Fermilab Library
0 1160 0038519 9

MURA-205
Internal

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION +

DEVELOPMENT OF ALGORISMS FOR FOROCYL POTENTIAL PROGRAM

L. Jackson Laslett*

November 10, 1956

ABSTRACT: Algorisms are developed for use in computations, on a two-dimensional grid, to determine the magnetic scalar potential characterizing the scaling field of a sprially-ridged FFAG accelerator and the resultant field-components to be used in computations of particle motion. In distinction to the case considered in MURA-LJL-8 (Revised), use of a rectangular mesh is permissible, rather than one in which the vertical and horizontal mesh dimensions are necessarily equal.

+ Supported by Contract AEC #AT(11-1)-384

* On leave from Iowa State College.

I. The Algorism for Ω, to be Solved by Relaxation:

The differential equation for Ω is taken to be [cf. MURA-LJL-8 (Revised), 21 February 1956]:

$$\frac{\ell^2}{8}(\Omega_{\xi\xi}+\Omega_{\eta\eta})+\frac{\pi^2\ell^2}{2\mathcal{D}_0}\eta^2\Omega_{\eta\eta}$$

$$-\frac{\pi(1/\omega)\ell^2}{2\mathcal{D}_0}\eta\Omega_{\xi\eta}+\frac{\pi k'(1/\omega)\ell^2}{2\mathcal{D}_0}\Omega_\xi \qquad (1)$$

$$-\frac{\pi^2(2k'-1)\ell^2}{2\mathcal{D}_0}\eta\Omega_\eta+\frac{\pi^2k'^2\ell^2}{2\mathcal{D}_0}\Omega=0,$$

where $\mathcal{D}_0$ denotes $(1/\omega)^2+N^2$, $k'=k+1$, and ℓ is taken to represent the mesh interval in the η direction. In practice the two terms $\Omega_{\xi\xi}+\Omega_{\eta\eta}$ strongly dominate and an algorism for this quantity may appropriately be developed in a manner which takes advantage of the fact that $\Omega_{\xi\xi}+\Omega_{\eta\eta}$ is very close to zero.

We now perform Taylor expansions for Ω_{10}, Ω_{-10}, Ω_{01}, Ω_{0-1}, Ω_{11}, Ω_{-11}, Ω_{1-1} and Ω_{-1-1} in terms of h and ℓ (the ξ and η mesh-dimensions, respectively), continuing the series through fourth order terms. The first two of these expansions are then each multiplied by $\frac{5\ell^2-h^2}{6}$, the next two by $\frac{5h^2-\ell^2}{6}$, the last four by $\frac{h^2+\ell^2}{12}$, and the results summed. We obtain

$$\frac{5\ell^2-h^2}{6}(\Omega_{10}+\Omega_{-10})-\frac{5}{3}(h^2+\ell^2)\Omega_{00}+\frac{5h^2-\ell^2}{6}(\Omega_{01}+\Omega_{0-1})$$

$$+\frac{h^2+\ell^2}{12}(\Omega_{11}+\Omega_{-11}+\Omega_{1-1}+\Omega_{-1-1})=h^2\ell^2(\Omega_{\xi\xi}+\Omega_{\eta\eta})$$

$$+\frac{h^2\ell^2}{12}\left[h^2(\Omega_{\xi\xi\xi\xi}+\Omega_{\xi\xi\eta\eta})+\ell^2(\Omega_{\xi\xi\eta\eta}+\Omega_{\eta\eta\eta\eta})\right].$$

