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NON-LINEAR RESONANCES IN ALTERNATING GRADIENT ACCELERATORS*

G. Parzen†

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ABSTRACT: A method is presented for treating the non-linear resonances which occur in the radial oscillations of the strong focussing alternating gradient type of high energy accelerator. The method predicts the location of the resonances, calculates the stability limits on the amplitude of the radial oscillations and also gives the stability limit orbit. The results of the theory are compared with those of a numerical calculation.

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†On leave from the University of Notre Dame.

I. Introduction

It has been found[1] that the strong focusing alternating gradient type of high energy accelerator has resonances which are due to non-linear terms in the equations of motion of the particle being accelerated. In this paper we will consider just the radial betatron oscillations of the particle about its equilibrium orbit. We will present a method for solving the non-linear equation which governs the radial oscillations. The method predicts the location of the resonances, calculates the stability limits on the amplitude of the radial oscillations and also gives the stability limit orbit. The results of the theory will be compared with those of a numerical calculation. The agreement seems quite good.

II. General Description of the Method

The general equations of motion are rather complicated and it is customary to expand them about the equilibrium orbit. The linear equation for the radial oscillations of the particle has the form[2]

$$\left\{ \frac{d^2}{d\theta^2} + n(\theta) \right\} u = 0 \tag{2.1}$$

u is the displacement of the particle from its equilibrium orbit. $n(\theta)$ is periodic in θ with the period $2\pi/N$, N being the number of sectors in the machine. If $n(\theta)$ is chosen properly[2], the solutions of Eq. (2.1) are stable and have the form

$$u = e^{i\nu\theta} h(\theta), \tag{2.2}$$

where $h(\theta)$ is periodic in θ with the period $2\pi/N$. ν gives the number of betatron oscillations the particle makes in going once around the machine.

It has been found that if ν happens to be near 1/3 N, then the motion may become unstable. This resonance is due to the quadratic term which was dropped in Eq. (2.1). There are many other possible resonances but we will limit ourselves to the $\nu \simeq 1/3$ N resonance. The methods can be easily extended to the other resonances.

If we include the quadratic term the equation of motion becomes[3]

$$\left\{ \frac{d^2}{d\theta^2} + N(\theta) \right\} u = B(\theta) u^2 \tag{2.3}$$

where $B(\theta)$ has the period $2\pi/N$.

It is more convenient for our method to re-write Eq. (2.3) as

$$\left\{ \frac{d^2}{d\theta^2} + E_o - g(\theta) \right\} u = B(\theta) u^2 \tag{2.4}$$

where we have put $n(\theta) = E_o - g(\theta)$. E_o and $g(\theta)$ are defined by requiring that the average value of $g(\theta)$ over a period be zero.

Before presenting our procedure for solving Eq. (2.4), we would like to review the properties of the linear equation. In the linear theory $u(\theta)$ obeys the equation

$$\left\{\frac{d^2}{d\theta^2} + E_o - g(\theta)\right\} u = 0 \tag{2.5}$$

For our purposes, we will discuss the solutions of the slightly more general equation

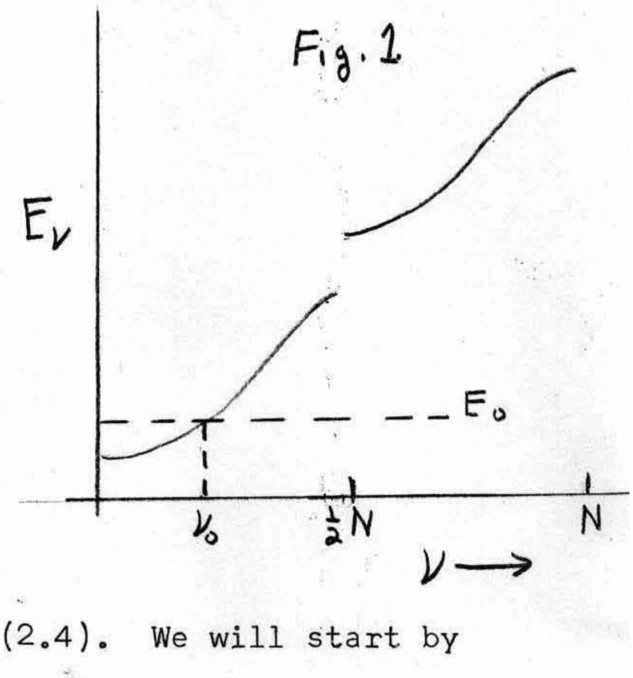
$$\left\{\frac{d^2}{d\theta^2} + E - g(\theta)\right\} u = 0 \tag{2.6}$$

and consider the solutions of Eq. (2.6) for all values of E.

It is known that Eq. (2.6) has stable solutions only for certain values of E. For these "allowed" values of E, the solutions have the form

$$u_\nu(\theta) = e^{i\nu\theta} h_\nu(\theta) \tag{2.7}$$

corresponding to the value of E, E_ν. $h(\theta)$ is periodic in θ with the period $2\pi/N$.



In Fig. 1 the allowed E-values, E_ν, is plotted against ν. Gaps appear at $\nu = 0$, $\frac{1}{2}$N, N, $\frac{3}{2}$N...etc. E_o is the operating value of E, and for the motion to be stable in the linear theory, E_o must not fall inside the gaps. In the linear theory, the motion is given by

$$u = a\, u_{\nu_o}(\theta) + a^* u^*_{\nu_o}(\theta).$$

Now let us return to the non-linear Eq. (2.4). We will start by considering a slightly more general equation

$$\left\{\frac{d^2}{d\theta^2} + W - g(\theta)\right\} \psi = B(\theta)\psi^2, \tag{2.8}$$

and let us ask for what values of W does Eq.(2.8) have stable solutions.

