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EX POST FACTO BETATRON FREQUENCY CALCULATIONS FOR THE MICHIGAN RADIAL SECTOR FFAG BETATRON

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ABSTRACT: Calculations of frequencies of linear betatron oscillations in the Michigan Model using measured field values are described. The discrepancies between design and measured values are resolved. These discrepancies are due to the difference in field shape as a function of azimuth caused by the finite cross-sectional area of the pole face current return windings.

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Introduction

When the design calculations for the Michigan Model[1] were performed,[2][3] magnetic field measurements were not available. Various conformal mappings for individual two-dimensional magnets were combined to give a field plot. This plot was used to find the pole geometry which gave the frequencies of linear betatron oscillations considered desirable. Table I shows these design frequencies and the frequencies measured experimentally[1] by two different methods, static (sigma-testing) and dynamic (R. F. knockout).

TABLE I

	Design	Static	Dynamic
ν_x	2.80	2.87	2.85
ν_z	1.80	2.12	2.18

The static experimental measurements are only accurate to approximately 2%, while the dynamic values are accurate to within 1% so that the two methods of measurement are consistent, but there is clearly a disagreement of the order of 15% between design and measured values of ν_z. The present note describes in detail the calculations done to understand this discrepancy.

I. Magnetic Field Measurements

Figure I shows in graphical form measurements performed by C. H. Pruett and R. O. Haxby's Purdue Group of the vertical component of magnetic field in the median plane as a function of azimuthal

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(1) MURA-LWJ/KMT-5
(2) MURA-FTC/DWK-1
(3) MURA /FTC-2

angle at constant radius for three different radii*. The field is symmetric about the centers of magnets. (The peak near the edge of the wide magnet occurs because the equilibrium orbit does not lie on a curve of constant radius and the field was designed to be constant along an equilibrium orbit.) The field may be written in the form

$$H_z = H_0 \left(\frac{r}{r_0}\right)^k f(\theta) = H_0 \left(\frac{r}{r_0}\right)^k \left[1 + \sum_{n=1}^{\infty} g_n \cos nN\theta\right] \qquad (1)$$

which is the same form as the spiral sector field with $\frac{1}{w} = 0$.

The parameter k can be computed easily from any two points at different radii and the same azimuth. Table II gives values of k averaged separately over the wide and narrow magnets. Points of low field near the center of the straight section are very inaccurate and are not included.

TABLE II

	k between r = 39.5 cm and r = 44.6 cm curves	k between r = 44.6 cm and r = 49.7 cm curves	Design Value
Wide Magnet	3.254	3.371	3.36
Narrow Magnet	3.346	3.346	3.36

The value at small radii in the wide magnet is conspicuously different, which is due to the details of pole-face windings, since this region includes the radius where the second layer of pole face winding begins, where fluctuations of k had been noted experimentally. This difference probably has a very small effect on frequencies, since

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* The measurements have been carried through only half of one wide magnet and half of the adjacent narrow magnet (with other magnets in place) leaving some doubt about whether this is a representative half-sector.

ν_z does not depend strongly on k, while $\nu_x \propto \sqrt{k+1}$ and the ratio of ν_x with minimum and maximum k (supposing they extend over the whole sector) is

$$\sqrt{\frac{1+3.371}{1+3.254}} = 1.014,$$

which might be masked by errors of measurement.

It is interesting for purposes of comparison with the spiral sector machine to Fourier-analyze $f(\theta)$. From the curve at r = 49.5 cm. we find

$g_1 = 4.426$	$g_5 = -0.225$
$g_2 = -2.115$	$g_6 = -0.005$
$g_3 = 0.024$	g_{10} 0.001
$g_4 = 0.473$	

The flutter ($=\sqrt{\sum_n g_n^2}$, neglecting orbit scalloping) is 4.933, in contrast to the values 1/4 in the sinusoidal and $\cong$ 1 in the separated sector spiral sector machines. For our purposes, it might be remarked that the usual analytical methods of approximating the equilibrium orbit converge rather slowly for such large flutters. In addition, higher approximations become laborious because of the large number of Fourier coefficients. Because of these difficulties, we have done the whole calculation numerically.

