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ENERGY DEPENDENCE OF SPACE CHARGE FORCES IN AN ACCELERATOR BEAM

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ABSTRACT: A brief derivation is given of two simplified formulas which give the energy dependence of the number of charged particles which can circulate in an accelerator.

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In a previous paper we discussed space charge limitations to the number of particles that can circulate in an accelerator. We give here a brief derivation of the formulas used in Ref. 1.

We first consider the limitation caused by the mutual repulsion of the particles in a direction transverse to the motion. The number of particles N that can circulate at an energy E (including the rest energy) is related to the number N_i that can circulate at injection energy E_i by

$$\frac{N}{N_i} = \frac{pE}{p_i E_i} \tag{1}$$

where p and p_i are the corresponding momenta of particles with energy E and E_i.[2] The proof of this formula goes as follows: The transverse force F_t on a particle in the beam is given by

$$F_t = e(\mathcal{E} - \frac{V}{c} B)$$

where e is the charge, $\mathcal{E}$ and B are the electric and magnetic fields, V is the velocity of the particle, and c is the velocity of light. If $\mathcal{E}$ and B are caused by a tube (uniform density) of moving charge of radius a, we have the result

$$\mathcal{E} = \frac{eN}{\pi R} \frac{x}{a^2} \qquad B = \frac{eN}{\pi R} \frac{V}{c} \frac{x}{a^2}, \qquad x < a$$

where x is the distance of the particle from the center of the tube and R is the radius of the equilibrium orbit (assumed constant). Then

$$F_t = \frac{e^2 N}{\pi R a^2 \gamma^2} x$$

where $\gamma = (1 - V^2/c^2)^{-\frac{1}{2}}$

Now the equation of motion satisfied by the betatron oscillations of the particle is

$$m \gamma \ddot{x} + f x + F_t = 0$$

where f is the restoring force of the accelerator magnet. Puting $\frac{f}{m\gamma} = w_o^2$ we get

$$\ddot{x} + \left(W_o^2 - \frac{e^2 N}{\pi R a^2 \gamma^3 m}\right) x = 0.$$

We can define a new angular frequency

$$w^2 = \omega_o^2 - \frac{e^2 N}{\pi R a^2 \gamma^3 m}$$

Now w is related to the number of betatron waves ν around the machine by $w = \frac{\nu V}{R}$. Putting this into the expression for w we get

$$\nu^2 = \nu_o^2 - \frac{e^2 N R}{\pi a^2 m \gamma^3 V^2}$$

But $\nu \simeq \nu_o$. Otherwise resonances are crossed. Then we obtain

$$\Delta\nu = -\frac{e^2 N R}{2\pi \nu_o a^2 m \gamma^3 V^2} \qquad (2)$$

where $\Delta\nu = \nu - \nu_o$. The derivation of Eq. (2) is due to Nielsen[3].

But adiabatic damping makes the radius a of the tube of charge decrease as the momentum of the particles increases according to the expression

$$a \sim \frac{1}{\sqrt{p}} \qquad (3)$$

Using Eq. (3) in Eq. (2) we get, apart from constants,

$$\Delta\nu \sim \frac{NP}{\gamma^3 V^2}$$

so that

$$\frac{\Delta\nu}{\Delta\nu_i} = \frac{NP}{\gamma^3 V^2} \Big/ \frac{N_i P_i}{\gamma_i^3 \nu_i^2} \qquad (4)$$

We now impose the condition $\Delta\nu/\Delta\nu_i = 1$. This amounts to saying that particles are injected until the space charge limit is reached.

Then since $E = m\gamma$ and $p = m v \gamma$, Eq. (1) directly follows from Eq. (4).

We next consider the limit caused by longitudinal space charge repulsion. The number of particles that can circulate

at an energy E is related to the number at injection by

$$\frac{N}{N_i} = \frac{E^2}{E_i^2} \tag{5}$$

Nielsen has made a detailed calculation of longitudinal space charge repulsion.[3] However, we do not need to make use of his results to derive (1).

We consider the forces in the rest system of the moving particles (denoted by primes). The repulsive force F' is simply the Coulomb repulsion[4]

$$F' \sim \frac{N}{r'^2} \tag{6}$$

The attractive force is due to the electric field $\mathcal{E}'$ of the oscillator. We demand that the ratio of repulsive to attractive forces be no greater than at injection. Then

$$\frac{N/r'^2}{\mathcal{E}'} = \frac{N_i/r_i'^2}{\mathcal{E}_i'} \tag{7}$$

We now write the expression for the distance between two particle in terms of the laboratory distance r

$$r' = \gamma\, r. \tag{8}$$

We also have

$$\mathcal{E}' = \mathcal{E} \tag{9}$$

since the longitudinal component of the electric field does not change in a Lorentz transformation. But

$$\mathcal{E} = \frac{V}{S} = \text{constant}$$

where V is the voltage across the gap of length S. Using Eqs. (8) and (9), Eq. (7) reduces to Eq. (5). Note that the particle is subject to the electric field only while in the gap. But the fraction of time the particle spends in the gap is $\frac{S'}{2\pi R'} = \frac{S}{2\pi R}$ = constant.

Whether Eq. (1) or Eq. (5) limits the number of particles in the beam depends on the strength of the magnetic focussing forces and the oscillator voltage. The foregoing results apply only if charges from the ionized gas are not trapped in the beam.

REFERENCES

1. MURA-DBL-2
2. The right hand sides of Eqs (1) and (5) were mistakenly inverted in Ref. 1.
3. Carl Nielsen, Private communication.
4. The actual expression for the force is a sum of terms each going as $(1/r'_j)^2$ (j going from 1 to N), but the situation is not changed by this slight complication.