We will solve Eq. (2.8) by using a perturbation procedure similar to that used in quantum mechanics. Every solution $u_\nu(\theta)$ of the linear equation (2.6) corresponding to the E-value E_ν will, because of the perturbation $B(\theta)\psi^2$, go over into the solution $\psi_\nu(\theta)$ of Eq. (2.8) corresponding to the W- value, W_ν. We will show how a little later. We can then draw a curve of the allowed W-values as in Fig. 2.

We will show that a new gap appears at $\nu = 1/3\ N$. However, the size of this gap will depend on the amplitude of the motion; and it will be larger for larger amplitudes

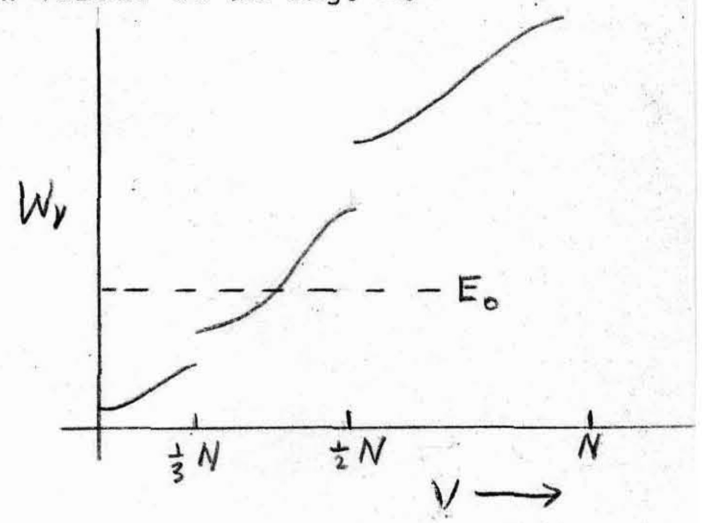


Fig. 2.

The question of stability is answered by where E_0 falls in Fig. 2, since our particle motion is given by solving Eq. (2.8) with $W = E_0$. If E_0 does not fall in the gap at $\nu = 1/3N$, the motion is stable. However, as the amplitude of the motion increases, the gap becomes larger and E_0 will eventually lie inside the gap and the motion becomes unstable. This will determine the stability limit amplitude.

To solve Eq.(2.8) for ψ we will expand ψ in terms of the solutions

$u_\nu(\theta)$ of Eq. (2.6). The solutions $u_\nu(\theta)$ are a complete orthogonal set. They are a continuous complete set. It is more convenient to make them a discrete set, which we can do by imposing the boundary condition on $u_\nu(\theta)$ that it be periodic between $\theta = 0$ and $\theta = T$ where T is some arbitrary large angle which we will eventually let become infinite.

The solutions of the linear equation (2.6) can then be written as

$$u_\nu(\theta) = \frac{1}{\sqrt{T}} e^{i\nu\theta} h_\nu(\theta) \tag{2.9}$$

where $\nu = \frac{2\pi}{T} q,\ q = 0, \pm 1, \pm 2, \pm 3 \ldots$

The values of ν are now discrete and the $u_\nu(\theta)$ form an orthonormal set,

$$\int_0^T d\theta\ u_{\nu'}^*(\theta)\, u_\nu(\theta) = \delta_{\nu\nu'}; \tag{2.10}$$

if $h_\nu(\theta)$ is so normalized that its average value over the period $2\pi/N$ is one,

$$\frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} d\theta\, |h_\nu|^2 = 1 \tag{2.11}$$

Now expand ψ in $u_\nu(\theta)$,

$$\psi = \sum_{\nu_i} a_{\nu_i}\ u_{\nu_i}(\theta), \tag{2.12}$$

and by putting this expansion in Eq. (2.8) we get an equation for the a_{ν_i}

$$(W - E_{\nu_i})\, a_{\nu_i} = \sum B_{\nu_i,\nu_j\nu_k}\, a_{\nu_j}\, a_{\nu_k}, \tag{2.13}$$

where

$$B_{\nu_i,\nu_j\nu_k} = \int_0^T d\theta\ u_{\nu_i}^*\ B\, u_{\nu_j}\, u_{\nu_k}. \tag{2.14}$$

Now we will solve Eq. (2.13) for the a_ν using a perturbation procedure which is very similar to the "weak binding approximation" in solid state theory.

Consider the particular real solution of the linear equation $a u_{\nu_s} + a^* u_{\nu_{-s}}$, which corresponds to the E-value E_{ν_s}. Through the perturbation $B(\theta)$, this solution will go over into a solution of Eq. (2.8) with a value of W which differs slightly from E_{ν_s}. Thus to find the solution of the non-linear Eq. (2.8) which corresponds to the solution $a u_{\nu_s} + a^* u_{-\nu_s}$ of the linear equation, we will first assume that in the expansion Eq. (2.12) for ψ, only a_{ν_s}, and $a_{-\nu_s}$ are large and all the other a_ν are small. We will see later that this is not always true.

Thus to the first approximation, Eq. (2.13) becomes

$$(W - E_i) a_i = B_{i,ss}\, a_s^2 + 2 B_{i,s\bar{s}}\, a_s a_{\bar{s}} + B_{i,\bar{s}\bar{s}}\, a_{\bar{s}}^2 \tag{2.15}$$

where we have written $E_{\nu_s} = E_s$, $u_{\nu_s} = u_s$ and $-\nu_s = \nu_{\bar{s}}$ for the sake of brevity.

