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RADIO FREQUENCY KNOCKOUT OF STACKED BEAMS

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An RF system which is frequency modulating its synchronous particles to the position of a previously deposited coasting or "stacked" beam has the likelihood of exciting betatron oscillations in this stacked beam. RF excitation of betatron oscillations has been observed experimentally in the Michigan synchrotron, the Berkeley 184"synchrocyclotron, and the Michigan model FFAG betatron, and must be considered in any beam stacking program. The following report gives the RF knockout frequencies for one or more RF cavities, finds the number of equivalent in phase kicks which the RF system imparts to the particle as the RF passes through a KO resonance, determines the amplitude resulting from a single kick by radial and slanted cavities, and discusses some numerical examples.

1. Frequencies of RF KO by one cavity.

The RF frequencies, ω_{RF}, from a single narrow cavity, which drive betatron oscillations in a beam of going around frequency ω_o have been calculated by a number of people. The simplest method is due to K. R. Symon.

The force acting on a particle as a function of time due to a single delta function RF cavity can be expressed as

$$F(t) = f(x,y) \quad \delta(\omega_o t) \sin \omega_{RF} t$$

The dependence of the force on the x and y betatron displacements is due to gradients in the RF field. Let $f(x,y) = x^p y^q$, p, q, integers, $x = \sin \omega_x t$, $y = \sin \omega_y t$, ω_x and ω_y the betatron oscillation frequencies. So f(x,y) has terms like $\sin p\omega_x t$ and $\sin q\omega_y t$. $\delta(\omega_o t)$, due to the narrow RF cavity, has terms $\cos(n\omega_o t)$, n an integer.

The force then has resultant terms as:

$$F(t) = \sin(\pm p\,\omega_x \pm q\,\omega_y \pm n\,\omega_o \pm \omega_{RF})t$$

If this term in the force has frequency ω_x it can drive the x betatron oscillation.

$$\omega_x = \pm p\,\omega_x \pm q\,\omega_y \pm n\,\omega_o \pm \omega_{RF},$$

or

$$\pm\omega_{RF} = (1 \pm p)\,\omega_x \pm q\,\omega_y \pm n\,\omega_o,$$

or

$$\pm\frac{\omega_{RF}}{\omega_o} = (1 \pm p)\,\nu_x \pm q\,\nu_y \pm n$$

Where $\omega_x = \nu_x\,\omega_o$ and p, q, and n are integers.

The result that an RF cavity with a linear radial gradient can drive the x betatron oscillations of synchronous particles has been found by a number of people and is again apparent from the above derivation ($1 + p = o$, $q = o$).

2. Frequencies of RF KO by a number of cavities.

If there are a number of cavities accelerating synchronous particles with harmonic number h, and the "ℓth" such cavity is at a machine angle θ_ℓ from the "o th" cavity, then, at one instant, the RF phase of the ℓ th cavity is h θ_ℓ different than the $_o$th (for each cavity to be at the same phase at the arrival of a synchronous particle). The total force on a stacked particle is the sum from the separate cavities and equals.

$$F(t) = f(x,y) \sum_{\ell} \sin(\omega_{RF}t - h\,\theta_\ell)\ \delta(\omega_o t - \theta_\ell)$$

$$= f(x,y) \sum_{\substack{n \\ 0 \to \infty}} \sum_{\ell} \sin(\omega_{RF}t - h\,\theta_\ell)\ \cos(n\,\omega_o t - n\,\theta_\ell)$$

$$= 1/2\ f(x,y) \sum_{\substack{n \\ -\infty \to \infty}} \sum_{\ell} \sin\left[(\omega_{RF} + n\,\omega_o)t - (h+n)\,\theta_\ell\right]$$

$$= 1/2\ f(x,y)\, I_m \left\{ \sum_{\substack{n \\ -\infty \to \infty}} \left(e^{i(\omega_{RF} + n\,\omega_o)t} \sum_{\ell} e^{-i(h+n)\theta_\ell} \right) \right\}$$

The force will be zero when

$$\sum_{\ell} e^{-i(h+n)\theta_\ell} = 0$$

which restricts the integral values of n for KO. The KO frequencies are:

$$\frac{\omega_{RF}}{\omega_o} = \pm(1 \pm p)\nu_x \pm q\,\nu_y - n$$ (n having restricted integral values between $-\infty \to \infty$)