Since, as mentioned above, Ω is nearly harmonic (in the Cartesian sense), we may with good accuracy ignore the fourth-order derivatives on the right and take $\Omega_{\xi\xi}+\Omega_{\eta\eta}$ to be well represented by the algorism:

$$\Omega_{\xi\xi}+\Omega_{\eta\eta} \doteq \frac{1}{12h^2\ell^2}\Big[2(5\ell^2-h^2)(\Omega_{10}+\Omega_{-10})$$
$$-20(h^2+\ell^2)\Omega_{00}+2(5h^2-\ell^2)(\Omega_{01}+\Omega_{0-1})$$
$$+(h^2+\ell^2)(\Omega_{11}+\Omega_{-11}+\Omega_{1-1}+\Omega_{-1-1})\Big]$$
$$=\frac{1}{12\ell^2}\Big[2(5r^2-1)(\Omega_{10}+\Omega_{-10})-20(r^2+1)\Omega_{00}$$
$$+2(5-r^2)(\Omega_{01}+\Omega_{0-1})+(r^2+1)(\Omega_{11}+\Omega_{-11}+\Omega_{1-1}+\Omega_{-1-1})\Big]$$

with $r\equiv\ell/h$.

For the remaining, smaller, terms simple algorisms should suffice:

$$\Omega_{\eta\eta}\doteq\frac{\Omega_{01}+\Omega_{0-1}-2\Omega_{00}}{\ell^2}$$

$$\Omega_{\xi\eta}\doteq\frac{\Omega_{11}+\Omega_{-1-1}-\Omega_{1-1}-\Omega_{-11}}{4\ell h}=\frac{r}{4}\,\frac{\Omega_{11}+\Omega_{-1-1}-\Omega_{1-1}-\Omega_{-11}}{\ell^2}$$

$$\Omega_{\xi}\doteq\frac{\Omega_{10}-\Omega_{-10}}{2h}=\frac{r}{2}\,\frac{\Omega_{10}-\Omega_{-10}}{\ell}$$

$$\Omega_{\eta}\doteq\frac{\Omega_{01}-\Omega_{0-1}}{2\ell}$$

$$\Omega=\Omega_{00}.$$

With these algorisms substituted into the differential equation (1), we obtain the working relation to be used in the iteration process:

$$\left\{\frac{5}{24}(r^2+1)+\frac{\pi^2}{D_0}\eta^2-\frac{\pi^2k'^2\ell^2}{2D_0}\right\}\Omega_{00}=$$
$$\Bigg[\left\{\frac{5-r^2}{48}+\frac{\pi^2}{2D_0}\eta^2-\frac{\pi^2(2k'-1)}{4D_0}\ell\eta\right\}\Omega_{01}$$
$$+\left\{\frac{5r^2-1}{48}-\frac{r}{4}\frac{\pi k'(1/\omega)\ell}{D_0}\right\}\Omega_{-10}+\left\{\frac{5r^2-1}{48}+\frac{r}{4}\frac{\pi k'(1/\omega)\ell}{D_0}\right\}\Omega_{10}$$
$$+\left\{\frac{5-r^2}{48}+\frac{\pi^2}{2D_0}\eta^2+\frac{\pi^2(2k'-1)}{4D_0}\ell\eta\right\}\Omega_{0-1}$$
$$+\left\{\frac{r^2+1}{96}+\frac{r}{8}\frac{\pi(1/\omega)}{D_0}\eta\right\}[\Omega_{-11}+\Omega_{1-1}]+\left\{\frac{r^2+1}{96}-\frac{r}{8}\frac{\pi(1/\omega)\eta}{D_0}\right\}[\Omega_{11}+\Omega_{-1-1}]\Bigg].$$

In this last expression, the coefficients of Ω_{00}, Ω_{01}, Ω_{-10}, Ω_{10}, Ω_{0-1}, $[\Omega_{-11}+\Omega_{1-1}]$, and $[\Omega_{11}+\Omega_{-1-1}]$ have been termed, respectively, $\mathcal{D}$, N_{01}, N_{-10}, N_{10}, N_{0-1}, N_{-11}, and N_{11}.