Let us notice that the matrix element $B_{i,jk}$ will vanish unless

$$\nu_i = \nu_j + \nu_k + \omega_n \tag{2.16}$$

where $\omega_n = nN$, $\omega_1 = N$, and n is any positive or negative integer. This follows if one expands $B(\theta)$, which is periodic, as $B(\theta) = \sum_n B_n e^{i\omega_n \theta}$ and puts the expansion into Eq. (2.14).

Thus the matrix elements $B_{i,ss}$, $B_{i,s\bar{s}}$, $B_{i,\bar{s}\bar{s}}$ will vanish unless $\nu_i = \pm 2\nu_s + \omega_n$ or $\nu_i = \omega_n$.

We will see now that the levels E_s do not get shifted unless $\nu_s \simeq \pm \frac{1}{3}\omega_n$. The two levels corresponding to $\nu_s = \pm\frac{1}{3}\omega_n$ will get split apart and a gap appears in the allowed E values of the non-linear Eq. (2.8) around $\nu_s = \pm\frac{1}{3}\omega_n$ as shown in Fig. 2.

The shift in the level E_s is given by Eq. (2.15) by putting $i = s$ and $i = \bar{s}$.

$$(W - E_s) a_s = B_{s,\bar{s}\bar{s}}\, a_{\bar{s}}^2 \tag{2.17a}$$

since $B_{s,s\bar{s}} = B_{s,ss} = 0$ by Eq. (2.16) unless $\nu_s = \omega_n$ which we assume is not so.

Also for $i = \bar{s}$, we get

$$(W - E_s) a_{\bar{s}} = B_{\bar{s},ss}\, a_s^2 \tag{2.17b}$$

Now $B_{s,\bar{s}\bar{s}} = 0$ unless $\nu_s = \frac{1}{3}\omega_n$ by Eq. (2.16). So if

$\nu_s \neq \frac{1}{3}\,\omega_n$, $W = E_s$; that is, there is no level shift.

For $\nu_s = \pm\frac{1}{3}\,\omega_n$, $B_{s,\bar{s}\bar{s}} \neq 0$, and we get a splitting of these levels which we can find by multiplying Eq. (2.17a) by Eq. (2.17b) and find that

$$(W - E_s) = \pm\,|B_{s,\bar{s}\bar{s}}|\,|a_s| \tag{2.18}$$

Note the splitting depends on $|a_s|$ the amplitude of the motion.

For $\nu_s \sim \frac{1}{3}\,\omega_n$ but $\nu_s \neq \frac{1}{3}\,\omega_n$ we also get a shift in the levels which is not shown by Eq. (2.15) since it turns out that for $\nu_s \sim \frac{1}{3}\,\omega_n$ but $\nu_s \neq \frac{1}{3}\,\omega_n$ Eq. (2.15) is not valid as some other a_i besides a_s and $a_{\bar{s}}$ can get large as was assumed. This case is treated in appendix A.

By Eq. (2.18) the gap shown in Fig. 2 has the width $2|B_{s,\bar{s}\bar{s}}|\,|a_s|$, $\nu_s = \frac{1}{3}\,\omega_n$. If E_0, the operating point, falls outside the gap, the motion is stable. Suppose E_0 lies below the gap. Then by increasing a_s we can bring the gap down until E_0 lies inside the gap and the motion becomes unstable. This will happen when $|a_s|$ becomes so large that

$$E_s - |B_{s,\bar{s}\bar{s}}|\,|a_s| = E_0$$

$$\text{or}\quad |a_s| = \frac{|E_s - E_0|}{|B_{s,\bar{s}\bar{s}}|} \tag{2.19}$$

where $\nu_s = \frac{1}{3}\,\omega_n$.

Eq. (2.19) gives the stability limit amplitude; that is, the largest permissible amplitude $|a_s|$ before the motion becomes unstable.

The phase of a_s is also determined by Eq. (2.17a)

$$\text{phase } a_s = 1/3\left\{\text{phase } B_{s,\bar{s}\bar{s}} - \text{phase } (E_0 - E_s)\right\} \tag{2.20}$$

It might be noticed that one can add $\pm 2\pi/3$ to the above phase. This is because there are three stability limit orbits which differ only by a shift along θ by $\pm 2\pi/3$.

Finally, let us notice that the equation of the stability limit orbit, which is the stable orbit having the largest amplitude, is given by

$$\psi = a_s u_s + a_{\bar{s}} u_{\bar{s}} \tag{2.21}$$

where $\nu_s = \frac{1}{3} \omega_n$.

Equations (2.19), (2.20) and (2.21) give all the general results about the stability limit orbit. In the next section we will give applications of the above general results.

One might notice that it is very easy to improve the result Eq. (2.21) for the stability limit orbit by computing the other a_i in the expansion $\psi = \sum_i a_i u_i(\theta)$ using the result Eq. (2.15) and the above results for a_s.

III. Discussion of Results

In this section we intend to illustrate our general results Eqs. (2.19) to (2.21) by treating a particular non-linear equation which has also been solved by numerical calculation. We will also discuss the points of the theory which are not entirely clear.

We will treat the differential equation which arises in the discussion of spiral sector type of FFAG machine.[4] For Eq. (2.8), we will write

$$\left\{\frac{d^2}{d\theta^2} + E_0 - 2g_1 \cos N\theta\right\}\psi = 2B_1 \sin N\theta \, \psi^2 \tag{3.1}$$

We will choose the following paramters which are characteristic of this type of machine, $N = 40$, $4E_0/N^2 = .135$, $g_1/N^2 \cong .180$ and $B_1/N^2 = .090$.

