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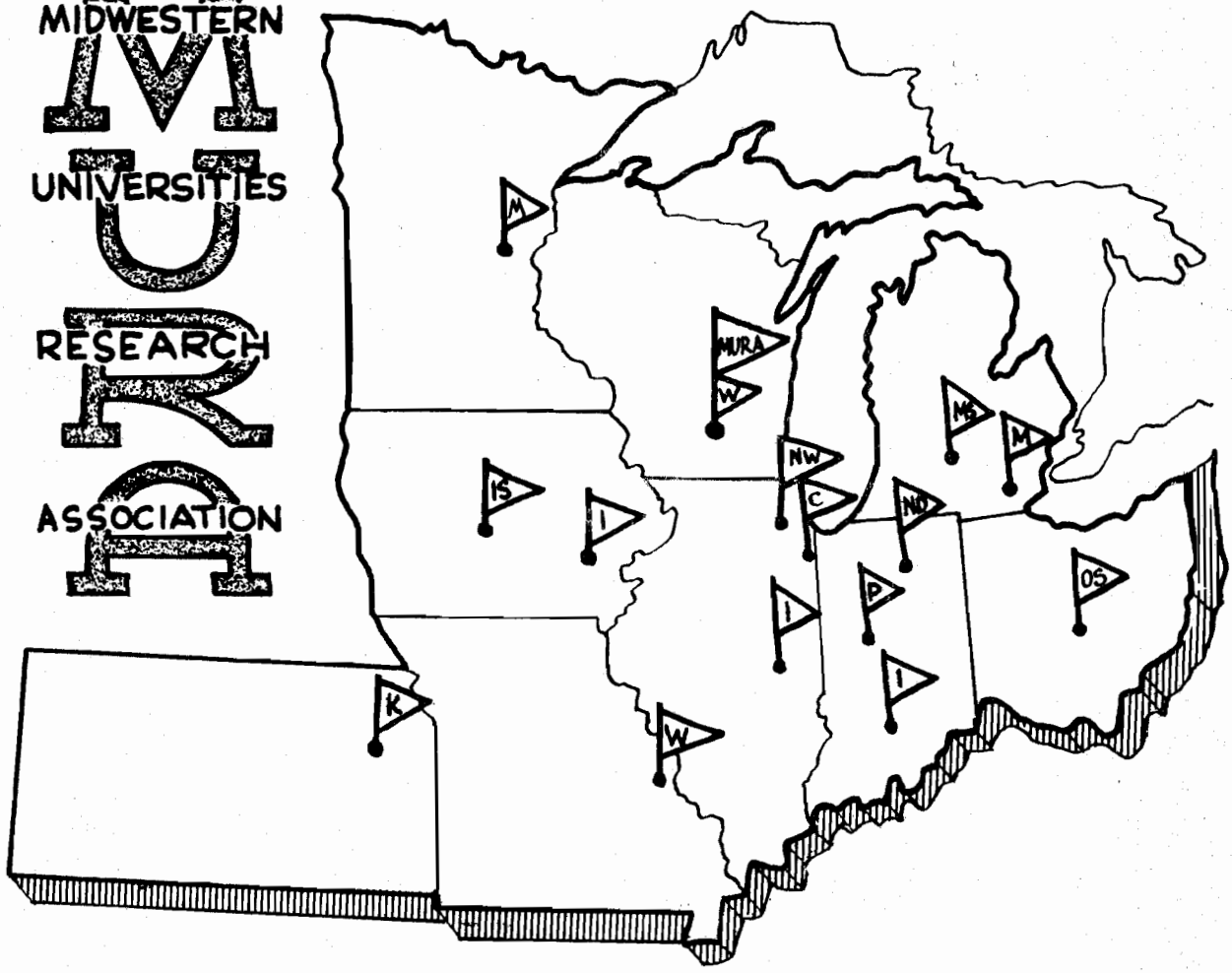

CONCERNING THE y-GROWTH PHENOMENON

EXHIBITED BY ALGEBRAIC TRANSFORMATIONS

REPORT **NUMBER** 246

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CONCERNING THE y-GROWTH PHENOMENON EXHIBITED BY ALGEBRAIC TRANSFORMATIONS

L. Jackson Laslett**

March 11, 1957

ABSTRACT:

Hamiltonian algebraic transformations which can lead to extensive exponential y-growth are discussed in regard to the threshold for y-growth. Computational examples are given.

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* Supported by Contract AEC No. AT(11-1)-384.

** On leave from Iowa State College, Ames, Iowa.

1. Motivation:

As is well known, non-linear coupling between the radial and axial motion in particle accelerators can lead to extensive exponential growth of the axial oscillations. The y-growth appears to be more rapid the further the x-amplitude is above a critical threshold value and the threshold becomes zero as a resonant relation between the radial and axial frequencies is approached. The amplitudes resulting from y-growth may differ sufficiently from those prevailing originally that instability is soon seen to develop, but in other cases the y-growth is found to "turn-over" and stability, for at least a limited interval of time, appears indicated.

Certain aspects of these phenomena have been studied both analytically and computationally. The computations may be based either on differential equations which represent closely those which govern particle motion in an actual accelerator or they may employ idealized differential equations which, it is supposed, contain the essential significant features of the exact equations. In either case, however, the computational time required for the integration of any particular problem is sufficiently great as normally to preclude carrying a single computation beyond a few hundred "sectors" -- i.e., through perhaps 100 oscillations.

It appears noteworthy that the y-growth and turn-over found by integration of differential equations for an AG (alternating-gradient) accelerator may be replicated fairly closely by a suitable non-AG problem and that, in the latter case, the particle does not appear to enter during the computation all regions of phase-space which are energetically available to it. Since some of the particles

which the computations thus indicate as "stable", in the equivalent non-AG structure under consideration, have sufficient energy to become unstable by traversal of a pass in the potential-energy surface, there is some interest in the ultimate fate of such particles.

