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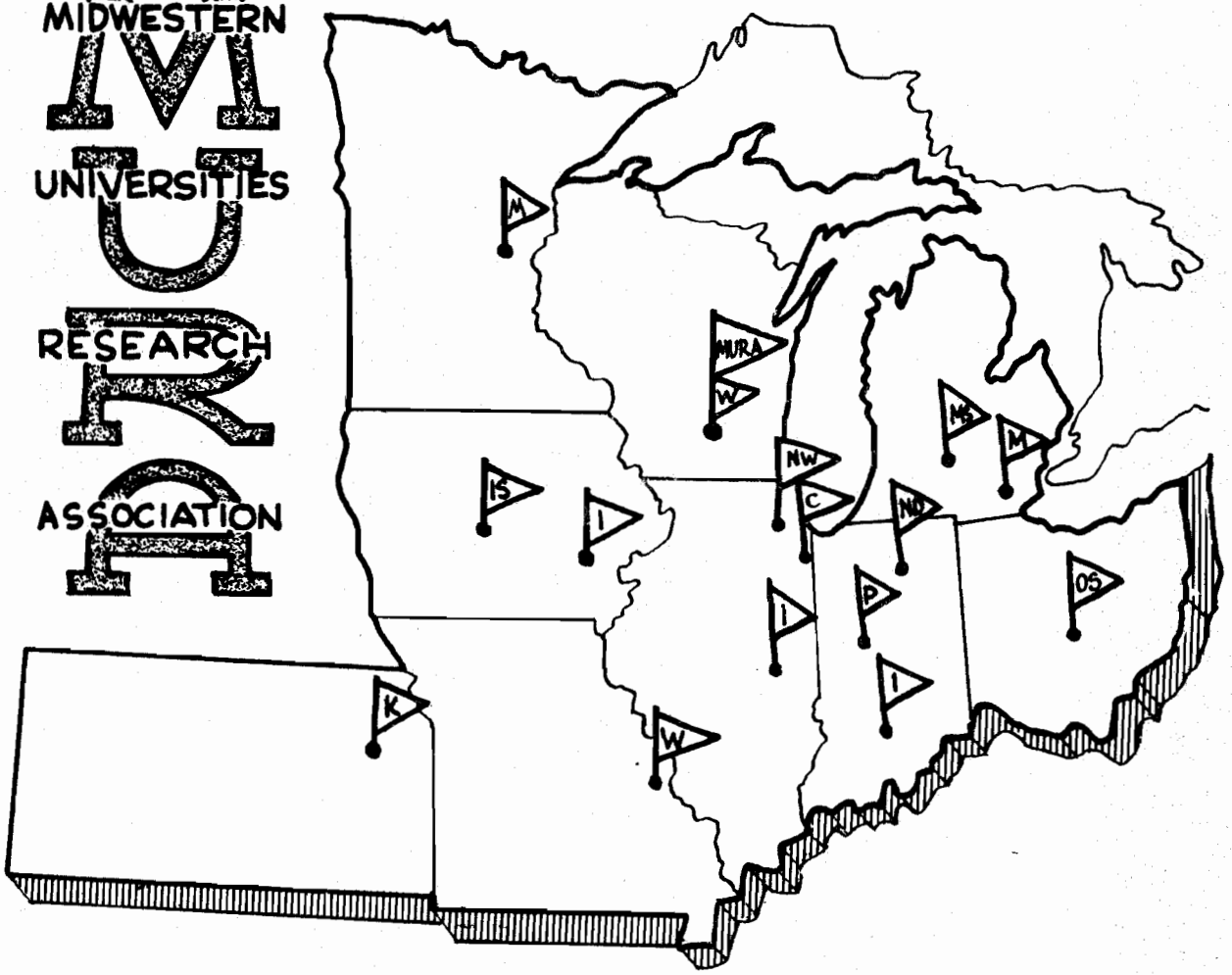

LAPSE-RATE CHARACTERIZING THE CONVERGENCE OF A FOROCYL POTENTIAL PROBLEM

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LAPSE-RATE CHARACTERIZING THE CONVERGENCE OF A FOROCYL POTENTIAL PROBLEM

L. Jackson Laslett**

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ABSTRACT: Data obtained concerning the convergence of a simple FOROCYL potential problem are analysized, in keeping with a suggestion of K.R. Symon, to indicate the lapse-rate which characterizes the convergence on meshes of various sizes. Working formulas are then given to indicate the number of iterations and the estimated computation time required to attain a desired degree of convergence in this particular example.

