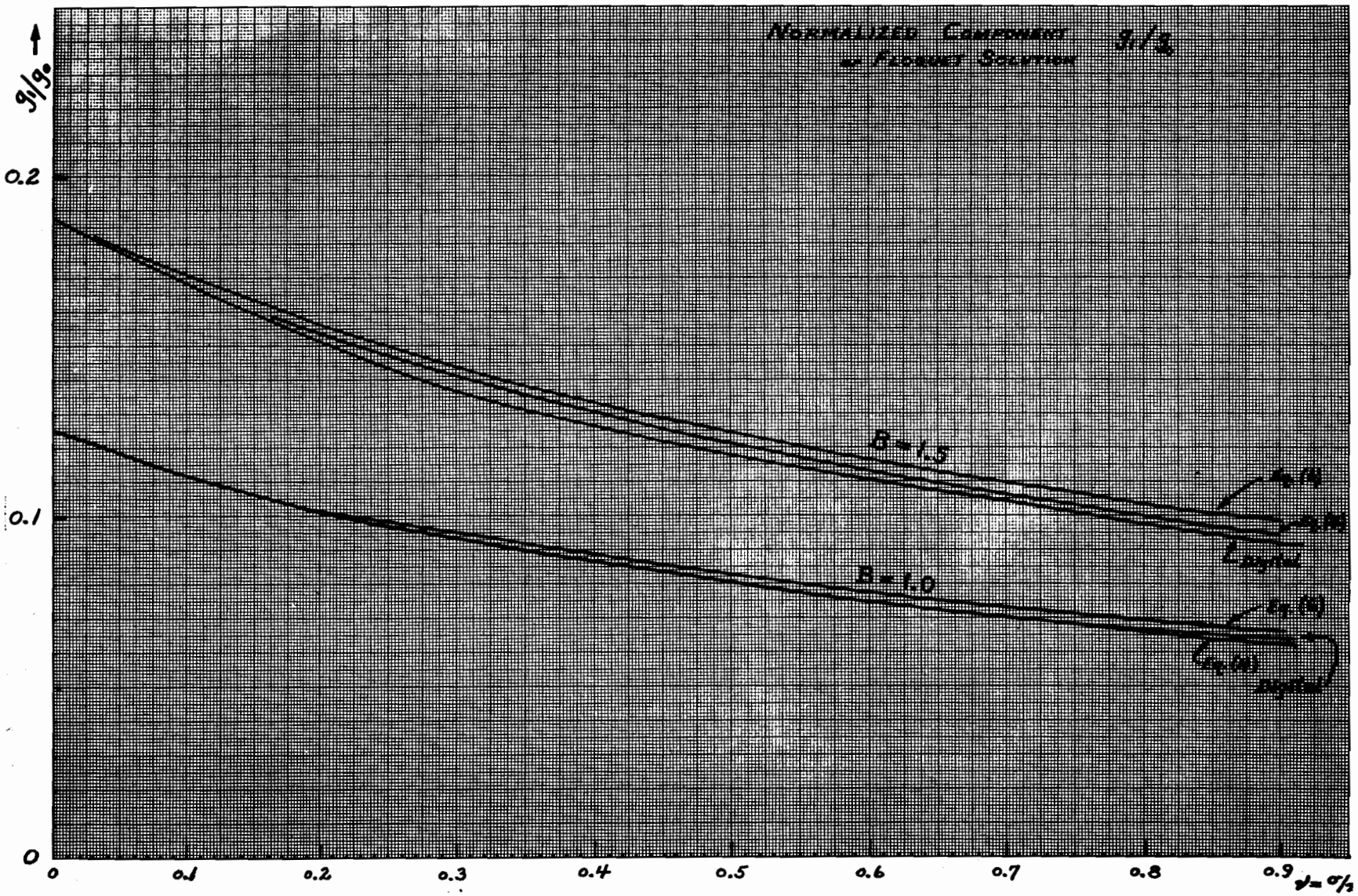
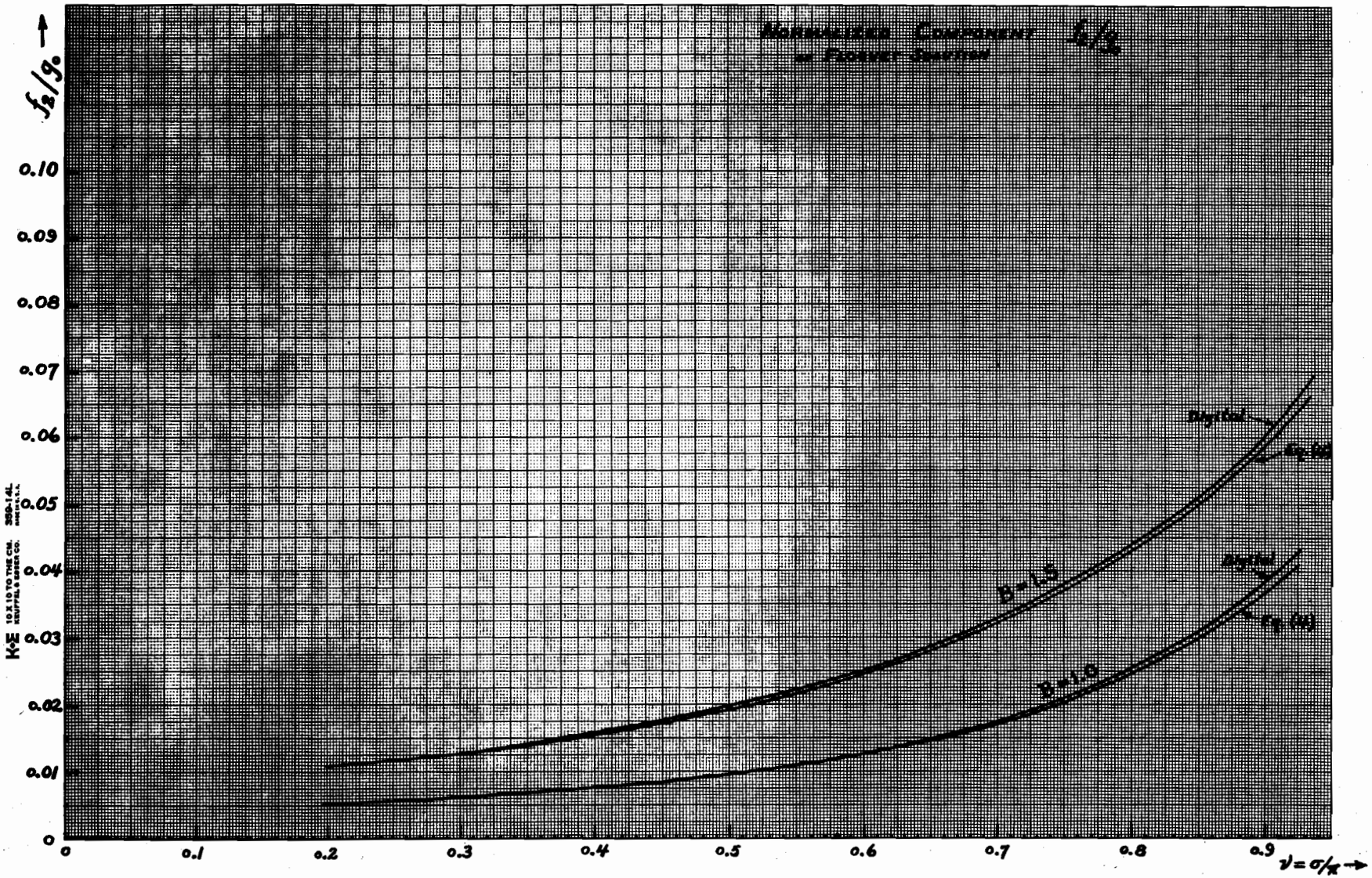

Normalized Component g_1/g_0
of Floquet Solution
g_1/g_0
0.2
0.1
0
$B = 1.5$
$B = 1.0$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$y = \sigma/\lambda$