According to the linear theory the motion of the particle is given by $u = a u_{\nu_0}(\theta) + a^* u^*_{\nu_0}(\theta)$ where $u_{\nu_0}(\theta)$ is the solution of the equation

$$\left\{\frac{d^2}{d\theta^2} + E_0 - 2g_1 \cos N\theta\right\} u = 0 \qquad)3.2)$$

For our parameters, we find that $\nu_0 / N = .387$.

Now to calculate the stability limit amplitude and orbit we use Eqs. (2.19) to (2.21). The stability limit orbit is given by Eq. (2.21).

$$\psi = a_{1/3}\, u_{1/3} + a^*_{1/3}\, u^*_{1/3} \qquad (3.3)$$

where $a_{1/3}$ is the stability limit amplitude given by Eq. (2.19) which we will calculate a little later, and $u_{1/3}$ is the solution of the linear equation for which $\nu/N = 1/3$. One should not confuse $u_{1/3}$ with u_{ν_0}, the solution of Eq. (3.2), which gives the motion for small amplitudes.

$u_{1/3}$ may be calculated with an accuracy of perhaps 5% or 10%, by the following formula[5] (see Appendix B)

$$u_\nu(\theta) = e^{i\nu\theta}\left\{1 - \frac{g_1}{N^2}\,\frac{1}{1-\frac{2\nu}{N}}\,e^{-iN\theta} - \frac{g_1}{N^2}\,\frac{1}{1+\frac{2\nu}{N}}\,e^{iN\theta}\right\} \qquad (3.4)$$

We will now compute the stability limit amplitude, $a_{1/3}$, which is given by Eq. (2.19). If $E_s = E_{1/3}$ is close to E_o, the operating E- value, we can write

$$\left| E_{1/3} - E_o \right| = \frac{\partial E}{\partial \nu} \left| \nu_0 - \tfrac{1}{3} N \right| \qquad (3.5)$$

where $\partial E_0/\partial \nu$ is evaluated at $\nu = 1/3N$. Note the E here is the $E(\nu)$ of the more general equation (2.6). To compute $\partial E/\partial \nu$, we may use the formula for $E(\nu)$ (See Appendix B).

$$E(\nu) = \left[1 - 8\left(\frac{g_1}{N^2}\right)^2\right]\nu^2 - \frac{1}{2}\,\frac{g_1^{\,2}}{N^2} \qquad (3.6)$$

Thus we find

$$\frac{\partial E}{\partial \nu} = 2\left[1 - 8\left(\frac{g_1}{N^2}\right)^2\right]\nu \qquad (3.7)$$

The matrix element $B_{\bar{s},ss}$ which appears in Eq. (2.19) remains to be calculated. $B_{\bar{s},ss}$ is given by.

$$B_{\bar{s},ss} = \int_0^T d\theta\, u_{1/3}(\theta)\; B(\theta)\, u_{1/3}(\theta)\, u_{1/3}(\theta) \tag{3.8}$$

It is worth recalling here that we have used the standard box normalization used in quantum mechanics to make energy levels discrete and T in Eq. (3.8) will eventually become infinite and will not appear in the final answer.

Using Eq. (3.4) we find that

$$B_{\bar{s},ss} = i\, B_1 \tag{3.9}$$

to about 10% accuracy. One might note that we dropped the factor of $\frac{1}{\sqrt{T}}$ which should be in Eq. (3.4) and we shall omit all these factors as they will not appear in the final answer.

Altogether, we may write for the stability limit amplitude $a_{1/3}$,

$$|a_{1/3}| = \frac{2}{3}\, \frac{\left[1 - 8\left(\frac{g_1}{N^2}\right)^2\right] N^2}{B_1} \left| \frac{\nu_0}{N} - \frac{1}{3} \right| \tag{3.10}$$

For our parameters we find that $|a_{1/3}|$ = .296.

The phase of $a_{1/3}$ is given by Eq. (2.20). Since $B_{s,\bar{s},s} = B^*_{\bar{s},ss} = -i\,B_1$, phase $B_{s,\bar{s}\bar{s}} = -90°$. Phase $(E_o - E_{1/3}) = 0$ since $E_o > E_{1/3}$ as $\nu_o/N > 1/3$. Thus by Eq. (2.20), phase $a_{1/3} = -30^o$.

We now can compute the stability limit orbit using Eq. (3.3). The orbit has also been computed numerically[6] using the Illiac Computer at the University of Illinois. However, the numerical calculation was done not with the Eq. (3.2) but for the exact equations of motion. This means that the numerical computation includes the effect of the higher order terms past the quadratic term kept in Eq. (3.3).

In Fig. 3 we have compared the theoretical and computed values of $\mu(\theta)$. In Fig. 4 we have the same comparison for $d\mu/d\theta$. The agreement seems fairly good and adds weight to the argument that near the $\frac{1}{3}N$ resonance it is sufficient to keep just the quadratic term in the equations of motion. It is possible that, in a similar manner, near the $\frac{1}{4}N$ resonance, it is sufficient to keep just the cubic term and similarly for other resonances. This last conjecture remains to be tested.