II. Methods of Calculation

The equations of motion can be derived from the variational principle

$$\delta \int_{Path} \left(\underline{p} + \frac{e}{c} \underline{A} \right) \cdot d\underline{s} = 0 \tag{2}$$

or

$$\delta \int_{\theta_1}^{\theta_2} \left(p \frac{ds}{d\theta} + \frac{e}{c} \frac{d\underline{s}}{d\theta} \cdot \underline{A} \right) d\theta = 0 \tag{3}$$

so that if θ is used as independent variable, the LaGrangian is

$$\mathcal{L} = p \frac{ds}{d\theta} + \frac{e}{c} \frac{d\underline{s}}{d\theta} \underline{A} \tag{4}$$

In the median plane,

$$s' = \frac{ds}{d\theta} = \sqrt{r^2 + r'^2} \tag{5}$$

and $\frac{d\underline{s}}{d\theta}$ has a θ-component equal to r. Since the field has only a z-component, the vector potential has only a θ- component

$$A_\theta = \frac{H_0 r_0}{k+2} \left(\frac{r}{r_0}\right)^{k+1} f(\theta) \tag{6}$$

so that

$$\mathcal{L} = p\left[\sqrt{r^2 + r'^2} + \frac{eH_0}{cp} \frac{r_0^2}{k+2} \left(\frac{r}{r_0}\right)^{k+2} f(\theta)\right]$$

or, introducing $\gamma = \frac{r}{r_0}$ and $\rho_0 = \frac{eH_0}{cp}$

$$\mathcal{L} = p r_0 \left[\sqrt{\gamma^2 + \gamma'^2} + \frac{r_0}{\rho_0} \frac{\gamma^{k+2}}{k+2} f(\theta)\right] \tag{7}$$

which is equivalent to the Hamiltonian

$$\mathcal{H} = p r_0 \left[\gamma\sqrt{1-\xi^2} - \frac{r_0}{\rho_0} \frac{\gamma^{k+2}}{k+2} f(\theta)\right] \tag{8}$$

The equations of motion are, from (8)

$$\begin{cases} \xi' = \frac{r_0}{\rho_0} \gamma^{k+1} f(\theta) - \sqrt{1-\xi^2} \\ \gamma' = -\frac{\xi}{\sqrt{1-\xi^2}} \gamma \end{cases} \tag{9}$$

The equations (9) are equivalent to equations (11) of reference 3 with $\xi = \sin\phi$. We put (9) in a more convenient form for numerical integration by the (non-canonical) transformation $\zeta = \gamma^{k+1}$. Then we have

$$\begin{cases} \xi' = \frac{r_0}{\rho_0} \zeta f(\theta) - \sqrt{1-\xi^2} \\ \zeta' = -(k+1) \frac{\xi}{\sqrt{1-\xi^2}} \zeta \end{cases} \tag{10}$$

We have integrated (10) numerically, first by hand using

Milne's Method[4] and later by a program prepared by one of us (H.K.M.) for the I.B.M. 650 of the University of Wisconsin, using the Runge-Kutta Method.[4] One may take advantage of symmetry; the equilibrium orbit is even about the centers of magnets. Thus we integrate through half a sector and the equilibrium orbit is that solution which has zero slope ($\xi = 0$) at both magnet centers.

We have used either 40 or 100 steps per half-sector. The values of $f(\theta)$ for each step were obtained by Aitken's Method of Interpolation[4] from the measurements of largest radius. Since the machine scales, one may vary either $\frac{r_0}{\rho_0}$ (the circumference factor) or $\zeta(0)$ on different trials.

The equilibrium orbit obtained from (10) is used to find values for the quantities in the betatron oscillation equations[2]

$$\begin{cases} \frac{d^2x}{ds^2} + \left[\frac{F^2(s)}{\rho_0^2} + \frac{n_0}{\rho_0^2}F(s)\frac{r_0}{r_e(s)}\sec\phi(s) + \frac{1}{\rho_0}\frac{dF}{ds}\tan\phi\right]x = 0 \\ \frac{d^2z}{ds^2} - \left[\frac{n_0}{\rho_0^2}F(s)\frac{r_0}{r_e(s)}\sec\phi(s) + \frac{1}{\rho_0}\frac{dF}{ds}\tan\phi\right]z = 0 \end{cases} \quad (11)$$