II. <u>The Algorism for Ω_ξ</u>:

In developing an algorism for Ω_ξ, to be used in forming the reduced fields required for the dynamical equations, we invoke a relation obtained by ξ - differentiation of the differential equation for Ω (equation (1)). This relation is of the form

$$\zeta\Omega_\xi + \alpha\frac{\ell^2}{2r^2}\Omega_{\xi\xi} + \beta\frac{\ell^2}{r}\Omega_{\xi\eta} + \gamma\frac{\ell^3}{6r^3}\Omega_{\xi\xi\xi} + \delta\frac{\ell^3}{2r^2}\Omega_{\xi\xi\eta} + \epsilon\frac{\ell^3}{2r}\Omega_{\xi\eta\eta} = 0, \tag{2}$$

where

$$\alpha = \frac{2r^2}{\ell^2}\,\frac{4\pi k'(1/\omega)}{\mathcal{D}_0} \qquad \delta = -\frac{2r^2}{\ell^3}\,\frac{4\pi(1/\omega)\eta}{\mathcal{D}_0}$$

$$\beta = -\frac{r}{\ell^2}\,\frac{4\pi^2(2k'-1)\eta}{\mathcal{D}_0} \qquad \epsilon = \frac{2r}{\ell^3}\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)$$

$$\gamma = \frac{6r^3}{\ell^3} \qquad \zeta = \frac{4\pi^2 k'^2}{\mathcal{D}_0}.$$

Again Taylor series for Ω_{10}, Ω_{-10}, Ω_{01}, Ω_{0-1}, Ω_{11}, Ω_{-11}, Ω_{1-1}, and Ω_{-1-1} are formed and the weighted sum constructed:

$$\frac{(6-\frac{1}{r^2})\alpha+6\gamma-6\epsilon}{12}(\Omega_{10}-\Omega_{00}) + \frac{(6-\frac{1}{r^2})\alpha-6\gamma+6\epsilon}{12}(\Omega_{-10}-\Omega_{00})$$
$$+\frac{-\alpha/r^2-6\delta}{12}(\Omega_{01}-\Omega_{00}) + \frac{-\alpha/r^2+6\delta}{12}(\Omega_{0-1}-\Omega_{00})$$
$$+\frac{\alpha/r^2+6\beta+6\delta+6\epsilon}{24}(\Omega_{11}-\Omega_{00}) + \frac{\alpha/r^2-6\beta+6\delta-6\epsilon}{24}(\Omega_{-11}-\Omega_{00})$$
$$+\frac{\alpha/r^2-6\beta-6\delta+6\epsilon}{24}(\Omega_{1-1}-\Omega_{00}) + \frac{\alpha/r^2+6\beta-6\delta-6\epsilon}{24}(\Omega_{-1-1}-\Omega_{00})$$

$$= \gamma \frac{l}{r} \Omega_\xi + \frac{\alpha l^2}{2r^2} \Omega_{\xi\xi} + \beta \frac{l^2}{r} \Omega_{\xi\eta} + \gamma \frac{l^3}{6r^3} \Omega_{\xi\xi\xi}$$

$$+ \delta \frac{l^3}{2r^2} \Omega_{\xi\xi\eta} + \epsilon \frac{l^3}{2r} \Omega_{\xi\eta\eta} + \alpha \frac{l^4}{24r^4} \left(\Omega_{\xi\xi\xi\xi} + \Omega_{\xi\xi\eta\eta} \right)$$

$$+ \beta \frac{l^4}{6r^3} \left(\Omega_{\xi\xi\xi\eta} + r^2 \Omega_{\xi\eta\eta\eta} \right);$$

accordingly, neglecting the fourth derivative terms on the right (which will be especially small when $r \approx 1$) and employing (2), we obtain