Because of the interest in more extended computations, attention is directed to the use of algebraic transformations, which may be performed with a speed perhaps two orders of magnitude greater than typical for solution of differential equations. Although a close equivalence between the differential equations and some specific transformation may be difficult to establish definitively, it appears possible to find transformations which describe well the general features of the solutions found computationally for the differential equations of interest.

We consider in this report a particular type of algebraic transformations which may be representative of motion influenced by the $\sigma_x = 2\sigma_y$ resonance. The scaling features and threshold for y-growth are discussed. Examples of computations through 1200 sectors, performed by the ALGYTEE program are also given.

2. Statement of the Algebraic Transformation Under Consideration:

We consider here a transformation in which the coupling is provided by the addition of y^2 terms to the equations for x, p_x and by the addition of xy terms to the equations for y, p_y:

$$\left.\begin{aligned} x_n &= a_x x_{n-1} + b_x p_{x_{n-1}} + (\lambda/2)(b_x/b_y)\, y_{n-1}^2 \\ p_{x_n} &= c_x x_{n-1} + d_x p_{x_{n-1}} + (\lambda/2)(d_x/b_y)\, y_{n-1}^2 \\ y_n &= a_y y_{n-1} + b_y p_{y_{n-1}} + \lambda\, x_{n-1}\, y_{n-1} \\ p_{y_n} &= c_y y_{n-1} + d_y p_{y_{n-1}} + \lambda (d_y/b_y)\, x_{n-1}\, y_{n-1} \end{aligned}\right\}, \qquad (1)$$

with $\begin{vmatrix} a_x & b_x \\ c_x & d_x \end{vmatrix} = 1, \quad \begin{vmatrix} a_y & b_y \\ c_y & d_y \end{vmatrix} = 1$, and the coupling terms selected to have coefficients which depend on a single parameter λ to insure that the transformation be Hamiltonian* (as adjudged from the bracket expressions).

If, for simplicity, we wish the diagonal members of the linear part of the x, p_x transformation to be equal and likewise for the y, p_y transformation (corresponding to the situation in which the amplitude functions for the associated Floquet solutions are stationary at the point of reference), we may put

$$\left.\begin{aligned} \begin{pmatrix} X \\ P_X \end{pmatrix}_j &= \begin{pmatrix} \alpha_x & 0 \\ \dfrac{a_x - d_x}{2\alpha_x b_x} & \dfrac{1}{\alpha_x} \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}_j \\ \text{and} \quad \begin{pmatrix} Y \\ P_Y \end{pmatrix}_j &= \begin{pmatrix} \alpha_y & 0 \\ \dfrac{a_y - d_y}{2\alpha_y b_y} & \dfrac{1}{\alpha_y} \end{pmatrix} \begin{pmatrix} y \\ p_y \end{pmatrix}_j . \end{aligned}\right\} \quad (2)$$

In terms of the upper-case variables the transformation then reads

$$\left.\begin{aligned} X_n &= \frac{a_x + d_x}{2} X_{n-1} + \alpha_x^2 b_x P_{X_{n-1}} + \frac{\lambda}{2} \frac{\alpha_x}{\alpha_y^2} \frac{b_x}{b_y} Y_{n-1}^2 \\ P_{X_n} &= \frac{\left[\frac{1}{2}(a_x + d_x)\right]^2 - 1}{\alpha_x^2 b_x} X_{n-1} + \frac{a_x + d_x}{2} P_{X_{n-1}} + \frac{\lambda}{2\alpha_x \alpha_y^2} \frac{a_x + d_x}{2 b_y} Y_{n-1}^2 \\ Y_n &= \frac{a_y + d_y}{2} Y_{n-1} + \alpha_y^2 b_y P_{Y_{n-1}} + \frac{\lambda}{\alpha_x} X_{n-1} Y_{n-1} \\ P_{Y_n} &= \frac{\left[\frac{1}{2}(a_y + d_y)\right]^2 - 1}{\alpha_y^2 b_y} Y_{n-1} + \frac{a_y + d_y}{2} P_{Y_{n-1}} + \frac{\lambda}{\alpha_x \alpha_y^2} \frac{a_y + d_y}{2 b_y} X_{n-1} Y_{n-1} . \end{aligned}\right\} \quad (3)$$

---

* The equations actually iterated on the computer in ALGYTEE runs 10-18 were strictly not Hamiltonian, but would become so by a trivial (non-canonical) transformation such that the y and p_Y values employed by the computer be each multiplied by the scale factor $\sqrt{-c_y b_y}$ to obtain the corresponding canonical quantities.

This transformation is seen to be of the same form as (1) ($\lambda' \equiv \lambda/\alpha_x$ now playing the role of λ), with symmetry in the diagonal elements of the linear portion. By suitably choosing α_x and α_y, the off-diagonal elements may also be made equal save for sign -- specifically, identifying $(1/2)(a_x + d_x) = \cos \nu$ and $(1/2)(a_y + d_y) = \cos \kappa$, we choose $\alpha_x^2 = \frac{\sin \nu}{b_x}$ and $\alpha_y^2 = \frac{\sin \kappa}{b_y}$. As a result we finally obtain

$$\left.\begin{aligned}
X_n &= (\cos \nu) X_{n-1} + (\sin \nu) P_{X_{n-1}} + \frac{\lambda'}{2} \frac{\sin \nu}{\sin \kappa} Y_{n-1}^2 \\
P_{X_n} &= -(\sin \nu) X_{n-1} + (\cos \nu) P_{X_{n-1}} + \frac{\lambda'}{2} \frac{\cos \nu}{\sin \kappa} Y_{n-1}^2 \\
Y_n &= (\cos \kappa) Y_{n-1} + (\sin \kappa) P_{Y_{n-1}} + \lambda' X_{n-1} Y_{n-1} \\
P_{Y_n} &= -(\sin \kappa) Y_{n-1} + (\cos \kappa) P_{Y_{n-1}} + \lambda' (\operatorname{ctn} \kappa) X_{n-1} Y_{n-1}
\end{aligned}\right\} \quad (4)$$

This conveniently simple form for the transformation is thus seen to be inherently as general as the original form (1) and will serve as the basis of the analysis to follow.