A simple, rough formula for the stability limit amplitude which gives a measure of the largest permitted radial displacement and which seems to work quite well can be obtained from Eq. (2.19). If in Eq. (2.19) we assume that $\mu_{1/3}(\theta) \simeq \exp(iN\theta/3)$ and that $E(\nu) \cong \nu^2$, which means we assume that the Flocquet solutions are just sine waves, then we find for A = $2\,|a_{1/3}|$

$$A = \frac{4}{3}\,\frac{N^2}{B_1}\left|\frac{\nu_0}{N} - \frac{1}{3}\right| \tag{3.11}$$

The result of Eq. (3.11) was previously obtained by Drs. L. J. Laslett and A. M. Sessler.

IV. Cubic Resonances

The method presented in the previous sections can be applied without much change to the other resonances in the radial motion. We will give here the results for the cubic term. We will treat the non-linear equation

$$\left\{\frac{d^2}{d\theta^2} + E_0 - g(\theta)\right\}\psi = c(\theta)\,\psi^3 \tag{4.1}$$

The method of Sect. II. when applied here will show there are resonances for $\nu = \frac{1}{4}N$ and for all integral multiples of $\frac{1}{4}N$. We will give the results only, and limit ourselves to the resonance $\nu = \frac{1}{4}N$.

The stability limit orbit is given by

$$\psi = a_{1/4}\, u_{1/4}(\theta) + a_{1/4}^{*}\, u_{1/4}^{*}(\theta) \tag{4.2}$$

where $u_{1/4}(\theta)$ is the solution of the linear equation (2.6) for ν /N = 1/4. The stability limit amplitude $a_{1/4}$ is given by

$$|a_{1/4}| = \left[\frac{|E_0 - E_{1/4}|}{|C|}\right]^{1/2} \tag{4.3}$$,

where $E_{1/4}$ is the E-value in the linear equation (2.6) which gives $\frac{\nu}{N}$ = 1/4, and $C = C_{\bar{s},sss}$ is defined by

$$C_{\bar{s},sss} = \int_0^T d\theta\, u_s(\theta)\; C(\theta)\, u_s(\theta)\, u_s(\theta)\, u_s(\theta) \tag{4.4}$$

with ν_s = 1/4N in complete analogy with $B_{\bar{s}ss}$ in the quadratic resonance case.

The phase of $a_{1/4}$ is given by

$$\text{phase } a_{1/4} = 1/4 \left\{\text{phase } C_{s,\bar{s}\bar{s}\bar{s}} - \text{phase}\,(E_0 - E_s)\right\} \tag{4.5}$$

where $\nu_s = \frac{1}{4}N$.

The results for the 1/4N resonance have not been treated numerically so far.

A simple rough formula analogous to (3.11) can also be obtained. The largest stable radial displacement is approximately given by

$$A = \left\{\frac{2N^2}{|C_1|}\left|\frac{\nu_0}{N} - \frac{1}{4}\right|\right\}^{1/2}, \tag{4.6}$$

where C_1 is the first Fourier component in the expansion

$$C(\theta) = \sum_n C_n e^{i w_n \theta}$$

I would like to thank Drs. Sessler, Laslett, Cole and Ohkawa of the MURA Group for much discussion of their results for the radial oscillations.

APPENDIX A

The method presented for finding the shift in the E values of the linear equation (2.5) caused by the non-linear perturbation $B(\theta)\psi^2$ is not quite right. This is because the levels of Eq. (2.5) are degenerate and we used a non-degenerate type perturbation theory. However, the method happens to be right when ν is exactly equal to $1/3\ \omega_n$ and when ν is far from $1/3\ \omega_n$.

We would like now to correct our method so as to obtain the correct equations for the level shifts for all values of ν. In perturbation theory, we have a degenerate case if there are two levels with E - values E_s snd $E_{s'}$ for which $E_s = E_{s'}$ and for which the matrix element of the perturbation which connects these two levels does not vanish. In our case the relevant matrix element is $B_{s',ss}$. If this situation occurs then in finding the shift in the level E_s, we must assume that not only a_s is large but also $a_{s'}$.

At first glance, all the levels seem degenerate since for any level E_s there is always the level $\nu_{s'} = -\nu_s$ for which $E_{s'} = E_s$. However $B_{s',ss}$ will always vanish unless $\nu_s = \frac{1}{3}\omega_n$. For $\nu_s = 1/3\ \omega_n$ the case is degenerate and therefore both a_s and $a_{s'}$ should be treated as large. However, both a_s and $a_{s'} = a_{\bar{s}}$ were considered as large in our treatment and our method for $\nu_s = \frac{1}{3}\omega_n$ is satisfactory.

It would seem that the case $\nu_s = 1/3\ \omega_n$ is the only degenerate case. However, the case where ν_s is close to $1/3\ \omega_n$ is also essentially degenerate for there exist other levels $E_{s'}$ for which $E_{s'}$ is close to E_s and for which the relevant matrix elements do not vanish. This situation is very similar to that which occurs in the weak binding approximation in solid state theory.

Indeed, it turns out that in this sense the levels $\nu_s \sim (1/3)\,\omega_n$ but $\nu_s \neq (1/3)\,\omega_n$ are not only degenerate but many-fold degenerate; that is, there are many levels $E_s{}'$ for which $a_s{}'$ must be considered large as well as a_s. This makes it very difficult to find the shift in the levels in this case.

Fortunately, it does not appear to be necessary for our purposes to be able to solve the case $\nu_s \sim (1/3)\omega_n$ but $\nu_s \neq (1/3)\omega_n$. Nevertheless we would like to present the following approximate treatment which should not be too far from the truth but whose accuracy is a little uncertain.

