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FUNDAMENTAL LIMITATIONS ON THE PERFORMANCE OF AN ADIABATIC ACCELERATOR

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We define an adiabatic accelerator as one in which all accelerating and guide fields vary slowly in time relative to the frequency of revolution of the particles. We shall show that the momentum gained by a particle in an adiabatic accelerator depends essentially only on the difference in total magnetic flux enclosed by the initial and final orbits. This result will apply in particular to acceleration via an outward travelling flux wave; and to methods involving switching of particles in and out of orbits which link changing flux paths, provided the switching is accomplished slowly.

This investigation was suggested by a treatment given by French[1] for the case of cylindrical symmetry, and by results obtained by Fulbright[2] for several special cases involving slow switching of

* Supported by AEC.

[1] Park French. Letter to A. M. Sessler, June 27, 1956.

[2] Private communication.

orbits. We first treat the case of cylindrical symmetry along the lines indicated by French, but with conclusions which differ somewhat from his. We then generalize to the case of arbitrary orbit shape.

I. Cylindrical Symmetry.

The Lagrangian for a particle in an electromagnetic field is

$$L = -mc^2 \sqrt{1 - v^2/c^2} + e\phi + \frac{e}{i} \vec{v} \cdot \vec{A} \tag{1}$$

If the fields have cylindrical symmetry, then L does not depend on θ, and the canonical angular momentum

$$P_\theta = p_\theta + \frac{e}{c} r A_\theta \tag{2}$$

is a constant of the motion, where p_θ is the mechanical angular momentum,

$$p_\theta = \frac{m r^2 \dot{\theta}}{\sqrt{1 - v^2/c^2}} . \tag{3}$$

It is easy to show that the total magnetic flux within a circle of radius r is

$$\Phi = 2\pi r A_\theta . \tag{4}$$

Hence from Equation (2) we have for the change in mechanical angular momentum between times t_o and t_1:

$$p_{\theta 1} - p_{\theta o} = - \frac{e}{2\pi c} (\Phi_1 - \Phi_o) , \tag{5}$$

where Φ_1, Φ_o are the total flux within circles of radii r_1, r_o.

The result (5) is correct, no matter how the fields may change with time, so long as they are cylindrically symmetric. If now the fields change adiabatically, then a particle moving initially on a circle will continue to move on a circle without betatron oscillations, and hence

$$p_\theta = rp = -\frac{e}{c} r^2 B_z \tag{6}$$

is a function only of r, when

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \tag{7}$$

is the magnitude of the linear momentum. We then have

$$p_1 - p_o = -\frac{e}{c}\left(\frac{\Phi_1}{2\pi r_1} - \frac{\Phi_o}{2\pi r_o}\right), \tag{8}$$

and

$$B_{z1} - B_{zo} = \frac{\Phi_1}{2\pi r_1^2} - \frac{\Phi_o}{2\pi r_o^2}), \tag{9}$$

which is the betatron flux condition.

If we neglect betatron oscillations, then the two terms on the right of Equation (2) are functions of r alone, and any given particle must always lie on a radius r which is a root of Equation (2) for a constant value of P_θ . If the fields vary periodically in time, then the particle must return periodically to the same radius r, unless Equation (2) has multiple roots; in that case, the particle may move progressively from one root to another during a cycle, as is the case in travelling flux wave acceleration. This possibility appears to have been overlooked by French, probably because he assumes a reference

time during the cycle when no accelerating fields are present, and no such time exists for travelling flux wave acceleration. At any rate, it is clear from Equation (8) or (9) that travelling flux wave acceleration is no more efficient than ordinary betatron acceleration with a flux core, in the sense that, for a given guide field, the total flux which must be caused to link the orbit during acceleration is the same in either case.

II. The General Case

We now assume an arbitrary system of guiding and accelerating fields. In order to generalize the results of Part I, we introduce a time dependent coordinate system R, Θ, Z, defined as follows. The curves $R = \text{Const.}$, $\Theta = Z = 0$ shall be at each instant t the closed equilibrium orbits for the magnetic field configuration at time t. We will define R, Θ, Z so that

$$2\pi R = \text{length of equilibrium orbit } R, \tag{10}$$

$$R\Theta = \text{distance measured along the orbit } R \tag{11}$$

$$Z = \text{distance measured along a curve } R = \text{const.}, \Theta = \text{const.} \tag{12}$$

We arrange that the curves $R = \text{const.}$, $\Theta = \text{const.}$ are orthogonal to the surfaces $Z = \text{constant}$. This coordinate system is in general not orthogonal, and slowly changing with time. The velocity $\vec{v}$ of a particle will be given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\frac{\partial \vec{r}}{\partial R} + \dot{\Theta}\frac{\partial r}{\partial \Theta} + \dot{Z}\frac{\partial r}{\partial Z} + \frac{\partial r}{\partial t}, \tag{13}$$

where $\vec{r}(R, \Theta, Z, t)$ is the position vector of the particle.