3. Analysis Concerning the Onset of Y-Growth:

A. Method:

If we direct our attention to cases in which the axial-amplitude is initially very small, we may analyze the transformation equations in the spirit of Walkinshaw. The Y^2 terms are accordingly ignored in the recursion relations for the radial motion, whereupon the radial motion becomes represented by <u>linear</u> difference equations whose solution may be entered as a prescribed function of n into the axial equations. It is recognized, of course, that this procedure destroys the Hamiltonian form of the equations treated and precludes drawing in this way any inferences concerning the eventual character

of the motion when Y may have grown to large amplitudes.

Proceeding to ignore the Y^2 term in the X and P_X equations, the solution for the radial motion becomes

$$X_n = (\cos n\nu) X_0 + (\sin n\nu) P_{X_0} .$$

This solution, when inserted into the remaining (axial) equations, then gives

$$Y_n = \left\{\cos\kappa + \lambda'\left[(\cos(n-1)\nu) X_0 + (\sin(n-1)\nu) P_{X_0}\right]\right\} Y_{n-1} + (\sin\kappa) P_{Y_{n-1}}$$

$$P_{Y_n} = \left\{-\sin\kappa + \lambda' \operatorname{ctn}\kappa\left[(\cos(n-1)\nu) X_0 + (\sin(n-1)\nu) P_{X_0}\right]\right\} Y_{n-1} + (\cos\kappa) P_{Y_{n-1}},$$

which, it may be noted, is a transformation with determinant unity. The two equations just written may, for the present purposes, be conveniently replaced by a second-order recursion relation involving only the quantities Y_j :

$$Y_{n+1} - \left\{2(\cos\kappa) + \lambda'\left[(\cos n\nu) X_0 + (\sin n\nu) P_{X_0}\right]\right\} Y_n + Y_{n-1} = 0.$$

Since the expression within the square brackets may be interpreted as the radial displacement, it is natural to replace it by A cos $(n\nu + \epsilon)$, in which A represents the amplitude of the (prescribed) radial motion and in which the phase-shift ϵ may be ignored for reasons of convenience. We accordingly direct our attention to the equation

$$Y_{n+1} - \left[2\cos\kappa + \lambda' A \cos n\nu\right] Y_n + Y_{n-1} = 0.$$

B. Solution for Threshold by Use of Corresponding Differential Equation:

It is informative to note that, if κ and $\lambda' A$ are taken as small, the equation just obtained at the end of the preceding subsection may be nicely approximated by a Mathieu differential equation of a type similar to that encountered in other treatments of y-growth. We note that

$$\frac{d^2 Y}{dn^2} \cong Y_{n+1} - 2Y_n + Y_{n-1}$$

and obtain

$$\frac{d^2Y}{dn^2} + \left[2(1-\cos\kappa) - \lambda' A \cos \nu n\right] Y \doteq 0$$

$$\frac{d^2Y}{dn^2} + \left[\kappa^2 - \lambda' A \cos \nu n\right] Y \doteq 0$$

or, with $\nu n \equiv 2\tau$,

$$\frac{d^2Y}{d\tau^2} + \left[\left(\frac{2\kappa}{\nu}\right)^2 - \frac{4\lambda' A}{\nu^2} \cos 2\tau\right] Y = 0.$$

The stability boundaries pertinent to the $\sigma_x = 2\sigma_y$ resonance ($\nu = 2\kappa$) are then, for $4\lambda' A/\nu^2$ small, of course given approximately by

$$\frac{2\lambda' A}{\nu^2} = \left|1 - \left(\frac{2\kappa}{\nu}\right)^2\right|$$

or

$$A = \frac{1}{2\lambda'} \left|\nu^2 - (2\kappa)^2\right|.$$

In terms of the quantities involved in our original transformation (1),

$$\begin{aligned} \text{amplitude of } x &= A/\alpha_x \\ &= \frac{1}{2\alpha_x \lambda'} \left|\nu^2 - (2\kappa)^2\right| \\ &= \frac{1}{2\lambda} \left|\nu^2 - (2\kappa)^2\right|. \end{aligned}$$

C. Threshold of Difference Equations:

It would be a more consistant procedure to derive directly the stability limits for the difference equations, without recourse to any allegedly-similar differential equation. It appears that this may be done by a variational method which closely parallels the method whereby we have elsewhere estimated stability limits for diverse Hill equations.

We imagine that ν is commensurate with the interval covered by the transformation, in that a whole number of radial oscillations

will fit into some whole number of transformation intervals. For convenience, then, we write

$$p\nu = m(2\pi), \quad \text{with } p, m \text{ integers } (p \text{ even}).$$

By employing the concept that periodic solutions of the difference equations correspond to stability boundaries, we then consider

$$Y_{j+1} - \left[2\cos\kappa + \lambda' A\left(\cos j\frac{2m\pi}{p}\right)\right] Y_j + Y_{j-1} = 0$$

with $\qquad Y_{j+p} = Y_j.$

(Solutions conforming to the aforementioned boundary condition are thus periodic in the interval $\Delta n = p$.)