$$\Omega_\xi = \frac{1}{\gamma \frac{l}{r} - \xi} \left[\begin{array}{l} \frac{\alpha}{r^2} - 6\beta + 6\delta - 6\epsilon \over 24 \Omega_{-11} + \frac{-\frac{\alpha}{r^2} - 6\delta}{12} \Omega_{01} + \frac{\frac{\alpha}{r^2} + 6\beta + 6\delta + 6\epsilon}{24} \Omega_{11} \\ + \frac{(6 - \frac{1}{r^2})\alpha - 6\gamma + 6\epsilon}{12} \Omega_{-10} - \left(1 - \frac{1}{6r^2}\right) \alpha \Omega_{00} + \frac{(6 - \frac{1}{r^2})\alpha + 6\gamma - 6\epsilon}{12} \Omega_{10} \\ + \frac{\frac{\alpha}{r^2} + 6\beta - 6\delta - 6\epsilon}{24} \Omega_{-1-1} + \frac{-\frac{\alpha}{r^2} + 6\delta}{12} \Omega_{0-1} + \frac{\frac{\alpha}{r^2} - 6\beta - 6\delta + 6\epsilon}{24} \Omega_{1-1} \end{array} \right]$$

$$= \frac{1}{l \left[6r^2 - \frac{4\pi^2 k'^2 l^2}{D_0} \right]} \left[\begin{array}{l} \left\{ \frac{\pi k' (1/\omega) l}{3D_0} + \frac{\pi^2 (2k'-1) \eta r l}{D_0} - \frac{2\pi (1/\omega) r^2 \eta}{D_0} - \frac{r}{2} \left(1 + \frac{4\pi^2 \eta^2}{D_0} \right) \right\} \Omega_{-11} \\ + \left\{ -\frac{2\pi k' (1/\omega) l}{3D_0} + \frac{4\pi (1/\omega) r^2 \eta}{D_0} \right\} \Omega_{01} \\ + \left\{ \frac{\pi k' (1/\omega) l}{3D_0} - \frac{\pi^2 (2k'-1) \eta r l}{D_0} - \frac{2\pi (1/\omega) r^2 \eta}{D_0} + \frac{r}{2} \left(1 + \frac{4\pi^2 \eta^2}{D_0} \right) \right\} \Omega_{11} \\ + \left\{ \frac{2(6r^2-1)\pi k' (1/\omega) l}{3D_0} - 3r^3 + r \left(1 + \frac{4\pi^2 \eta^2}{D_0} \right) \right\} \Omega_{-10} \\ - \left\{ \frac{4(6r^2-1)\pi k' (1/\omega) l}{3D_0} \right\} \Omega_{00} \\ + \left\{ \frac{2(6r^2-1)\pi k' (1/\omega) l}{3D_0} + 3r^3 - r \left(1 + \frac{4\pi^2 \eta^2}{D_0} \right) \right] \Omega_{10} \end{array} \right. \quad (3)$$

$$+\left\{\frac{\pi k'(1/\omega)l}{3D_0}-\frac{\pi^2(2k'-1)\eta r l}{D_0}+\frac{2\pi(1/\omega)r^2\eta}{D_0}-\frac{r}{2}\left(1+\frac{4\pi^2\eta^2}{D_0}\right)\right\}\Omega_{-1-1}$$

$$+\left\{-\frac{2\pi k'(1/\omega)l}{3D_0}-\frac{4\pi(1/\omega)r^2\eta}{D_0}\right\}\Omega_{0-1}$$

$$+\left\{\frac{\pi k'(1/\omega)l}{3D_0}+\frac{\pi^2(2k'-1)\eta r l}{D_0}+\frac{2\pi(1/\omega)r^2\eta}{D_0}+\frac{r}{2}\left(1+\frac{4\pi^2\eta^2}{D_0}\right)\right\}\Omega_{1-1}$$