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** On leave from Iowa State College, Ames, Iowa

I. Introduction:

We have previously reported, at a MURA sesquimonthly technical meeting[1], the results of certain tests of the FOROCYL computational program.[2] In these tests data were obtained concerning the rate of convergence, with meshes of various sizes, for an azimuthally-independent problem with the exact analytic solution (k+ 1 odd):

$$\Omega \propto \left(R/r\right)^{k+1} P_n(\cos\gamma)$$
$$\propto (\csc\gamma)^{k+1} P_n(\cos\gamma)$$
$$\propto \left[1 + \frac{(2\pi\omega\eta)^2}{1+(\omega N)^2}\right]^{\frac{k+1}{2}} P_{k+1}\left(1 + \frac{1+(\omega N)^2}{(2\pi\omega\eta)^2}\right)^{-1/2},$$

k + 1 being 3 in this work. Specifically, $\Omega = \dfrac{\eta(1 - 8\eta^2/39)}{.3978994872}$.

As a result of considerations concerning "eigenpoles"[1], Dr. Symon has pointed out that after a relatively small number of iterations a solution may be expected to converge exponentially to the final solution, with a lapse-rate characteristic of a single eigen-solution. It was suggested that it would be of interest to present the existing data in this light and that knowledge of the prevailing lapse-rate for a given situation would permit predictions of the iterations required to attain a given accuracy in other problems involing the same geometrical configuration. As Symon has observed, the basic lapse-rate associated with a given pole-configuration might, in fact, be most conveniently obtained in a short preliminary run with zero potential on the boundaries (and some non-zero initial values within the mesh), since then the convergence limit (o) is known in advance.

It should be noted that the limiting solution—i. e., the solution approached as

the number of iterations becomes large--is the theoretical solution of the computational algorism and may differ noticeably from the true solution of the differential equation when a coarse mesh is employed.

II. Results:

The available convergence data, for tests involving meshes of dimensions (a x b) 25 x 5, 50 x 9, 100 x 17, and 200 x 33, were analysized by semi-logarithmic plots of the residual error vs. the number of iterations. The lapse-rates indicated by such plots appeared, moreover, to be inversely proportional to the mesh-area, taken as a x (b -1), as indicated in the following table.

Mesh a x b	Mesh Area a x (b-1)	Lapse-Rate Decades/Iteration	Product (Lapse-Rate) x Area
25 x 5	100	0.1541	15.4
50 x 9	400	0.0393	15.7
100 x 17	1600	0.010	16
200 x 33	6400	0.0024	15.4

III. Correlation With ϵ:

Ignoring any transient behaviour prior to the onset of a purely exponential decrease of error, an initial error E_0 is reduced, after I iterations, to

$$E = E_0 \, 10^{-\mu I},$$

μ representing the lapse-rate in decades per iteration. The computation proceeds until the change of value in a single iteration is less than a prescribed ϵ. The requisite number of iterations may thus be estimated for the present configuration as:

$$I = \frac{\log(\ln 10) + \log(\mu E_0/\epsilon)}{\mu}$$

$$= \frac{0.36222 + \log(\mu E_0/\epsilon)}{\mu}$$

$$= 0.065\left[a \times (b-1)\right]\left[0.36222 + \log(\mu E_0/\epsilon)\right].$$

Similarly, supposing the speed of computation to be roughly $0.7 \times 10^{-4} \frac{\text{min}}{\text{Iteration} \cdot \text{Area}}$

the requisite time becomes

$$T = 0.7 \times 10^{-4}\left[a \times (b-1)\right] \frac{0.36222 + \log(\mu E_0/\epsilon)}{\mu} \text{ min.}$$

$$= 4.5 \times 10^{-6}\left[a \times (b-1)\right]^2\left[0.36222 + \log(\mu E_0/\epsilon)\right] \text{ min.,}$$

with $\mu = 15.4/\left[a \times (b-1)\right]$ decades per iteration.