Consider two examples. First, the case where there are s equally spaced cavities around the machine. Then

$$\theta_\ell = \frac{2\pi\ell}{s}$$

and

$$\sum_{\substack{\ell \\ 0 \to s-1}} e^{-i\,2\pi(h+n)\frac{\ell}{s}} = \frac{1-e^{-i\,2\pi(h+n)}}{1-e^{-i2\frac{\pi(h+n)}{s}}}$$

which equals zero unless

$$\frac{h+n}{s} = \pm m \text{ an integer}$$

So Ko will occur only when

$$n = \pm ms - h$$

or

$$\frac{\omega_{RF}}{\omega_o} = \pm(1 \pm p)\nu_x \pm q\,\nu_y \pm ms + h$$

The larger s the fewer KO frequencies.

Second, consider just two cavities spaced θ apart. There will be <u>no</u> KO when

$$(h+n)\theta = \pm(2m-1)\pi$$

or

$$n = \pm(2m-1)\frac{\pi}{\theta} - h$$

So for two cavities fewest n can cause Ko when $\theta = \pi$.

It is clear that particular frequencies can be legislated away. To design out all relevant ones becomes difficult as the frequency modulation range increases.

3. Buildup of betatron oscillation amplitudes going through a resonance.

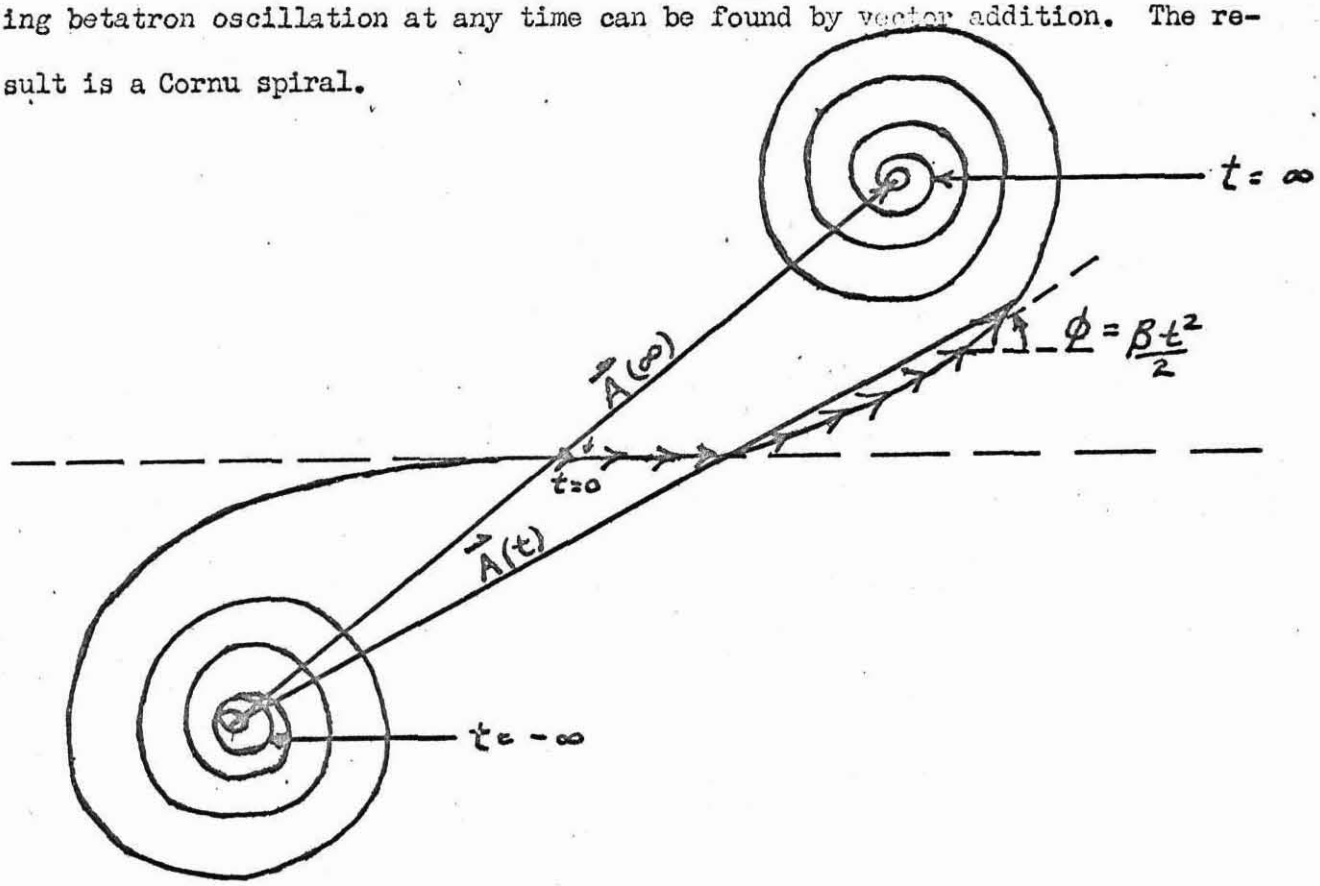
Let a betatron oscillation of frequency ω_x be driven by an impulsive kick of frequency ω where