III. <u>The Algorisms for Ω_η</u>:

Similarly, in developing an algorism for Ω_η, we make use of a result obtained by η- differerentiation of (1):

$$f\Omega_\eta + a\frac{l^2}{2}\Omega_{\eta\eta} + b\frac{l^2}{r}\Omega_{\xi\eta} + c\frac{l^3}{6}\Omega_{\eta\eta\eta} + d\frac{l^3}{2r}\Omega_{\xi\eta\eta} + e\frac{l^3}{2r^2}\Omega_{\xi\xi\eta} = 0,$$

with

$$a = -\frac{8\pi^2(2k-1)\eta}{l^2 D_0} \qquad d = -\frac{8\pi(1/\omega)r\eta}{l^3 D_0} \qquad (4)$$

$$b = \frac{4\pi k(1/\omega)r}{l^2 D_0} \qquad e = \frac{2r^2}{l^3}$$

$$c = \frac{6}{l^3}\left(1+\frac{4\pi^2\eta^2}{D_0}\right) \qquad f = \frac{4\pi^2 k^2}{D_0},$$

with $k = k'-1$ and (as before) $D_0 = (1/\omega)^2 + N^2$.

Taylor's series for $\Omega_{10}, \Omega_{-10}, \Omega_{01}, \Omega_{0-1}, \Omega_{11}, \Omega_{-11}, \Omega_{1-1}$, and Ω_{-1-1}, are again employed and the weighted sum constructed:

$$\frac{-r^2a-6d}{12}(\Omega_{10}-\Omega_{00}) + \frac{-r^2a+6d}{12}(\Omega_{-10}-\Omega_{00})$$

$$+\frac{(6-r^2)a+6b-6e}{12}(\Omega_{01}-\Omega_{00}) + \frac{(6-r^2)a-6b+6e}{12}(\Omega_{0-1}-\Omega_{00})$$

$$+\frac{r^2a+6b+6d+6e}{24}(\Omega_{11}-\Omega_{00}) + \frac{r^2a-6b-6d+6e}{24}(\Omega_{-11}-\Omega_{00})$$

$$+\frac{r^2a-6b+6d-6e}{24}(\Omega_{1-1}-\Omega_{00}) + \frac{r^2a+6b-6d-6e}{24}(\Omega_{-1-1}-\Omega_{00})$$

$$= c\,l\,\Omega_\eta + a\frac{l^2}{2}\Omega_{\eta\eta} + b\frac{l^2}{r}\Omega_{\xi\eta} + c\frac{l^3}{6}\Omega_{\eta\eta\eta}$$
$$+ d\frac{l^3}{2r}\Omega_{\xi\eta\eta} + e\frac{l^3}{2r^2}\Omega_{\xi\xi\eta} + a\frac{l^4}{24}\left(\Omega_{\eta\eta\eta\eta} + \Omega_{\xi\xi\eta\eta}\right)$$
$$+ b\frac{l^4}{6r^3}\left(r^2\Omega_{\xi\eta\eta\eta} + \Omega_{\xi\xi\xi\eta}\right);$$

neglecting the combinations of fourth-derivative terms and employing (4) we now obtain

$$\Omega_\eta = \frac{1}{c\,l - f}\left[\frac{r^2 a - 6b - 6d + 6e}{24}\Omega_{-11} + \frac{(6-r^2)a + 6c - 6e}{12}\Omega_{01} + \frac{r^2 a + 6b + 6d + 6e}{24}\Omega_{11}\right.$$
$$+ \frac{-r^2 a + 6d}{12}\Omega_{-10} - \left(1 - \frac{r^2}{6}\right)a\,\Omega_{00} + \frac{-r^2 a - 6d}{12}\Omega_{10}$$
$$\left. + \frac{r^2 a + 6b - 6d - 6e}{24}\Omega_{-1-1} + \frac{(6-r^2)a - 6c + 6e}{12}\Omega_{0-1} + \frac{r^2 a - 6b + 6d - 6e}{24}\Omega_{1-1}\right]$$