We give below some examples of the application of these formulas, with E_0 at the check-point(η = 0.08) taken to be $E_0 = 7.9185 \times 10^{-4}$.

<u>Example 1:</u>

100×17 Mesh, $\epsilon = 1 \times 10^{-10}$.

With $a \times (b-1) = 1600$, $E_0 = 7.9185 \times 10^{-4}$, and $\mu \cong 0.01$,

$$I_{calc} = 0.065 \times 1600 \times 5.26 = 557 \text{ iterations;}$$

$$I_{observed} = 506.$$

$$T_{calc} = 4.5 \times 10^{-6} \times (1600)^2 \times 5.26 = 60 \text{ min;}$$

$$T_{observed} \cong 50 \text{ min.}$$

Example 2:

200 x 33 Mesh, $\epsilon = 3 \times 10^{-8}$.

With a x (b-1) = 6400, $E_0 = 7.9185 \times 10^{-4}$, and $\mu = 2.4 \times 10^{-3}$,

$$I_{calc} = 0.065 \times 6400 \times 2.16 = 900 \text{ iterations};$$
$$I_{observed} = 887.$$

$$T_{calc} = 4.5 \times 10^{-6} \times (6400)^2 \times 2.16 = 400 \text{ min.};$$
$$T_{observed} \cong 360 \text{ min.}$$

IV. Conclusion:

It should be emphasised in closing that the particular case considered here represents an especially simple configuration. It is believed that Dr. Symon is acquiring additional information concerning the lapse-rate which will prevail for other configurations in connection with his study of eigen-poles.

V. References:

[1] MURA Sesquimonthly Technical Meeting, Minutes #13 (January 4 and 5, 1957).

[2] MURA Report-221, Internal, I.B.M. Program (February-March, 1957).

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Semi-Logarithmic, 5 Cycles X 5 to the inch.
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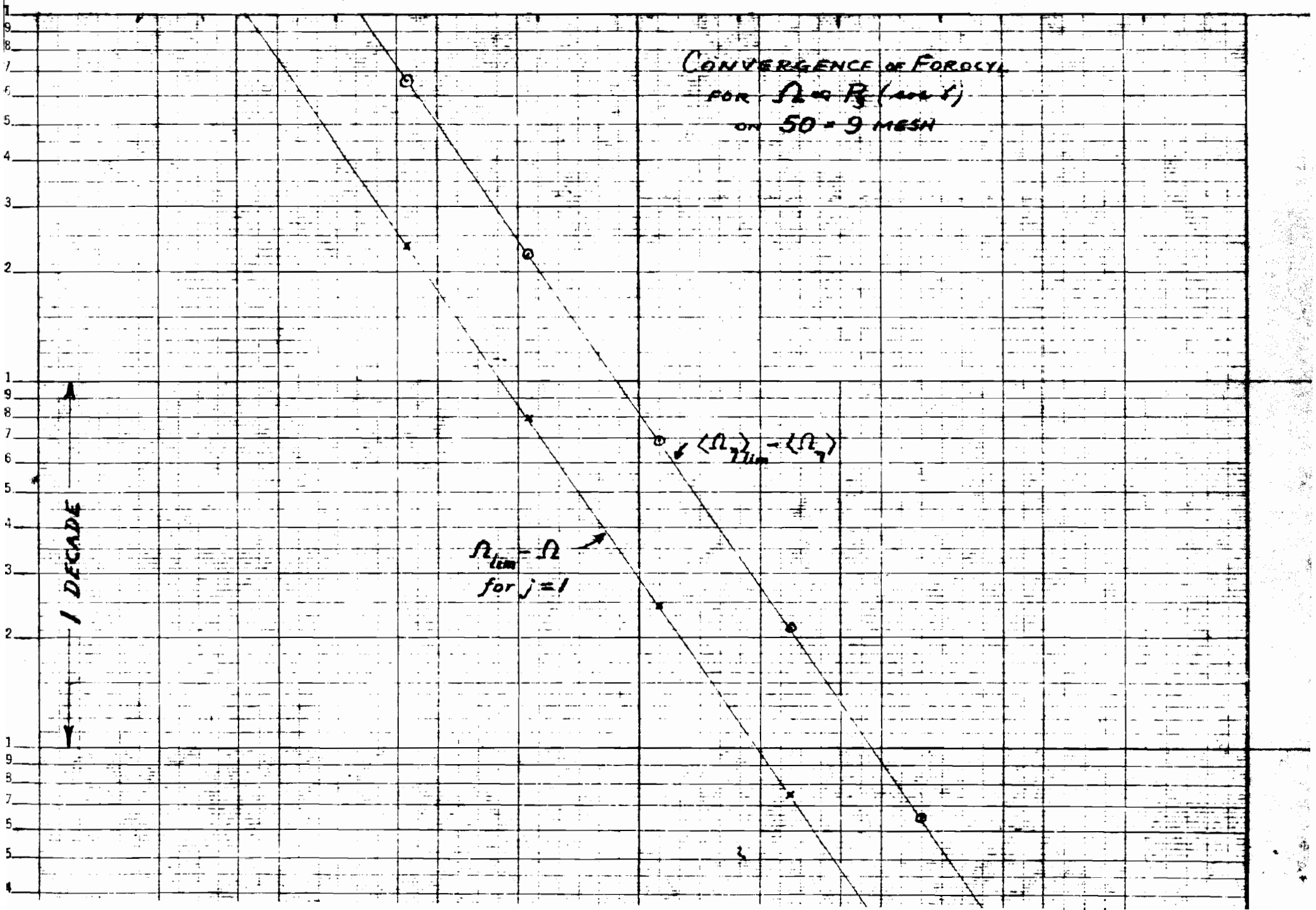