or their equivalent

$$\begin{cases} \frac{d^2x}{d\theta^2} + \left[2 - \bar{F}(\theta)\frac{r_e(\theta)}{\rho_0}\sec\phi(\theta)\right]\frac{dx}{d\theta} = -\left(\frac{r_e}{\rho_0}\sec\phi\right)^2\left[\bar{F}^2(\theta) + \frac{n_0 r_0}{r_e}\bar{F}\sec\phi + \rho_0\frac{d\bar{F}}{d\theta}\frac{\sin\phi}{r_e}\right]x \\ \frac{d^2z}{d\theta^2} + \left[2 - \bar{F}(\theta)\frac{r_e(\theta)}{\rho_0}\sec\phi(\theta)\right]\frac{dz}{d\theta} = \left(\frac{r_e}{\rho_0}\sec\phi\right)^2\left[\frac{n_0 r_0}{r_e}\bar{F}\sec\phi + \rho_0\frac{d\bar{F}}{d\theta}\frac{\sin\phi}{r_e}\right]z \end{cases} \quad (12)$$

where n_0 and r_0 are the values of $n = +\frac{\rho}{H}\frac{dH}{dx}$ and r at the center of the positive magnet, $r_e(\theta) = r_e(s)$ is the radius of the equilibrium orbit and $F(s) = \bar{F}(\theta) = \left(\frac{r_e}{r_0}\right)^k f(\theta)$ is the field along the equilibrium orbit normalized to F(0) = 1.

Equations (11) or (12) were integrated numerically by the "Coleslaw" program on Illiac, which is described elsewhere[5].

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(4) See "Introduction to Numerical Analysis", F. B. Hildebrand, McGraw Hill (1956)

(5) The MURA Programs for Illiac - J.N. Snyder, p. 85

Both (11) and (12) have been used, with identical results. (12) is easier to use because the equilibrium orbit data are given in terms of θ .

III. Results

The results are very sensitive to certain errors in the field measurements.* If the measurements in Figure I had been carried through a complete sector, it would have been possible to measure the azimuthal angle more accurately from the symmetry. As it is, there is an experimental uncertainty of 0.1 degree in the position of the center of the straight section. This changes the magnetic length of the sector and therefore changes the scalloping of the equilibrium orbit, which affects the vertical focusing. Table III below gives values of ν for maximum shift of θ in both directions and for zero shift, all relative to the experimental data.

TABLE III

Shift of θ	ν_x	ν_z
- 0.1	2.94	2.22
0	2.88	2.12
+ 0.1	2.84	2.02

The best calculated values may be taken as

$$\nu_x = 2.88 \pm 0.05$$

$$\nu_z = 2.12 \pm 0.1 ,$$

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* Lack of realization of this sensitivity on the part of both experimenters and theoreticians concerned has been responsible for the beclouded situation in recent months with regard to the betatron frequencies in the Michigan Model.

which are in agreement with the measured values.

The agreement between design and measured values of the radial frequency is considered good, since the field shape, which is not well known for the design values, affects the radial frequency. The reason for the large discrepancy in ν_z becomes evident when one plots F(s), as is done in Figure II. The measured field falls off much more steeply in the straight section than the design field. This makes the $\frac{dF}{ds}$ terms of (11) larger by about 20%, averaging over the region where $\frac{dF}{ds} \neq 0$. This edge-focusing term dominates the vertical focusing, so that one would expect this same order of magnitude increase in the vertical frequency, which is what is found. It may also be pointed out that the greater magnetic length of each magnet in the measured field gives greater gradient length, so that the radial frequency should be larger than the design value, as it is.

It appears that the field falls off differently from the design field because of the finite size of the pole face current return windings. In the design, these currents were assumed concentrated at the magnet edges, whereas they take up about half of the straight section, thus making the field fall off less steeply near the magnets and then more rapidly at the center.

We can estimate roughly the effect on ν_z of differences among the magnets. The most important effect should be the variation in thickness of the return winding bundles at magnet edges, which are about 5/8" in diameter at the maximum radius, with a variation of about 1/8" among them. Changing all the bundles from 0 (the design value) to 5/8" changes ν_z from

1.80 to 2.12. We might then estimate by interpolating linearly that the maximum error in ν_z due to variation in thickness is about $\pm$ 0.07. The r.m.s. error is certainly no larger than this and has a small effect when added to the greater error due to uncertainty in azimuthal angle.

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