The recursion equations written above are those which formally result from minimizing* the expression

$$\begin{aligned} S = {} & Y_0 Y_1 + Y_1 Y_2 + \cdots + Y_{j-1} Y_j + Y_j Y_{j+1} + \cdots + Y_{p-2} Y_{p-1} + Y_{p-1} Y_0 \\ & - \tfrac{1}{2}\left[2\cos\kappa + \lambda' A\right] Y_0^{\,2} \\ & - \tfrac{1}{2}\left[2\cos\kappa + \lambda' A\cos\frac{2m\pi}{p}\right] Y_1^{\,2} \\ & - \tfrac{1}{2}\left[2\cos\kappa + \lambda' A\cos\left(2\frac{2m\pi}{p}\right)\right] Y_2^{\,2} \\ & \cdots \\ & - \tfrac{1}{2}\left[2\cos\kappa + \lambda' A\cos\left(j\frac{2m\pi}{p}\right)\right] Y_j^{\,2} \\ & \cdots \\ & - \tfrac{1}{2}\left[2\cos\kappa + \lambda' A\cos\left((p-1)\frac{2m\pi}{p}\right)\right] Y_{p-1}^{\,2}. \end{aligned}$$

The connection between κ, ν and A at the stability boundaries can then be sought by the introduction of suitable trial solutions into S. For the purpose of this report it may be sufficient to consider, in turn, the simple forms

$$Y_j = B_1 \cos j\frac{\pi m}{p}$$

---

* A rigorous development of this method might better regard the "minimization" as causing a sum to be stationary subject to an auxiliary (isoperimetric) condition. The use of a Lagrange multiplier should then result in the equations with which we are concerned here.

and, alternatively $Y_j = C_1 \sin j \frac{\pi m}{p}$,

which represent the dominant terms of solutions appropriate in the neighborhood of the $\sigma_x = 2\sigma_y$ resonance.

With the first of these trial solutions, S becomes given by

$$(1/p)S = \left[\tfrac{1}{2}\cos\frac{\pi m}{p} - \tfrac{1}{2}\cos\kappa - \tfrac{1}{8}\lambda' A\right] B_1^2$$

and is "stationary" (for $B_1 \neq 0$) when

$$A = \frac{4}{\lambda'}\left[\cos\frac{\pi m}{p} - \cos\kappa\right];$$

similarly, with the second trial solution,

$$(1/p)\,S = \left[\tfrac{1}{2}\cos\kappa - \tfrac{1}{2}\cos\frac{\pi m}{p} - \tfrac{1}{8}\lambda' A\right] C_1^2$$

and

$$A = \frac{4}{\lambda'}\left[\cos\kappa - \cos\frac{\pi m}{p}\right].$$

Recalling that $\pi m/p = \nu/2$, we thus find the stability boundaries to be approximately located at

$$A \doteq \frac{4}{\lambda'}\left|\cos\kappa - \cos(\nu/2)\right|$$
$$= \frac{8}{\lambda'}\left|\sin^2(\nu/4) - \sin^2(\kappa/2)\right|,$$

or

$$\text{amplitude of } x = \frac{8}{\lambda}\left|\sin^2(\nu/4) - \sin^2(\kappa/2)\right|.^{*}$$

With κ small one notes that this result for the threshold reduces to that obtained from consideration of a differential equation (cited at the end of sub-section B):

$$\text{amplitude of } x \cong \frac{1}{2\lambda}\left|\nu^2 - (2\kappa)^2\right|.$$

---

* In terms of the quantities $\cos\nu$ and $\cos\kappa$ most directly available from the original transformation, this result may be written perhaps most conveniently for calculation as

$$\text{amplitude of } x = \frac{4}{\lambda}\left|\sqrt{\frac{1+\cos\nu}{2}} - \cos\kappa\right|.$$

and, alternatively $Y_j = C_1 \sin j \frac{\pi m}{p}$,

which represent the dominant terms of solutions appropriate in the neighborhood of the $\sigma_x = 2\sigma_y$ resonance.

With the first of these trial solutions, S becomes given by

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* In terms of the quantities $\cos \nu$ and $\cos \kappa$ most directly available from the original transformation, this result may be written perhaps most conveniently for calculation as

$$\text{amplitude of } x = \frac{4}{\lambda} \left|\sqrt{\frac{1+\cos\nu}{2}} - \cos \kappa\right|.$$

4. Estimate of Lapse-Rate:

An estimate for the lapse-rate to be expected when the initial radial amplitude is above threshold may be readily obtained by reference to the differential equation cited in Section 3B. The general procedure for obtaining such an estimate has been outlined by McLachlan ["Theory and Application of Mathieu Functions" (Claredon Press, Oxford, 1947), Sects. 4.90 - 4.91] and has been applied in previous discussions of y-growth.

In this way the lapse-rate associated with the Mathieu equation cited is found to be

$$(\lambda'/\nu^2)\sqrt{A^2 - A_{thr.}^2} \quad \text{nepers per unit increment of } \tau ,$$

or

$$\frac{\lambda'}{2\nu}\sqrt{A^2 - A_{thr.}^2} \quad \text{nepers per iteration.}$$

In terms of the amplitude "a" for our initial variable "x", the corresponding lapse-rate is

$$\frac{\lambda}{2\nu}\sqrt{a^2 - a_{thr.}^2} \quad \text{nepers per iteration}$$

or

$$0.21715\,(\lambda/\nu)\sqrt{a^2 - a_{thr.}^2} \quad \text{decades per iteration.}$$

A procedure parallel to that outlined by McLachlan, if applied to the difference equations, suggests a lapse rate which, when small, is

$$\frac{\lambda}{4\sin(\nu/2)}\sqrt{a^2 - a_{thr.}^2} \quad \text{nepers per iteration}$$

or

$$0.10857\,\frac{\lambda}{\sin(\nu/2)}\sqrt{a^2 - a_{thr.}^2} \quad \text{decades per iteration.}$$

This formula, which for ν small reduces to the result found for the differential equation, is presumably preferable for predicting the lapse-rate developed by the transformation.