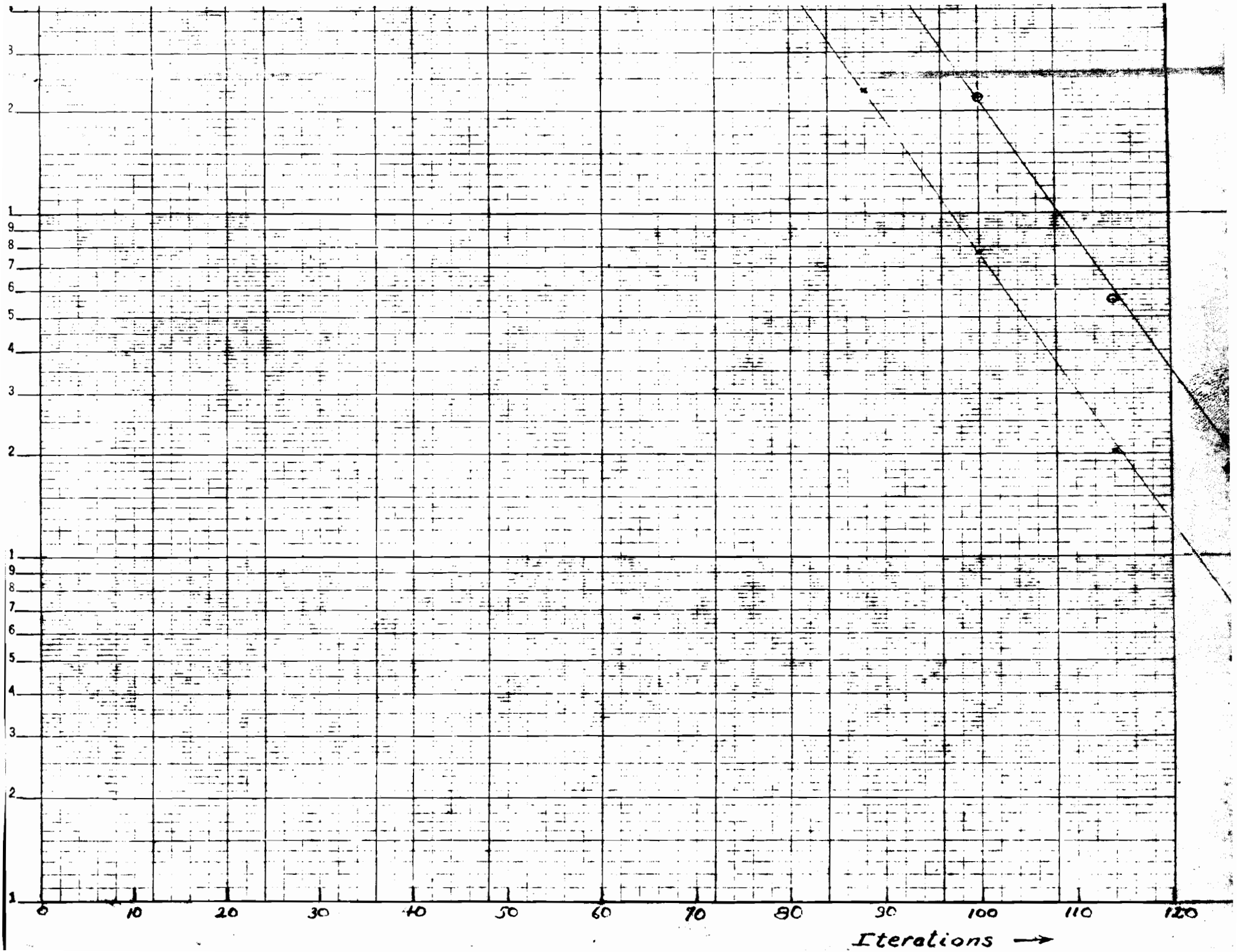
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Semi-Logarithmic, 5 