$$= \frac{1}{l\left[6\left(1 + \frac{4\pi^2\eta^2}{D_0}\right) - \frac{4\pi^2 k^2 l^2}{D_0}\right]}\left[\left\{-\frac{\pi^2(2k-1)\eta r^2 l}{3D_0} - \frac{\pi k (1/\omega) r l}{D_0} + \frac{2\pi(1/\omega)r\eta}{D_0} + \frac{r^2}{2}\right\}\Omega_{-11}\right.$$
$$+ \left\{-\frac{\pi^2(2k-1)\eta r^2 l}{3D_0} + \frac{\pi k (1/\omega) r l}{D_0} - \frac{2\pi(1/\omega)r\eta}{D_0} + \frac{r^2}{2}\right\}\Omega_{11}$$
$$+ \left\{-\frac{2\pi^2(2k-1)\eta(6-r^2)l}{3D_0} + 3\left(1 + \frac{4\pi^2\eta^2}{D_0}\right) - r^2\right\}\Omega_{01}$$
$$+ \left\{\frac{2\pi^2(2k-1)\eta r^2 l}{3D_0} - \frac{4\pi(1/\omega)r\eta}{D_0}\right\}\Omega_{-10}$$
$$+ \left\{\frac{4\pi^2(2k-1)\eta(6-r^2)l}{3D_0}\right\}\Omega_{00}$$
$$+ \left\{\frac{2\pi^2(2k-1)\eta r^2 l}{3D_0} + \frac{4\pi(1/\omega)r\eta}{D_0}\right\}\Omega_{10}$$

$$
\left.
\begin{aligned}
&+\left\{-\frac{2\pi^2(2k-1)\eta(6-r^2)\ell}{3\mathcal{D}_0}-3\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+r^2\right\}\Omega_{0-1}\\
&+\left\{-\frac{\pi^2(2k-1)\eta r^2\ell}{3\mathcal{D}_0}+\frac{\pi k(1/\omega)r\ell}{\mathcal{D}_0}+\frac{2\pi(1/\omega)r\eta}{\mathcal{D}_0}-\frac{r^2}{2}\right\}\Omega_{-1-1}\\
&+\left\{-\frac{\pi^2(2k-1)\eta r^2\ell}{3\mathcal{D}_0}-\frac{\pi k(1/\omega)r\ell}{\mathcal{D}_0}-\frac{2\pi(1/\omega)r\eta}{\mathcal{D}_0}-\frac{r^2}{2}\right\}\Omega_{1-1}
\end{aligned}
\right] \quad (5)
$$

IV. Ω_η in the Median Plane, and its Average Value:

For normalization of the scaled fields we require the average value of Ω_η evaluated in the median plane. Making use of the fact that Ω is an odd function, eq. (5) reduces, when $\eta = 0$, to:

$$
\Omega_\eta\Big|_{\eta=0} = \frac{\left[r^2-\frac{2\pi k(1/\omega)r\ell}{\mathcal{D}_0}\right]\Omega_{-11}+\left[6-2r^2\right]\Omega_{01}+\left[r^2+\frac{2\pi k(1/\omega)r\ell}{\mathcal{D}_0}\right]\Omega_{11}}{\ell\left[6-\frac{4\pi^2k^2\ell^2}{\mathcal{D}_0}\right]} .
$$

With $r = 1/k$ intervals in the period $\Delta\xi = 1$, the average value of this quantity is

$$
\langle\Omega_\eta|_{\eta=0}\rangle_{Av} = \frac{\sum_{i=0}^{n-1}\Omega_{i1}}{n\ell\left[1-\frac{2\pi^2k^2\ell^2}{3\mathcal{D}_0}\right]} = \frac{\sum_{i=0}^{n-1}\Omega_{i1}}{r\left[1-\frac{2\pi^2k^2\ell^2}{3\mathcal{D}_0}\right]} .
$$