For hand calculations we write

$$\text{Lapse-Rate} = \frac{\lambda}{4}\,\frac{\sqrt{a^2 - a_{thr.}^2}}{\sqrt{\frac{1}{2}(1-\cos\nu)}} \qquad \text{nepers per iteration}$$

$$= 0.15355\,\lambda\,\frac{\sqrt{a^2 - a_{thr.}^2}}{\sqrt{1-\cos\nu}} \qquad \text{decades per iteration.}$$

5. Generating Function:

The transformation (4) may be written

$$X_n = (\sec\nu)X_{n-1} + (\tan\nu)P_{X_n}$$

$$P_{X_{n-1}} = (\tan\nu)X_{n-1} + (\sec\nu)P_{X_n} - (\lambda'/2)(\csc\kappa)Y_{n-1}^2$$

$$Y_n = (\sec\kappa)Y_{n-1} + (\tan\kappa)P_{Y_n}$$

$$P_{Y_{n-1}} = (\tan\kappa)Y_{n-1} + (\sec\kappa)P_{Y_n} - \lambda'(\csc\kappa)X_{n-1}Y_{n-1}.$$

These relations may be derived from a generating function

$$W(P_{X_n}, P_{Y_n}; X_{n-1}, Y_{n-1})$$
$$= \frac{1}{2}(\tan\nu)X_{n-1}^2 + (\sec\nu)X_{n-1}P_{X_n} + \frac{1}{2}(\tan\nu)P_{X_n}^2$$
$$-(\lambda'/2)(\csc\kappa)X_{n-1}Y_{n-1}^2$$
$$+\frac{1}{2}(\tan\kappa)Y_{n-1}^2 + (\sec\kappa)Y_{n-1}P_{Y_n} + \frac{1}{2}(\tan\kappa)P_{Y_n}^2,$$

employing

$$X_n = \partial W/\partial P_{X_n} \qquad Y_n = \partial W/\partial P_{Y_n}$$

$$P_{X_{n-1}} = \partial W/\partial X_{n-1} \qquad P_{Y_{n-1}} = \partial W/\partial Y_{n-1}.$$

It is possible that this generating function will be found of use in the further application of dynamical theory to transformations reduceable to the form represented by equations (4).

6. The Inverse Transformation:

The inverse of transformation (4) is found to be

$$X_{n-1} = (\cos\nu)X_n - (\sin\nu)P_{X_n}$$

(continued)

$$P_{X_{n-1}} = (\sin v) X_n + (\cos v) P_{X_n} - (\lambda'/2)(\csc \varkappa)[(\cos \varkappa) Y_n - (\sin \varkappa) P_{Y_n}]^2$$

$$Y_{n-1} = (\cos \varkappa) Y_n - (\sin \varkappa) P_{Y_n}$$

$$P_{Y_{n-1}} = (\sin \varkappa) Y_n + (\cos \varkappa) P_{Y_n} - \lambda'(\csc \varkappa)[(\cos v) X_n - (\sin v) P_{X_n}][(\cos \varkappa) Y_n - (\sin \varkappa) P_{Y_n}].$$

As with the forward transformation, this inverse transformation is again a rational algebraic transformation of degree not exceeding two. It would appear that transformations of this degree could be synthesized so that a closer similarity of form would obtain between the direct and inverse forms.

7. Computational Example:

A. Discussion:

A transformation equivalent in form to (4) has been run on the I.B.M.-704 computer by aid of the ALGYTEE program. Denoting the variables employed by the computer as ρ, p_ρ, Ψ, and p_Ψ, the equations directly iterated (Runs 10-18) were

$$\rho_n = -.128\,\rho_{n-1} + 1.744\,p_{\rho_{n-1}} + .11834784\,\Psi_{n-1}^2$$

$$p_{\rho_n} = -.564\,\rho_{n-1} - .128\,p_{\rho_{n-1}} - .00868608\,\Psi_{n-1}^2$$

$$\Psi_n = .74\,\Psi_{n-1} + 2.60\,p_{\Psi_{n-1}} + .78\,\rho_{n-1}\,\Psi_{n-1}$$

$$p_{\Psi_n} = -.174\,\Psi_{n-1} + .74\,p_{\Psi_{n-1}} + .222\,\rho_{n-1}\,\Psi_{n-1}.$$

These equations may be put into the form (1) [see footnote, section 2] by the substitution (change of scale)

$$\rho_j = x_j \qquad \Psi_j = y_j/\sqrt{.4524}$$

$$p_{\rho_j} = p_{x_j} \qquad p_{\Psi_j} = p_{y_j}/\sqrt{.4524}$$

to become of the Hamiltonian form:

$$x_n = -.128\,x_{n-1} + 1.744\,p_{x\,n-1} + .2616\,y_{n-1}^2$$

$$p_{x_n} = -.564\,x_{n-1} - .128\,p_{x_{n-1}} - .0192\,y^2_{n-1}$$

$$y_n = .74\, y_{n-1} + 2.60\, p_{y_{n-1}} + .78\, x_{n-1}\, y_{n-1}$$
$$p_{y_n} = -.174\, y_{n-1} + .74\, p_{y_{n-1}} + .222\, x_{n-1}\, y_{n-1},$$

with $\lambda = .78$, $\cos v = -.128$, and $\cos \kappa = .74$.

From the results of Section 3C we expect the threshold x-amplitude for this problem to be