Cycles X 6 to the Inch.
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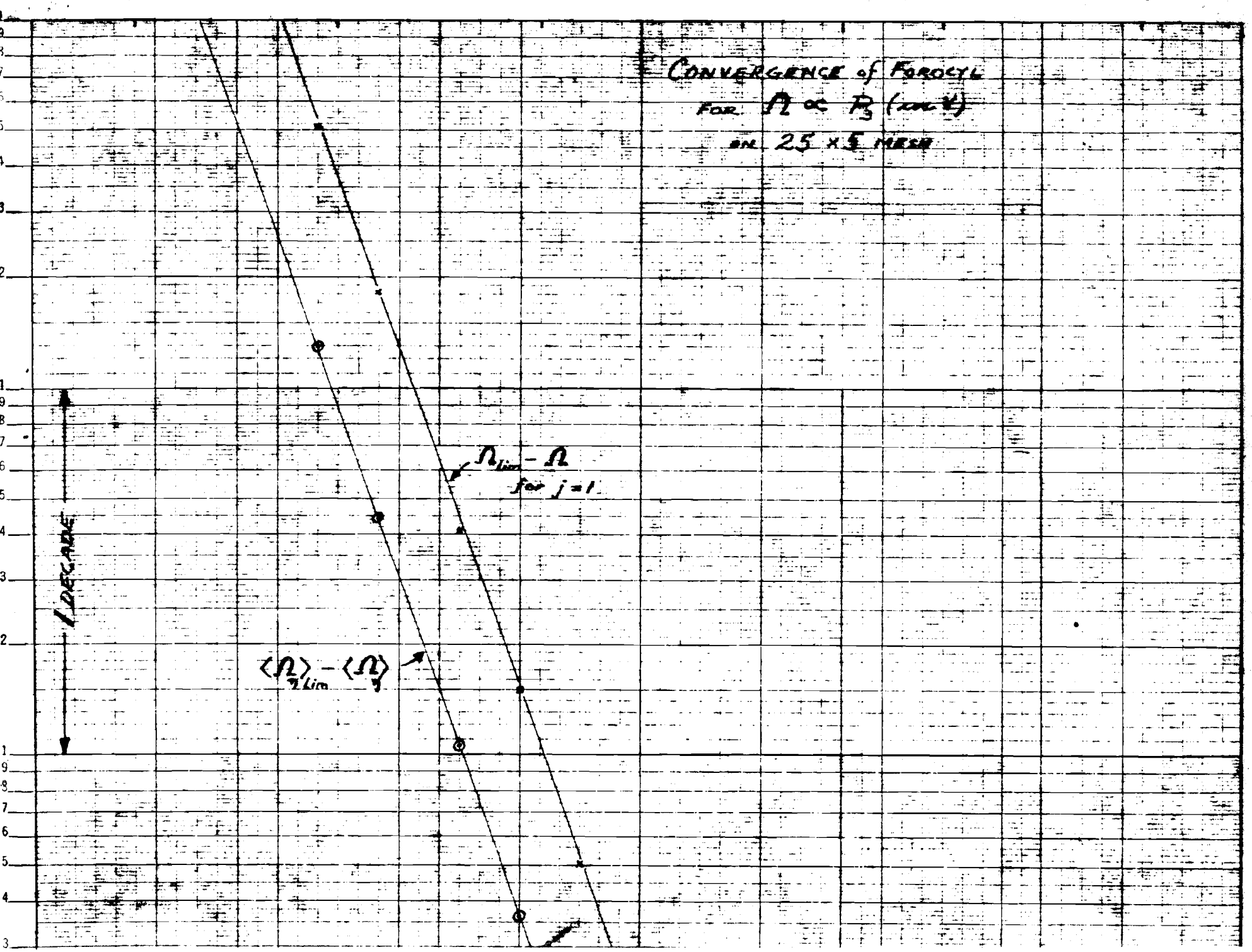