Since by Eq. (11), $\left|\frac{\partial \vec{r}}{\partial \theta}\right| = R$, (14)

we have $P_\theta = p_\theta + \frac{e}{c} R A_\theta$, (15)

where A_θ is the component of $\vec{A}$ along the equilibrium orbit, and

$$\begin{aligned} p_\theta &= \frac{\partial}{\partial \dot{\theta}}\left[-mc^2\sqrt{1-\frac{v^2}{c^2}}\right] \\ &= \frac{m\left(R^2\dot{\theta} + \dot{R}\frac{\partial \vec{r}}{\partial R}\cdot\frac{\partial \vec{r}}{\partial \theta} + \frac{\partial \vec{r}}{\partial t}\cdot\frac{\partial \vec{r}}{\partial \theta}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{mRv_\theta}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned} \tag{16}$$

where v_θ is the component of velocity along the equilibrium orbit.

The time rate of change of P_θ is

$$\frac{dP_\theta}{dt} = \frac{\partial L}{\partial \theta} \tag{17}$$

We have, therefore, for the change of P_θ during one revolution

$$\Delta P_\theta = \int_{\theta_0}^{\theta_0+2\pi} \frac{\partial L(R,\theta,Z,t)}{\partial \theta}\, d\theta \tag{18}$$

If we now assume that L varies slowly with t and neglect betatron oscillations, we can treat R, Z, t as constant during one revolution and Equation (18) then gives

$$\begin{aligned} \Delta P_\theta &= L(R, \theta_0+2\pi, 0, t) - L(R, \theta_0, 0, t) \\ &= 0 \end{aligned} \tag{19}$$

Hence P_θ is a periodic invariant under adiabatically changing fields; that is, it has a constant value at each point along the equilibrium orbit. In particular, $\bar{P}_\theta$ averaged around the orbit is a constant of the motion. The generalized form of Equation (2) is then

$$\bar{P}_\theta = \bar{p}_\theta + \frac{e}{c} R \bar{A}_\theta \,. \tag{20}$$

(All bars denote averages around the equilibrium orbit)

Since
$$\vec{B} = \nabla \times \vec{A} \tag{21}$$

we have for the flux encircled by the orbit R

$$\Phi = \iint_{(R)} \vec{B} \cdot d\vec{S} = \oint_{(R)} \vec{A} \cdot d\vec{R}$$

$$= 2\pi R \bar{A}_\theta \,, \tag{22}$$

where $\bar{A}_\theta$ is the average component of $\vec{A}$ along the orbit, averaged around the orbit. We can now generalize Equation (5):

$$\bar{p}_{\theta_1} - \bar{p}_{\theta_0} = -\frac{e}{2\pi c}\left(\Phi_1 - \Phi_0\right). \tag{23}$$

For a particle moving along an equilibrium orbit, Equation (16) becomes

$$p_\theta = R p \,. \tag{24}$$

Hence we can generalize Equation (8):

$$\bar{p}_1 - \bar{p}_0 = -\frac{e}{c}\left(\frac{\Phi_1}{2\pi R_1} - \frac{\Phi_0}{2\pi R_0}\right). \tag{25}$$

If we set
$$\mu(R, \theta) = \frac{R}{\rho} \tag{26}$$

where ρ is the radius of curvature of the equilibrium orbit at the point R, θ then we can show that[2]

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[2] Symon, Kerst, Jones, Laslett and Terwilliger, "Fixed Field Alternating Gradient Acceleration". To be published in Physical Review See Equation (4.4)

$$\bar{\mu} = 1 \tag{27}$$

Hence we have

$$\overline{B_z} = \left(-\overline{\frac{p_c}{e\rho}}\right) = -\left(\frac{p_c \mu}{eR}\right) = -\frac{p_c}{eR}, \tag{28}$$

provided p is constant around the orbit, i.e. in the absence of electrostatic fields. We then have a generalized betatron flux condition

$$\overline{B_{z1}} - \overline{B_{z0}} = \frac{\Phi_1}{2\pi R_1^2} - \frac{\Phi_0}{2\pi R_0^2} . \tag{29}$$

If there are electrostatic fields, then Equation (29) holds for some suitably defined average field.