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REMARKS ON THE INVARIANT QUADRATIC FORMS PERTAINING TO MOTION CHARACTERIZED BY A LINEAR DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENT

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ABSTRACT: The nature and interpretation of the invariant quadratic forms are reviewed and means for determining the coefficients are outlined. Computation of such quadratic forms can be helpful in following the secular growth of "amplitude" in certain cases involving coupling resonances.

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1. The solutions of the linear equation (forced motion absent)

$$x'' + F(\theta)\, x = 0,$$

with $F(\theta)$ periodic with period T, may be conveniently expressed

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\theta+T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\theta}.$$

Here A, B, C, D are periodic functions of θ (period T) such that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1 \qquad \text{(constancy of the Wronskin)}$$

$$\frac{dA}{d\theta} = BF + C \qquad \frac{dB}{d\theta} = D - A$$

$$\frac{dC}{d\theta} = (D - A)F \qquad \frac{dD}{d\theta} = -(BF + C) \qquad \text{(Appendix A)}$$

$$1/2 \text{ Trace} \equiv 1/2\,(A + D) = \cos\sigma \text{ , an invariant.}$$

Solutions at homologous points may be related by

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\theta+nT} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\theta} = \begin{pmatrix} \cos n\sigma - \frac{1}{2}\frac{B_0'}{\sin\sigma}\sin n\sigma & \frac{B_0}{\sin\sigma}\sin n\sigma \\ \frac{1}{B_0 \sin\sigma}\left[\sin^2\sigma + \frac{1}{4}B_0'\right]\sin n\sigma & \cos n\sigma + \frac{1}{2}\frac{B_0'}{\sin\sigma}\sin n\sigma \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\theta};$$

the solution at an <u>any</u> θ may moreover be expressed in terms of the initial values x_0 and x_0'

$$y(\theta) = \sqrt{\frac{B(\theta)}{B_0}}\left\{\left[\cos\phi - \frac{1}{2}\frac{B_0'}{\sin\sigma}\sin\phi\right]x_0 + \left[\frac{B_0}{\sin\sigma}\sin\phi\right]x_0'\right\},$$

where $\varphi = (\sin\sigma)\int_0^{\theta}\frac{d\theta}{B}$.

2. The quantity

$$I^2 = -Cx^2 + (A - D)\,x x' + B\,x'^2$$

constitutes an invariant of the motion. (Appendix B)

In particular it is of interest to construct from this the two quantities

$$R^2 = \frac{-C}{\sin\sigma}x^2 + \frac{A-D}{\sin\sigma}x x' + \frac{B}{\sin\sigma}x'^2,$$

an invariant throughout the motion,

and

$$K^2 = \frac{-CB}{\sin^2\sigma} x^2 + \frac{(A-D)B}{\sin^2\sigma} x x' + \frac{B^2}{\sin^2\sigma} x'^2 ,$$

an invariant at homologous points.

The first of these gives the invariant R^2 which is $1/\pi$ times the area of the ellipse described by the phase-point (x, x') plotted at homologous points; the second gives the quantity K representing the maximum displacement for the particular set of homologous points chosen.

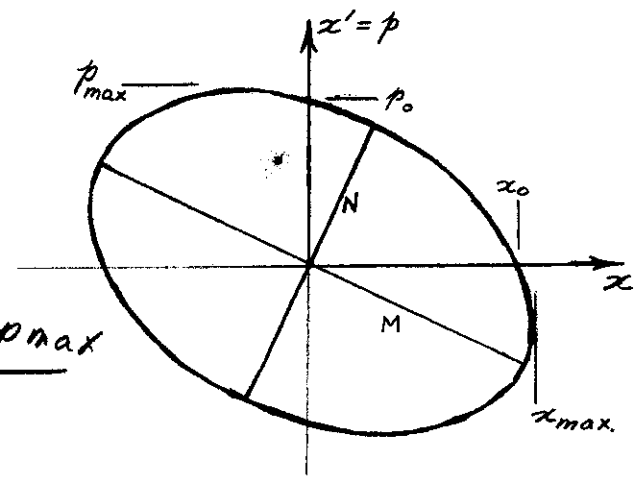
3. Writing

$$R^2 = a x^2 + b x x' + c x'^2$$

and

$$K^2 = \xi x^2 + \eta x x' + \zeta x'^2 ,$$

the coefficients may be expressed in terms of the parameters of the ellipse described by the phase-point (x, $x' = y$) (Appendix C):