It may prove convenient to employ this constant as a scaling factor, to form

$$
\frac{\Omega}{\langle\Omega_\eta|_0\rangle} = \frac{r\left[1-\frac{2\pi^2k^2\ell^2}{3\mathcal{D}_0}\right]\Omega}{\sum_{i=0}^{n-1}\Omega_{i1}} \qquad (6)
$$

for the values at the individual points of the mesh. In this same vein

$$
\frac{\Omega_\xi}{\langle\Omega_\eta|_0\rangle} = \frac{r\left[1-\frac{2\pi^2k^2\ell^2}{3\mathcal{D}_0}\right]\Omega_\xi}{\sum_{i=0}^{n-1}\Omega_{i1}}
$$

$$
\begin{aligned}
&\left\{-\frac{r}{2}\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+\frac{\pi k'(1/\omega)l}{3\mathcal{D}_0}-\frac{2\pi(1/\omega)r^2\eta}{\mathcal{D}_0}+\frac{\pi^2(2k'-1)\eta r l}{\mathcal{D}_0}\right\}\Omega_{-11}\\
&+\left\{\frac{r}{2}\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+\frac{\pi k'(1/\omega)l}{3\mathcal{D}_0}-\frac{2\pi(1/\omega)r^2\eta}{\mathcal{D}_0}-\frac{\pi^2(2k'-1)\eta r l}{\mathcal{D}_0}\right\}\Omega_{11}\\
&\qquad+\left\{\frac{4\pi(1/\omega)r^2\eta}{\mathcal{D}_0}-\frac{2\pi k'(1/\omega)l}{3\mathcal{D}_0}\right\}\Omega_{01}\\
&+\left\{-3r^3+r\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+\frac{2\pi(6r^2-1)k'(1/\omega)l}{3\mathcal{D}_0}\right\}\Omega_{-10}\quad\frac{4\pi(6r^2-1)k'(1/\omega)l}{3\mathcal{D}_0}\Omega_{00}\\
&+\left\{3r^3-r\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+\frac{2\pi(6r^2-1)k'(1/\omega)l}{3\mathcal{D}_0}\right\}\Omega_{10}\\
&\qquad+\left\{-\frac{4\pi(1/\omega)r^2\eta}{\mathcal{D}_0}-\frac{2\pi k'(1/\omega)l}{3\mathcal{D}_0}\right\}\Omega_{0-1}\\
&+\left\{-\frac{r}{2}\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+\frac{\pi k'(1/\omega)l}{3\mathcal{D}_0}+\frac{2\pi(1/\omega)r^2\eta}{\mathcal{D}_0}-\frac{\pi^2(2k'-1)\eta r l}{\mathcal{D}_0}\right\}\Omega_{-1-1}\\
&+\left\{\frac{r}{2}\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+\frac{\pi k'(1/\omega)l}{3\mathcal{D}_0}+\frac{2\pi(1/\omega)r^2\eta}{\mathcal{D}_0}+\frac{\pi^2(2k'-1)\eta r l}{\mathcal{D}_0}\right\}\Omega_{1-1}
\end{aligned}
$$

$$
= \frac{r\left[1-\frac{2\pi^2k^2l^2}{3\mathcal{D}_0}\right]}{l\left[6r^2-\frac{4\pi^2k'^2l^2}{\mathcal{D}_0}\right]}\cdot\frac{}{\sum_{i=0}^{n-1}\Omega_{i1}} \qquad (7)
$$

(Note that k' and $k\equiv k'-1$ are to be distinguished here, in that the latter quantity is used in the factor $\left[1-\frac{2\pi^2k^2l^2}{3\mathcal{D}_0}\right]$)

and

$$
\frac{\Omega_\eta}{\langle\Omega_\eta|_0\rangle} = \frac{r\left[1-\frac{2\pi^2k^2l^2}{3\mathcal{D}_0}\right]\Omega_\eta}{\sum_{i=0}^{n-1}\Omega_{i1}}
$$