$$a_{thr.} = \frac{4}{0.78}\left[0.74 - \sqrt{0.436}\right] = 0.41 .$$

The computational results to be reported suggest

$$a_{thr.} = 0.388 \cong 0.39$$

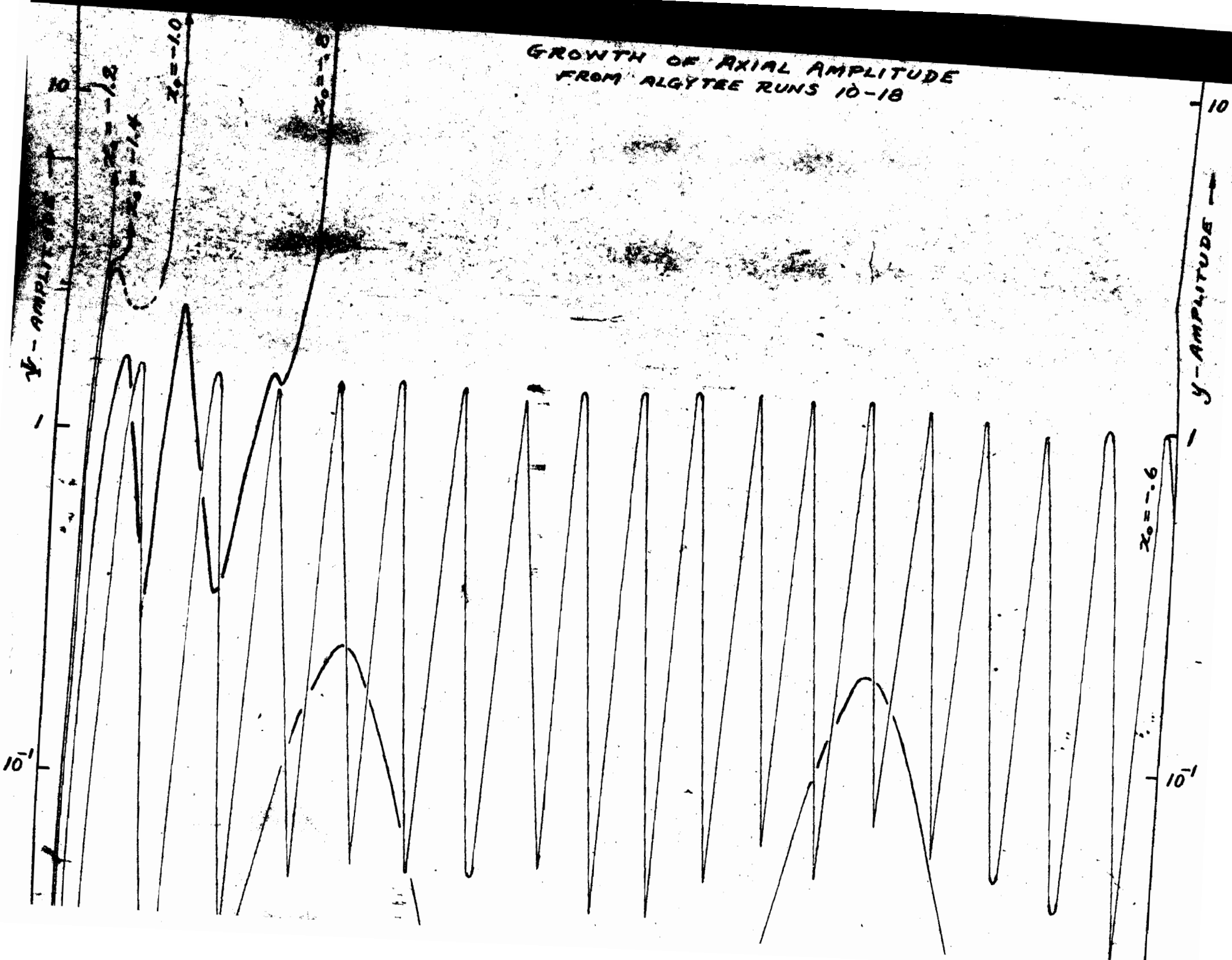
for this transformation, affording what may be regarded as a satisfactory check of the theory. (The approximate theoretical result, obtained from a <u>differential</u> equation in the limiting case of small κ, is $a_{thr.} = 0.455$, in somewhat poorer agreement with the computational result.)

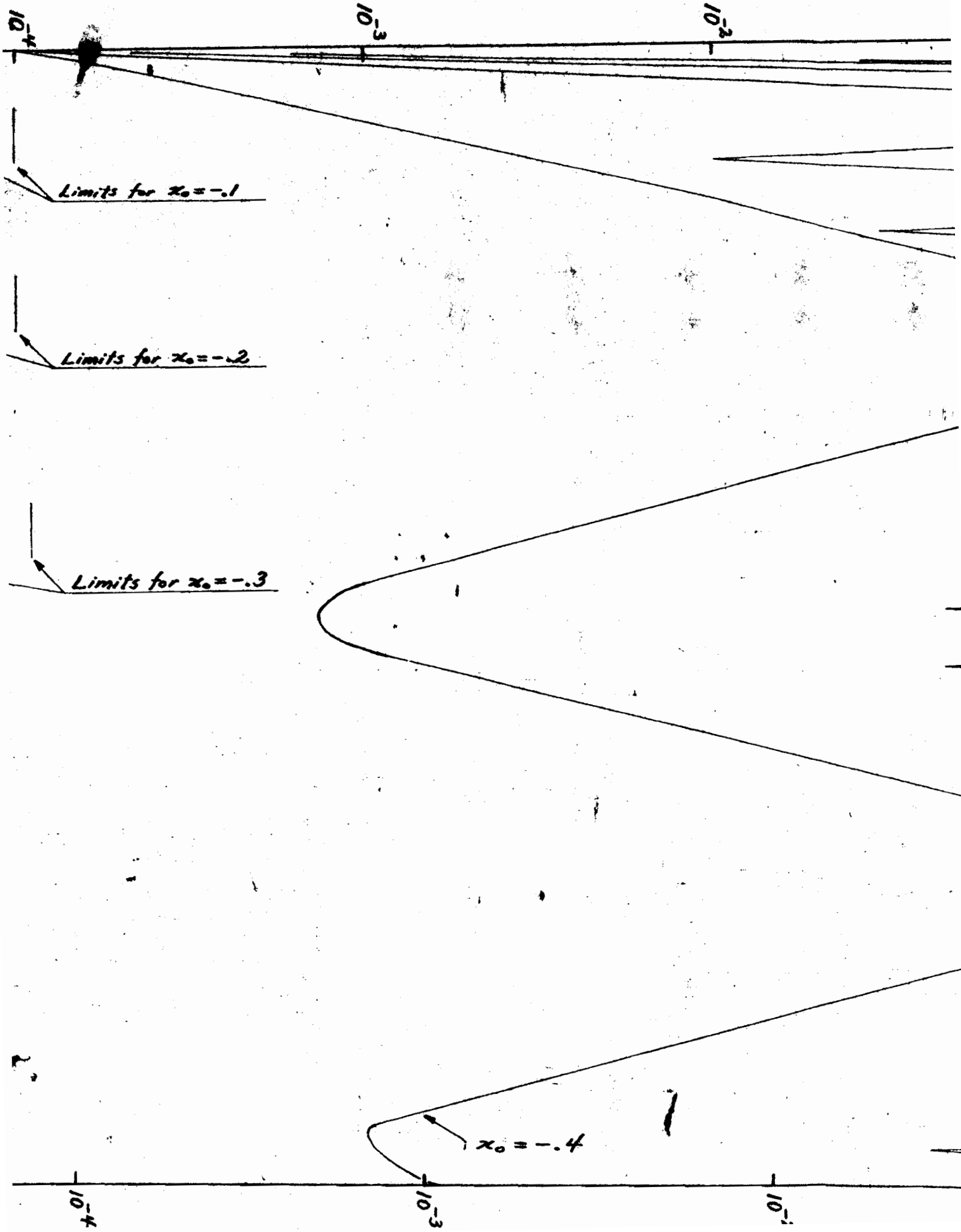
Likewise, for the lapse-rate, the results at the end of Section 4 suggest