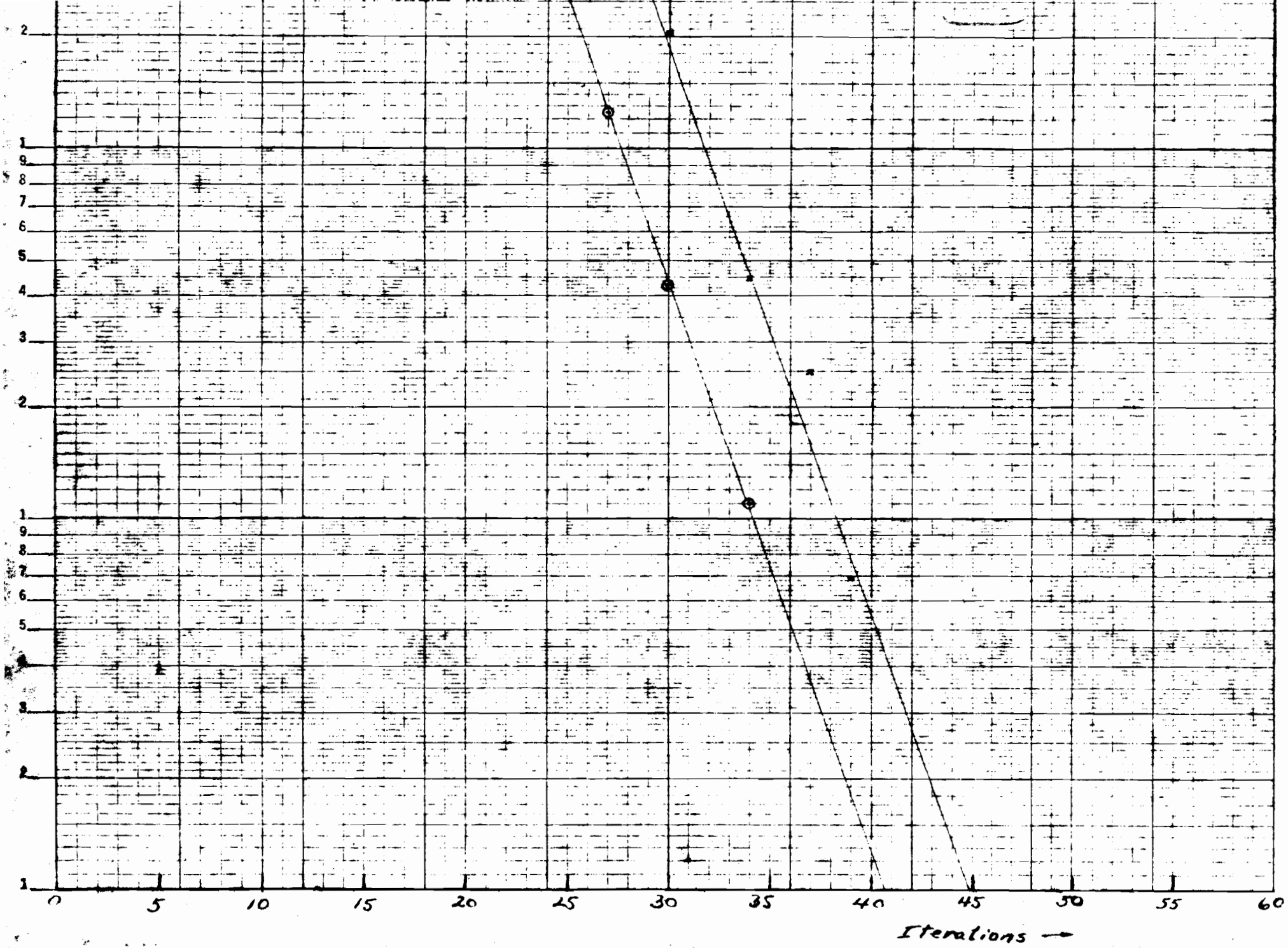
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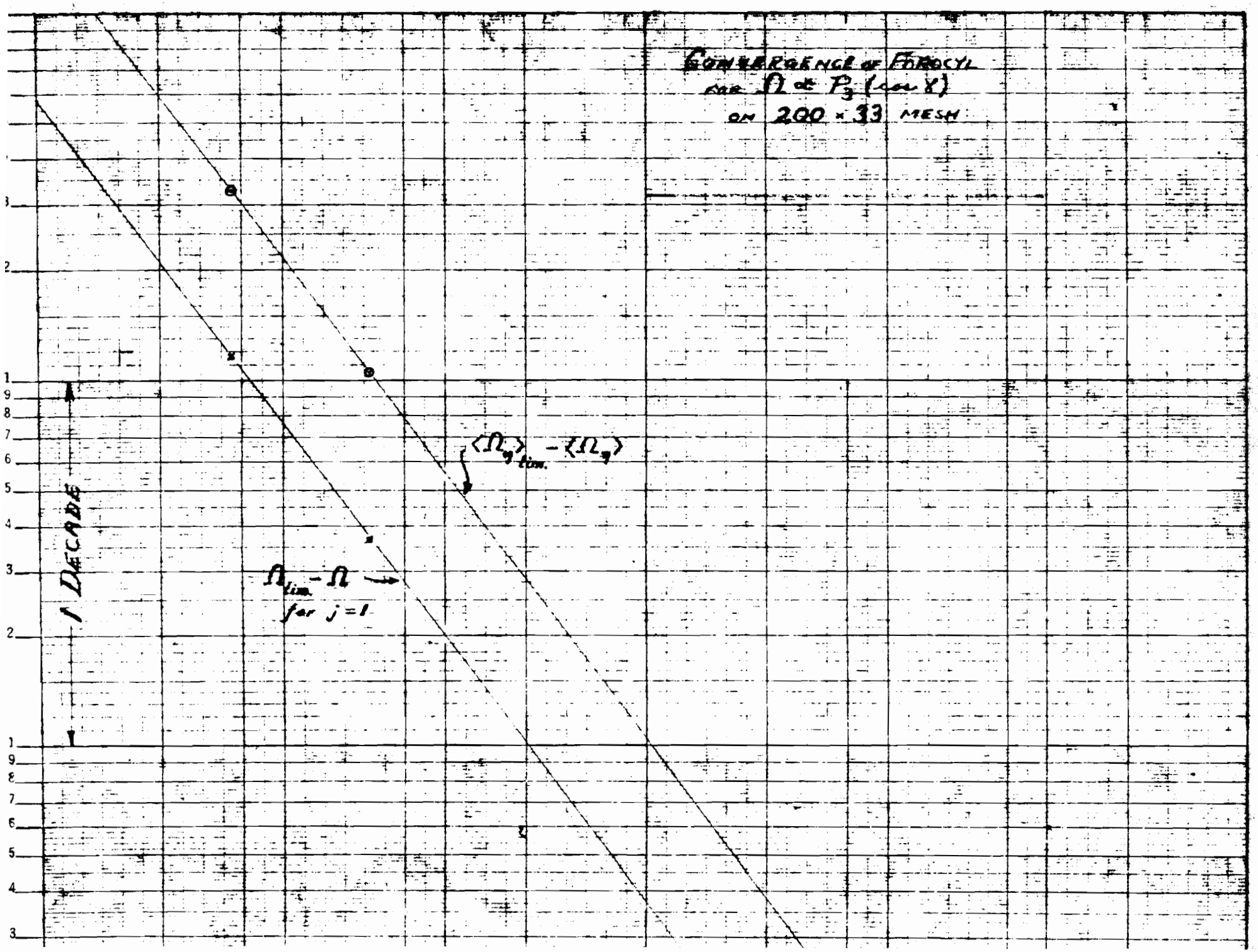