Since we are interested in finding the shift in those levels for which ν_s is close to $1/3\ \omega_n$, we may assume that $\psi_s \sim \psi_r$ where $\nu_r = 1/3\ \omega_n$. Now we will linearize our non-linear equation (2.8) by replacing one ψ in the ψ^2 term by ψ_r, the solution for $\nu_r = 1/3\ \omega_n$. Our linearized equation becomes

$$\left\{\frac{d^2}{d\theta^2} + W - g(\theta)\right\}\psi = C(\theta)\,\psi \tag{A.1}$$

where $C(\theta) = B(\theta)\,\psi_r(\theta)$, and $\psi_r = a_r\, u_r(\theta) + a_{\bar{r}}\, u_{\bar{r}}(\theta)$

Now we proceed as before and expand $\psi = \sum_i a_i u_i(\theta)$ and get the equation for the a_i,

$$(W - E_i)\,a_i = \sum_j C_{ij}\, a_j \tag{A.2}$$

where

$$C_{ij} = \int d\theta\, u_i^{*}\, C\, u_j. \tag{A.3}$$

Note that $C_{ij} = 0$, unless $\nu_i = \nu_j \pm \nu_r + \omega_n$

We wish to find the shift in the level ν_s where $\nu_s \sim \nu_r = 1/3\,\omega_n$. First let us notice that this level is degenerate. Let us restrict our attention to the case $\nu_r = 1/3\ \omega_1 = 1/3N$. Then it is clear that the level $\nu_{s'} = -\nu_s - \nu_r + \omega_1$ is degenerate with ν_s and we must consider not only a_s and $a_{\bar{s}}$ ($\nu_{\bar{s}} = -\nu_s$) as large but also $a_{s'}$ and $a_{\bar{s}'}$. E_s and E_s' are not exactly equal but they may be very close and the levels are essentially degenerate.

By retaining only a_s and $a_{s'}$ in the right side of Eq. (A.2) we get our perturbation equations for the a_i.
If in these equations we put $i = s,\ s',\ \bar{s},\ \bar{s}'$ we will get the equations for the level shifts. Thus we get for i = s,

$$(W - E_s)a_s = C_{s\bar{s}'}\ a_{\bar{s}'}, \tag{A.4a}$$

and for $i = \bar{s}'$,

$$(W - E_{s'})\,a_{\bar{s}'} = C_{\bar{s}'s}\ a_s \tag{A.4b}$$

Note that $C_{s\bar{s}'} = B_{s,\bar{r}\bar{s}'}\ a_{\bar{r}}$

and that $C_{\bar{s}'s} = B_{\bar{s}',rs}\ a_r$.

Thus the level shift is given by

$$(W - E_s)(W - E_{s'}) = |B_{r,\bar{r}\bar{r}}|^2 |a_r|^2, \tag{A.5}$$

where we have put $B_{s,\bar{r}\bar{s}'} \simeq B_{r,\bar{r}\bar{r}}$.

Eq. (A.5) gives the shift in the levels E_s and E_s', one of which lies above E_r and the other lies below E_r, for a given amplitude of the motion a_r.

The accuracy to be expected by using (A.5) is probably not very good. Linearizing is a dangerous and tricky procedure. However the results do give a correct picture of the level shifts. As we mentioned previously, we do not have to be able to solve this particular aspect of the problem to establish the main results of this paper.

There is one point in the method which is not clear and is worth pointing out. In solving the non-linear equation (2.8) we say that if E_0 happens to fall in the gaps shown in Fig. 2, then the motion is unstable and if it does not fall in one of the gaps the motion is stable. It is not clear what is meant by unstable here. It does not necessarily mean that the solution will grow expontially as is true for the linear equation. All the theory can say at present is that if E_0 falls in a gap then the solution can not be represented by a simple combination of only a few of the solutions of the linear equation, and thus it is very likely the motion has departed greatly from the motion given by the linear equation.

Appendix B

In this appendix we would like to obtain the formula for the Flocquet functions used in Section III. Various people, Drs. Laslett, Vogt-Nilsen, and Adler have developed methods for calculating the Flocquet functions with good accuracy. The formulas we will give have the particular advantage of being simple in form which being reasonably accurate when ν /N is not near .5. which is usually true in our calculations.

We wish to solve the equation

$$\left\{\frac{d^2}{d\theta^2} + E - g(\theta)\right\} u = 0 . \tag{B.1}$$

We will first give the results. The solutions $u_\nu(\theta)$ are given by.

$$u_\nu = e^{i\nu\theta}\left\{1 - \sum_{n=-\infty}^{\infty} \frac{g_n}{\omega_n^2} \frac{1}{1+\frac{2\nu}{\omega_n}} e^{i\omega_n\theta}\right\} \tag{B.2}$$

Where g_n are the fourier coefficients in the expansion $g(\theta) = \sum_{n-\infty}^{\infty} g_n \exp(i\omega_n\theta)$ and ω_n = nN. The formula (B.2) holds best when ν (N << 1) and will do farily well for $\nu/N \lesssim .3$.

The allowed E - values, $E(\nu)$, are given by

$$E(\nu) = \left\{1 - 8\sum_{n=1}^{\infty} \left|\frac{g_n}{W_n^2}\right|^2\right\}\nu^2 - 2\sum_{n=0}^{\infty} \frac{|g_n|^2}{W_n^{\ 2}} \qquad (B.3)$$

Formulas (B.1) and (B.2) are easily established by using the standard perturbation theory of quantum machanics. We take as the unperturbed equation,

$$\left\{\frac{d^2}{d\theta^2} + \mathcal{E}\right\} \varphi = 0 \qquad (B.4)$$

which has the solutions $\varphi_\nu = \exp(i\nu\theta)$

and the $\mathcal{E}$ - values $\mathcal{E}_\nu = \nu^2$

We can make these solutions discrete by box normalization and by first order perturbation theory u_ν is given by

$$u_\nu = \varphi_\nu + \sum_{\nu'} \frac{g_{\nu'\nu}}{\mathcal{E}_\nu - \mathcal{E}_{\nu'}} \varphi_{\nu'} \qquad (B.5)$$

which gives Eq. (B.2) immediately.