$$\frac{0.15355 \times 0.78}{\sqrt{0.872}} \sqrt{a^2 - a_{thr.}^2} \quad \text{decades per iteration}$$

or $\quad 0.12826 \sqrt{a^2 - 0.1505} \quad$ decades per iteration

(to employ the computational result for the threshold amplitude). We tabulate below the lapse-rates calculated from this last formula and the corresponding values observed from the computations. It may be noted that the form of the theoretical equation suggests that the <u>square</u> of the lapse-rate will grow linearly with a^2, for values of $a > a_{thr.}$, a prediction which appears to be substantiated by the computations. The theoretical and computational results for $d(\mu^2)/da^2$ are, respectively, 0.016 and 0.014 (decades/iteration)^2.


GROWTH OF AXIAL AMPLITUDE
FROM ALGYTEE RUNS 10-18
Ψ - AMPLITUDE
y - AMPLITUDE
10
1
10^{-1}
$x_0 = -1.2$
$x_0 = -1.4$
$x_0 = -1.0$
$x_0 = -.8$
$x_0 = -.6$


10^{-4}
10^{-3}
10^{-2}
Limits for $x_0 = -.1$
Limits for $x_0 = -.2$
Limits for $x_0 = -.3$
$x_0 = -.4$
10^{-4}
10^{-3}
10^{-2}

10^{-2}
10^{-3}
10^{-4}
$x_0 = -.4$
Limits for $x_0 = -.3$
Limits for $x_0 = -.2$
Limits for $x_0 = -.1$
100
200
300
400
500
600
700
800
900
1000
1100
Number of Iterations