$$
= \frac{r\left[1-\frac{2\pi^2 k^2 l^2}{3\mathcal{D}_0}\right]}{l\left[6\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)-\frac{4\pi^2 k^2 l^2}{\mathcal{D}_0}\right]} \cdot \frac{\left[\begin{array}{l}
\left\{\frac{r^2}{2}-\frac{\pi k(1/w)rl}{\mathcal{D}_0}+\frac{2\pi(1/w)r\eta}{\mathcal{D}_0}-\frac{\pi^2(2k-1)\eta r^2 l}{3\mathcal{D}_0}\right\}\Omega_{-11} \\
+\left\{\frac{r^2}{2}+\frac{\pi k(1/w)rl}{\mathcal{D}_0}-\frac{2\pi(1/w)r\eta}{\mathcal{D}_0}-\frac{\pi^2(2k-1)\eta r^2 l}{3\mathcal{D}_0}\right\}\Omega_{11} \\
\qquad +\left\{3\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)-r^2-\frac{2\pi^2(2k-1)(6-r^2)\eta l}{3\mathcal{D}_0}\right\}\Omega_{01} \\
+\left\{-\frac{4\pi(1/w)r\eta}{\mathcal{D}_0}+\frac{2\pi^2(2k-1)\eta r^2 l}{3\mathcal{D}_0}\right\}\Omega_{-10}\ \frac{4\pi^2(2k-1)(6-r^2)\eta l}{3\mathcal{D}_0}\Omega_{00} \\
+\left\{\frac{4\pi(1/w)r\eta}{\mathcal{D}_0}+\frac{2\pi^2(2k-1)\eta r^2 l}{3\mathcal{D}_0}\right\}\Omega_{10} \\
\qquad +\left\{-3\left(1+\frac{4\pi^2\eta^2}{\mathcal{D}_0}\right)+r^2-\frac{2\pi^2(2k-1)(6-r^2)}{3\mathcal{D}_0}\eta l\right\}\Omega_{0-1} \\
+\left\{-\frac{r^2}{2}+\frac{\pi k(1/w)rl}{\mathcal{D}_0}+\frac{2\pi(1/w)r\eta}{\mathcal{D}_0}-\frac{\pi^2(2k-1)\eta r^2 l}{3\mathcal{D}_0}\right\}\Omega_{-1-1} \\
+\left\{-\frac{r^2}{2}-\frac{\pi k(1/w)rl}{\mathcal{D}_0}-\frac{2\pi(1/w)r\eta}{\mathcal{D}_0}-\frac{\pi^2(2k-1)\eta r^2 l}{3\mathcal{D}_0}\right\}\Omega_{1-1}
\end{array}\right]}{\sum_{i=0}^{n-1}\Omega_{i1}} \qquad (8)
$$

V. <u>The Scaled Fields</u>:

For use in the dynamical equations, with $\frac{e}{AP}=\frac{2\pi}{\mathcal{D}_0^{1/2}\langle\Omega_\eta|_0\rangle}$, we make use of the scaled quantities ("fields") formed from expressions (6), (7) and (8):

$$
h_z \equiv \frac{e r_1}{p}\ \frac{B_z}{(1+x)k} = -\frac{\Omega_\eta}{\langle\Omega_\eta|_0\rangle} \qquad (9)
$$

$$h_r \equiv \frac{e r_1}{p} \frac{B_r}{(1+x)^k} = -\frac{2\pi}{\sqrt{(1/\omega)^2 + N^2}} \frac{k'\Omega + \frac{1}{2\pi\omega}\Omega_\xi - \eta\Omega_\eta}{\langle \Omega_\eta |_0 \rangle} \tag{10}$$

$$h_\theta \equiv \frac{e r_1}{p} \frac{B_\theta}{(1+x)^k} = +\frac{N}{\sqrt{(1/\omega)^2 + N^2}} \frac{\Omega_\xi}{\langle \Omega_\eta |_0 \rangle} . \tag{11}$$

Of these quantities, h_z will be an even function of η and both h_r and h_θ will be odd.

VI. <u>Acknowledgement</u>:

The work reported here was stimulated by helpful discussions with Dr. J. N. Snyder. The assistance of Mr. Homer Meier in careful perusal of the manuscript is also gratefully acknowledged.