The allowed $E(\nu)$ is given by second order perturbation theory as

$$E(\nu) = \mathcal{E}_\nu + \sum \frac{|g_{\nu'\nu}|^2}{\mathcal{E}_\nu - \mathcal{E}_{\nu'}} . \qquad (B.6)$$

(B.6) says then that

$$E(\nu) = \nu^2 + \sum_{n=-\infty}^{\infty} \frac{|g_n|^2}{\nu^2 - (\nu + W_n)^2} \qquad (B.7)$$

In (B.7) let us combine the terms of n and -n. Then

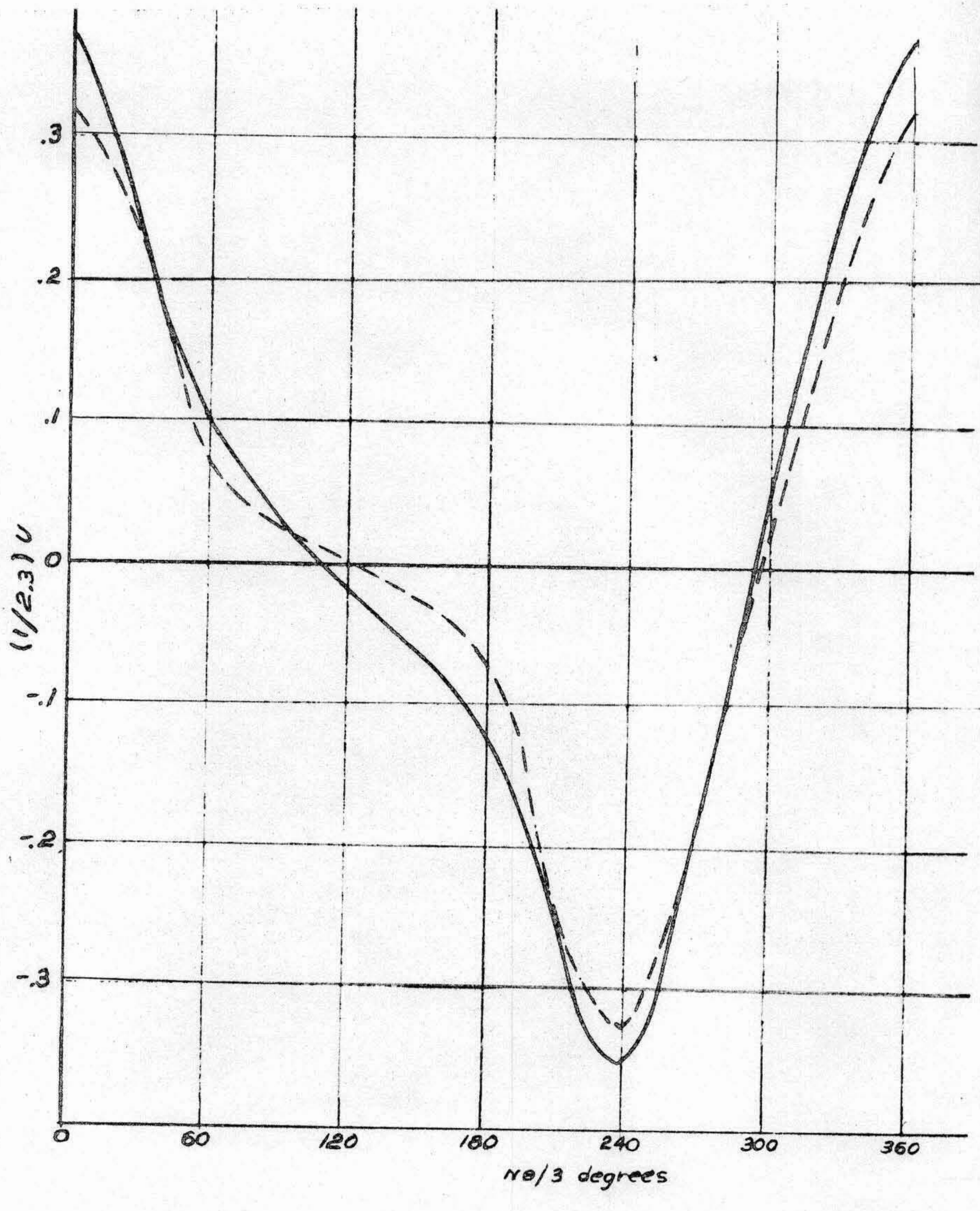
$$E(\nu) = \nu^2 + \sum_{n=1}^{\infty} |g_n|^2 \left\{\frac{1}{\nu^2 - (\nu + W_n)^2} + \frac{1}{\nu^2 - (\nu - W_n)^2}\right\} \qquad (B.8)$$

If we expand in powers of $\frac{\nu}{W_n}$ and keep terms up to $(\nu/W_n)^2$, we get the result Eq. (B.3).