Normalized Component f_2/g_0
of Floquet Solution
f_2/g_0
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\nu = \sigma/\pi$
B=1.5
B=1.0
Digital
Eq. (4)
Digital
Eq. (4)
K+E 10 X 10 TO THE CM. 359-14L
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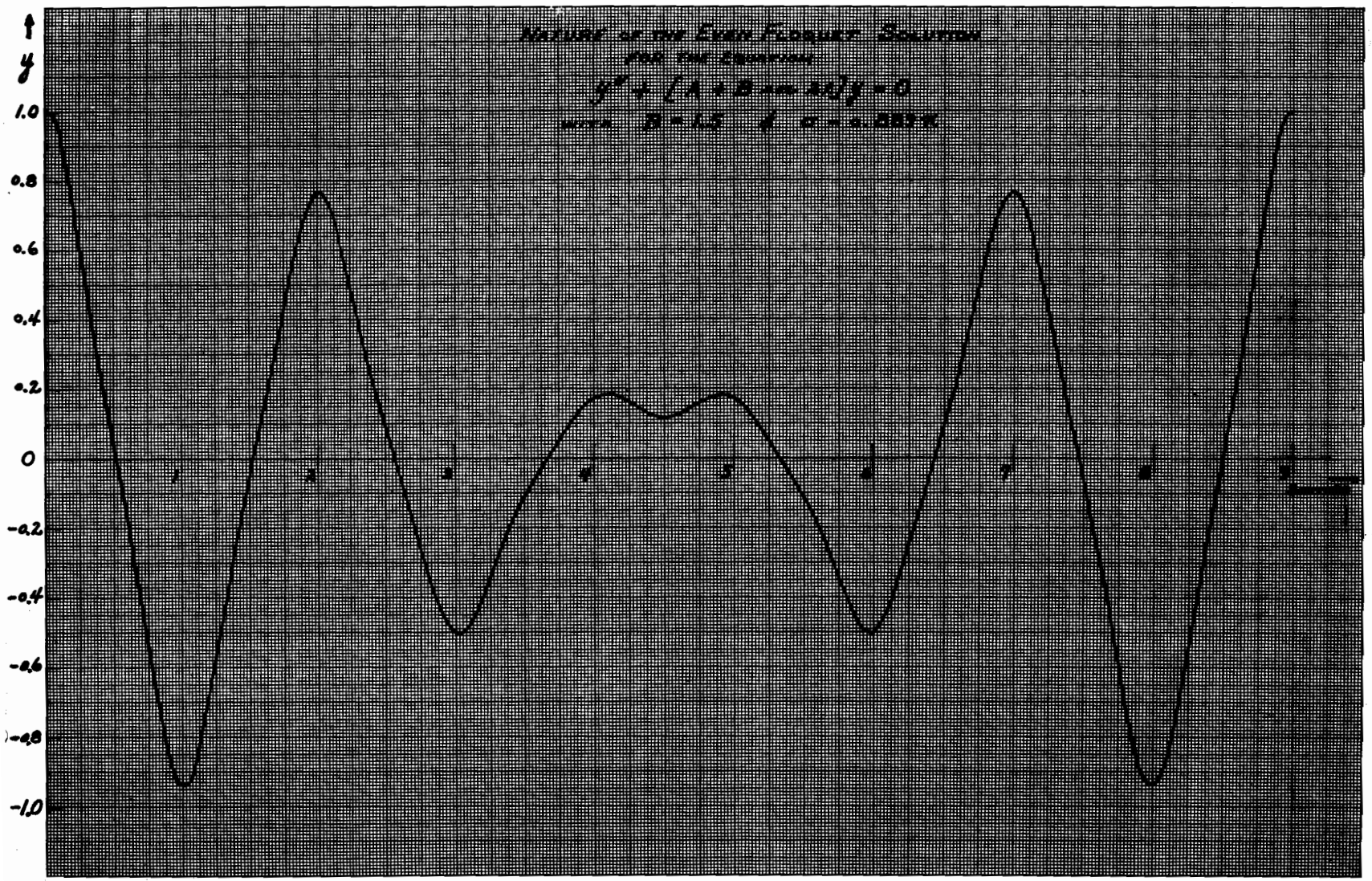
NATURE OF THE EVEN FLOQUET SOLUTION
FOR THE EQUATION
y
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
1
2
3
4
5
6
7
8
9