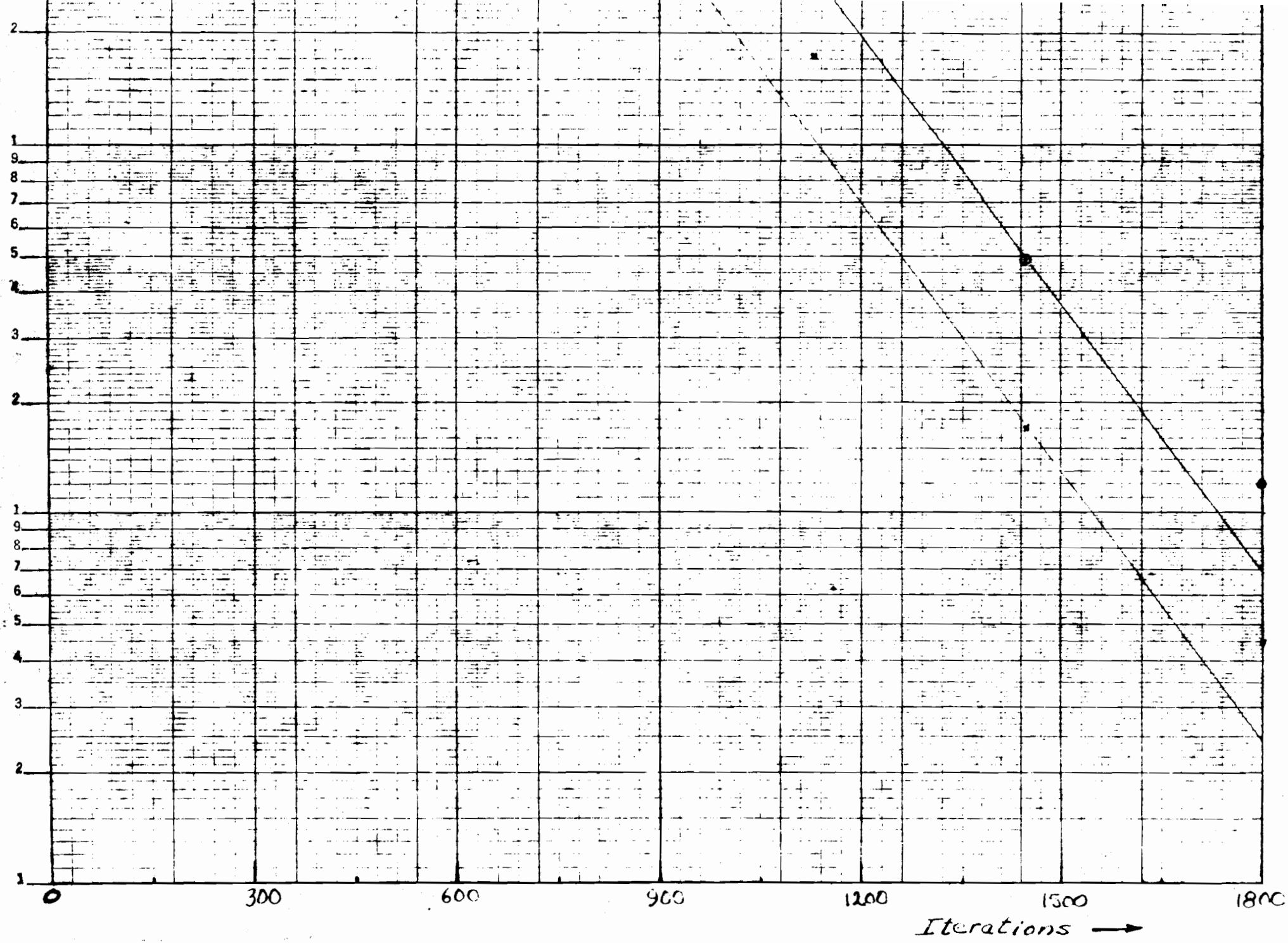
Iterations →


CONVERGENCE OF FOROCYL
FOR Ω AT P3 (LOW Y)
ON 200 × 33 MESH
⟨Ω_q⟩_lim. − ⟨Ω_q⟩
Ω_lim. − Ω
for j = 1
1 DECADE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1200
1500
1800
Iterations