The accuracy of Eq. (B.3) for $E(\nu)$ can be observed in Table I. In this table, $E(\nu)$ has been computed both by Eq. (B.3) and exactly for the equation

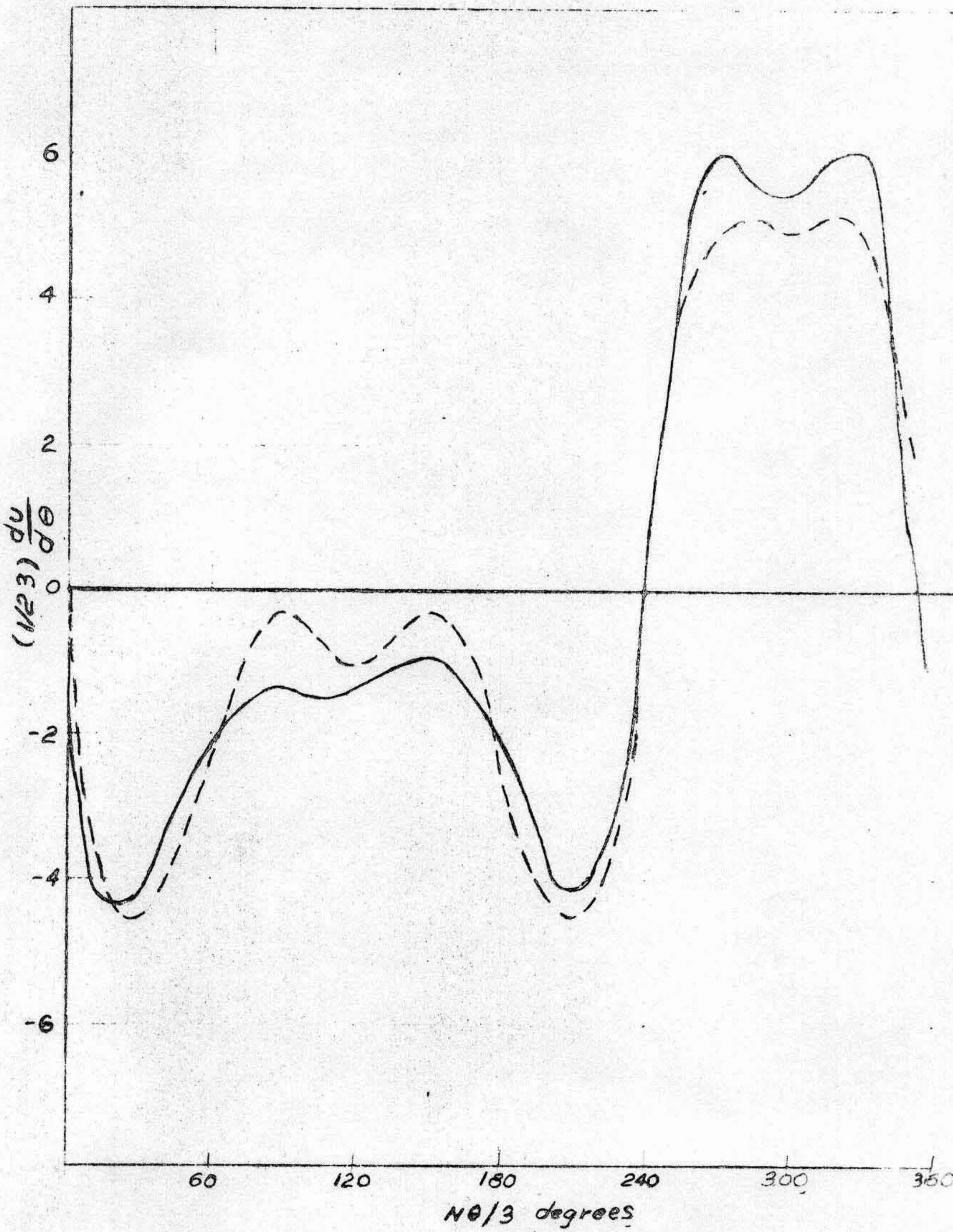
$$\left\{\frac{d^2}{d\theta^2} + E - 2g_1 \cos N\theta\right\} u = 0 \qquad (B.9)$$

Fig. 1. A comparison of the stability limit orbit as found by the theory and by direct numerical calculation. $(1/2.3)\mu$ is plotted against $N\theta/3$. The solid line is the numerical result. The broken line is the theoretical result.



.3
.2
.1
0
-.1
-.2
-.3
(1/2.3) U
0
60
120
180
240
300
360
Nθ/3 degrees

Fig. 2. A comparison of the stability limit orbit as found by the theory and by direct numerical calculation. $(1/2.3)\,du/d\theta$ is plotted against $N\theta/3$. The solid line is the numerical result. The broken line is the theoretical result.



6
4
2
0
-2
-4
-6
$(1/23)\frac{du}{d\theta}$
60
120
180
240
300
360
$N\theta/3$ degrees

$8g_1/N^2$	ν/N	$4E/N^2$	
		Theory	Exact
	.1278	.0600	.06
0.2	.1974	.1501	.15
	.3196	.4014	.40
	.3571	.5025	.50
	.0948	- .0935	- .09
	.2092	.0281	.03
1.00	.3632	.3366	.30
	.4058	.4510	.50
	.1474	- .2188	- .20
1.5	.2461	- .1070	- .09
	.3526	.0763	.06
	.4270	.2431	.15

Table I. The E - values of Eq. (B.9) as computed by the theory, Eq. (B.3) and exactly. The exact values were taken from the tables of Belford, Laslett and Snyder, Tables pertaining to Solutions of a Hill Equation, MURA Reports.

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