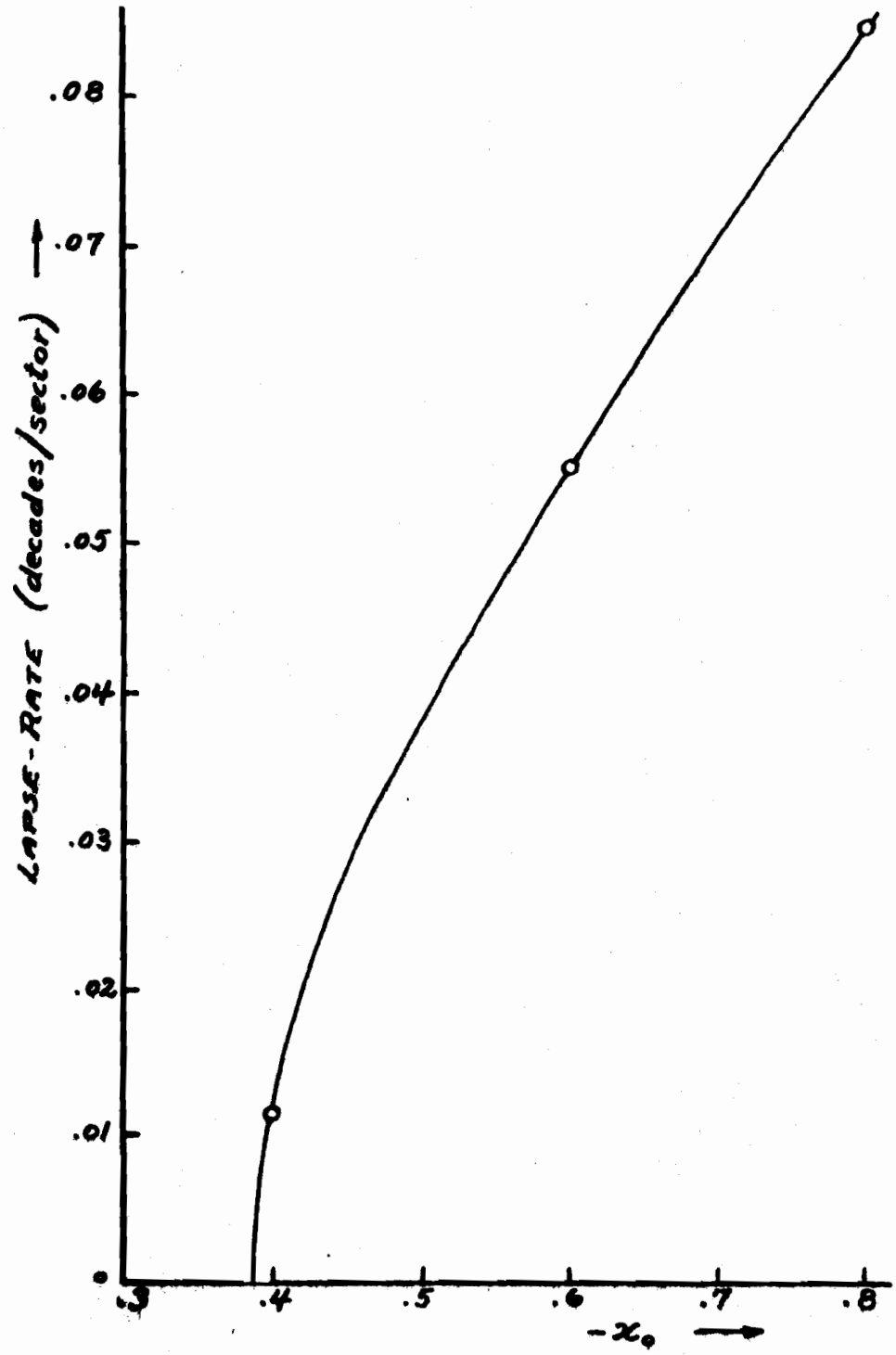


Fig. 2

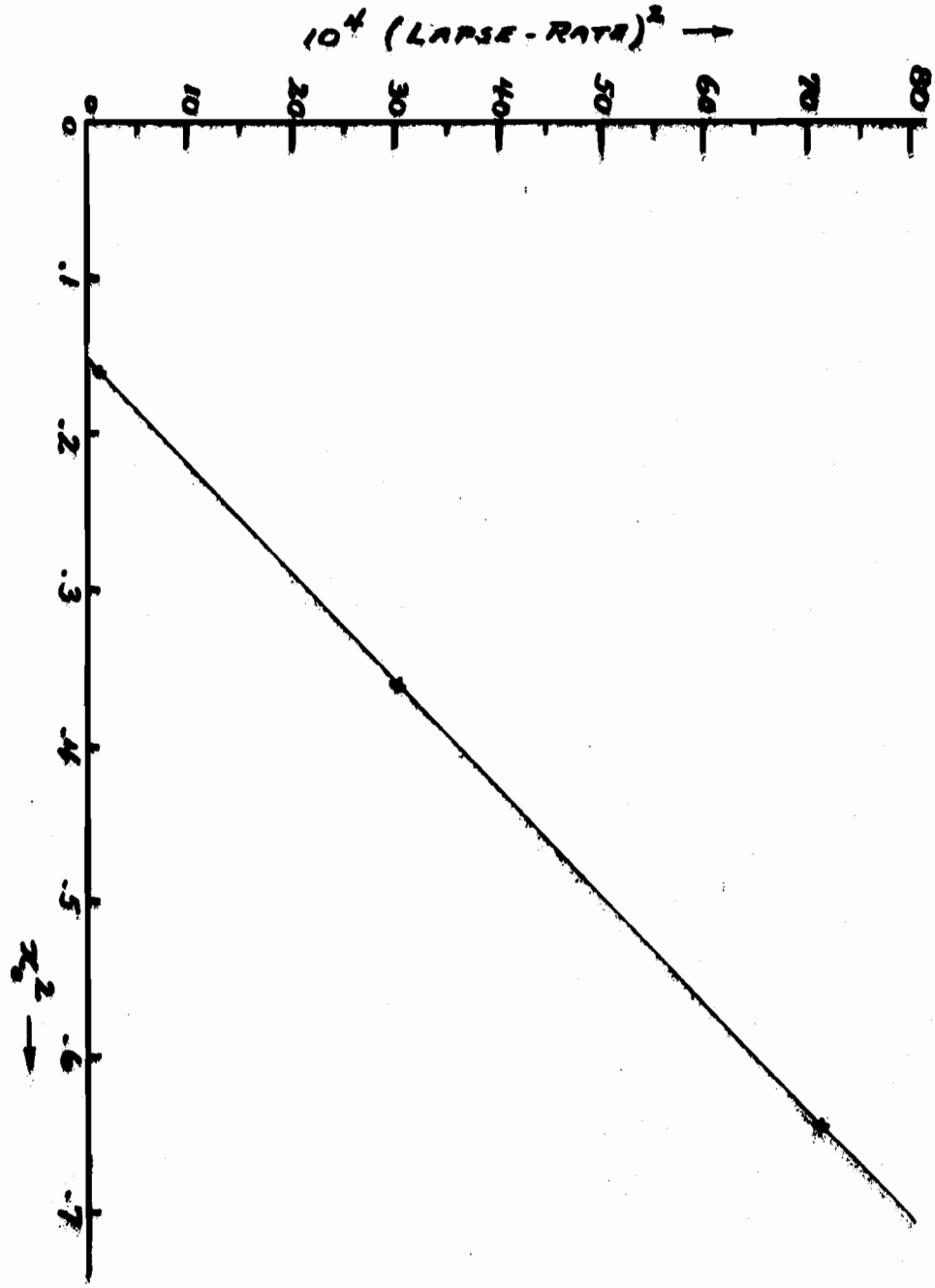


FIG. 3

CORRECTION - To MURA-246 (Int.)

"CONCERNING THE y-GROWTH PHENOMENON

EXHIBITED BY ALGEBRAIC TRANSFORMATIONS"

1. We have detected a slight numerical error in the calculations to an example given in Section 7A of MURA-246 (Int.). On p. 13, the expected lapse-rate should read

$$\frac{0.15355 \times 0.78}{\sqrt{1.128}} \sqrt{a^2 - a^2_{thr.}} \quad \text{decades per iteration}$$

or $0.11277 \sqrt{a^2 - 0.1505}$ decades per iteration when the computational result for the threshold amplitude is employed.

2. The theoretical and computational results for $d(\mu^2)/d(a^2)$ accordingly are 0.013 and 0.014 (decades/iteration)^2, respectively.

3. The table on p. 14 should read

x_o	$\lvert x_o - a_{thr.} \rvert$	Lapse-Rate (decades per iteration)	
		Calc. from Theory	From IBM Computations
- 0.4	0.01	0.011	0.0116
- 0.6	0.21	0.052	0.055
- 0.8	0.41	0.079	0.084
- 1.0	0.61	0.104	0.11

4. Similar results, giving a computational value of $d(\mu^2)/d(a^2)$ just slightly greater than the theoretical value, have also been obtained in subsequent computations with a similar transformation for which $\cos \nu = -0.125$, $\cos \kappa = 0.75$, and $\lambda = 1$.

L. Jackson Laslett
March 25, 1957