$$a = \frac{p_{max}}{x_o}$$

$$b = \pm 2\sqrt{\frac{x_{max}\, p_{max}}{x_o p_o} - 1}$$

$$c = \frac{x_{max}}{p_o}$$

$$ac - 1 = \left(\frac{b}{2}\right)^2$$

$$\xi = \left(\frac{x_{max}}{x_o}\right)^2 = \left(\frac{p_{max}}{p_o}\right)^2 = \frac{x_{max}\, p_{max}}{x_o\, p_o}$$

$$\eta = \pm 2\,\frac{x_{max}}{p_o}\sqrt{\frac{x_{max}\, p_{max}}{x_o\, p_o} - 1}$$

$$\zeta = \left(\frac{x_{max}}{p_o}\right)^2$$

$$\xi\zeta - 1 = \frac{\eta^2}{4}$$

with the sign of the square root selected to be positive if the major axis lies in the IInd & IVth quadrants, and conversely.

4. Alternatively, of course, the coefficients a,b,c; ξ, η, ζ may be computed from the matrix elements A,B,C,D by aid of two one-sector runs between successive homologous points of the type of interest.

Thus, most simply, if a run is commenced with $y' = 0$,

$$A = y_1/y_0$$

$$C = y_1'/y_0 ;$$

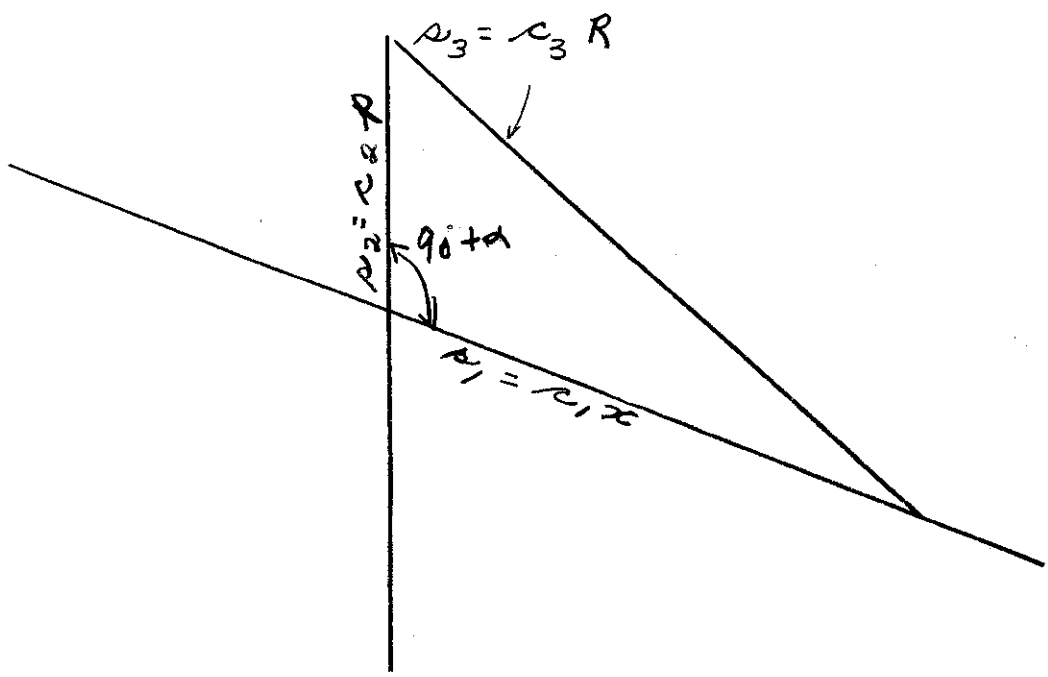
and a run commenced with $y = 0$,

$$B = y_1/y_0'$$

$$D = y_1'/y_0' .$$

(<u>If</u> $F(\theta)$ possesses <u>symmetry</u> about the reference point, $A = D = \cos\sigma$ & $-C = \frac{\sin^2\sigma}{B}$; The matrix elements and $\cos\sigma$ may then be obtained, if desired, through the use of formulas pertaining to two runs each of length T/2.)

5. The square root of a quadratic form, such as R, may be evaluated by a convenient construction:



$$c_3^2 R^2 = c_1^2 x^2 + 2 c_1 c_2 (\sin\alpha) x p + c_2^2 p^2 .$$

Choose $\left(\frac{c_1}{c_3}\right)^2 = a$, i.e., $c_1 = c_3\sqrt{a}$;

$\left(\frac{c_2}{c_3}\right)^2 = c$, $c_2 = c_3\sqrt{c}$;

$2\frac{c_1 c_2}{c_3^2}\sin\alpha = b$, $\sin\alpha = \frac{1}{2}\frac{b}{\sqrt{ac}}$.

APPENDIX A -- Proof of the differential relations for A, B, C, D:

To first order we expand

$$((M[\theta|\theta+\delta\theta]))((M)) = ((M+\delta M))((M[\theta|\theta+\delta\theta]))$$

$$\begin{pmatrix} 1 & \delta\theta \\ -F\delta\theta & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A+\delta A & B+\delta B \\ C+\delta C & D+\delta D \end{pmatrix}\begin{pmatrix} 1 & \delta\theta \\ -F\delta\theta & 1 \end{pmatrix}$$

$$\begin{pmatrix} A+C\delta\theta & B+D\delta\theta \\ C-AF\delta\theta & D-BF\delta\theta \end{pmatrix} = \begin{pmatrix} A+\delta A-BF\delta\theta & B+\delta B+A\delta\theta \\ C+\delta C-DF\delta\theta & D+\delta D+C\delta\theta \end{pmatrix}$$

$$\delta A = (BF+C)\,\delta\theta \qquad \delta B = (D-A)\,\delta\theta$$

$$\delta C = (D-A)F\,\delta\theta \qquad \delta D = -(BF+C)\,\delta\theta .$$

APPENDIX B -- Proof of the invariance of I:

Differentiation of the expression given for I^2 leads to:

$$\frac{dI^2}{d\theta} = -2Cxx' + (A-D)(x'^2 + xx'') + 2Bx'x'' - C'x^2 + (A'-D')xx' + B'x'^2 .$$

The terms $(A - D)\, x'^2$ and $B'x'^2$ cancel by virtue of the relation $B' = D - A$. Employing the relation $x'' = -Fx$,

$$\frac{dI^2}{d\theta} = (-2C + A' - D' - 2BF)xx' - [C' + F(A-D)]x^2,$$

which vanishes by virtue of the relation

$$A' = -D' = BF + C$$

$$C' = (D-A)F.$$

APPENDIX C -- The interpretation of R^2 and K^2:

In the ellipse

$$\left(\frac{x\cos\beta - p\sin\beta}{M}\right)^2 + \left(\frac{x\sin\beta + p\cos\beta}{N}\right)^2 = 1,$$

$$x_0 = \left[\left(\frac{\cos\beta}{M}\right)^2 + \left(\frac{\sin\beta}{N}\right)^2\right]^{-1/2}$$

$$p_0 = \left[\left(\frac{\sin\beta}{M}\right)^2 + \left(\frac{\cos\beta}{N}\right)^2\right]^{-1/2}$$

$$x_{max} = MN/p_0$$

$$p_{max} = MN/x_0 .$$

1. We form

$$R^2 = \frac{p_{max}}{x_0}x^2 \pm 2\left[\frac{x_{max}\,p_{max}}{x_0\,p_0} - 1\right]^{1/2} xp + \frac{x_{max}}{p_0}p^2$$

$$= MN\left\{\left[\left(\frac{\cos\beta}{M}\right)^2 + \left(\frac{\sin\beta}{N}\right)^2\right]x^2 \pm 2\left[\sin^2\beta\cos^2\beta\left(\frac{1}{M^4}+\frac{1}{N^4}\right) - \frac{1-\sin^4\beta-\cos^4\beta}{M^2N^2}\right]^{1/2} xp + \left[\left(\frac{\sin\beta}{M}\right)^2 + \left(\frac{\cos\beta}{N}\right)^2\right]p^2\right\}$$

$$= MN\left\{\left[\left(\frac{\cos\beta}{M}\right)^2 + \left(\frac{\sin\beta}{N}\right)^2\right]x^2 + 2\sin\beta\cos\beta\left(\frac{1}{N^2}-\frac{1}{M^2}\right)xp + \left[\left(\frac{\sin\beta}{M}\right)^2 + \left(\frac{\cos\beta}{N}\right)^2\right]p^2\right\}$$

$$= MN = \frac{\text{area}}{\pi}.$$

2. The quantity K^2 is $\frac{x_{max}}{p_0}R^2$ and hence is invariant for any particular set of homologous points.

x_{max} occurs when $p = -\frac{\eta}{2\xi}x$ and

$$K^2 = \left(\xi - \frac{\eta^2}{4\xi}\right)x_{max}^2 = x_{max}^2 .$$

3. The maximum amplitude at <u>any</u> point along a given orbit may be expressed in terms of K at some reference point, or in terms of R or I:

$$x_{max,max} = K_{max} = \sqrt{\frac{B_{max}}{B_{ref}}}\,K_{ref}$$

$$= \sqrt{\frac{B_{max}}{\sin\sigma}}\,R$$

$$= \frac{\sqrt{B_{max}}}{\sin\sigma}\,I .$$