$$\omega - \omega_x = \beta t,$$

a linear modulation of the relative frequency. Let the amplitude of betatron oscillation generated by a single kick be a, a constant. The relative phase of a betatron oscillation excited by a kick at time t to one which was excited at t = o is

$$\phi(t) = \int_0^t (\omega - \omega_x)dt = \frac{\beta t^2}{2}$$

Letting the vector $\longrightarrow$ represent the amplitude and phase at time t of the oscillation excited at t = o, and $\phi = \frac{\beta t^2}{2}$ that at t, the resulting betatron oscillation at any time can be found by vector addition. The result is a Cornu spiral.



The magnitude of $\vec{A}(\infty)$ can be found analytically, assuming differential kicks,

$$\frac{d\vec{A}}{dt} = f\vec{a}(t) \qquad \text{where} \quad \vec{a}(t) = ae^{i\phi(t)},$$

the betatron oscillation vector from one kick at t, and f is the number of kicks/second.

$$\frac{d\vec{A}}{dt} = fae^{i\phi(t)} = fae^{i\frac{\beta t^2}{2}}$$

$$\vec{A}(\infty) = fa\int_{-\infty}^{\infty} e^{i\frac{\beta t^2}{2}}\,dt$$

$$= fa\sqrt{2/\beta}\int_{-\infty}^{\infty} e^{iu^2}\,du$$

$$= fa\sqrt{2/\beta}\left[\sqrt{\frac{\pi}{2}} + i\sqrt{\pi/2}\right]$$

So $$|A(\vec{\infty})| = fa\sqrt{2\pi/\beta} = \frac{fa}{\sqrt{\frac{d\nu}{dt}}}$$ where ν is the relative frequency in cycles/sec.

I understand that F. T. Cole and D. W. Kerst have made similar calculations in connection with the buildup of betatron oscillations going through a magnetic field bump resonance.

In RF driving of a stacked beam oscillation, the driving frequency is that seen by the particles, obtained in sections 1 & 2 to be

$$\omega = \pm(1 \pm p)\,\omega_x \pm q\,\omega_y, \pm(\omega_{RF} + n\,\omega_0)$$

where ω goes through ω_x. Only a single value of n is considered here. The effects of any other resonances are assumed calculable by multiplying by the square root of the number passed through, i.e, assuming random phases.

So $$\frac{d\nu}{dt} = \frac{1}{2\pi}\frac{d}{dt}(\omega - \omega_x) = \frac{d\nu_{RF}}{dt}$$

So the equivalent number of in phase kicks obtained going through the resonance, is

$$\frac{f}{\sqrt{\frac{d\nu_{RF}}{dt}}}$$

where f is the number of kicks/second

and $\frac{d\nu_{RF}}{dt}$ is the frequency modulation rate of the RF in cycles/sec.^{2} The RF cavity of course does not give constant magnitude kicks but sinusoidal ones, the peak of the sine wave occuring at the relative phase $\phi = \frac{\beta_t^2}{2}$. The average value of a group of kicks is also at phase ϕ and has a value $\overline{\cos^2\theta} = 1/2$ of the peak kick. This assumes that the phase ϕ is slowly varying compared to other frequencies involved, certainly correct over the region of interest.

4. Betatron oscillation from a single traversal of an RF cavity.

a) Radial (standard) cavities.

A particle passing through an accelerating cavity has a fast change in the position of its equilibrium orbit; since its actual radial position has not changed a betatron oscillation is excited whose amplitude equals the change in position of the equilibrium orbit. So since

$$p = p_o\left(\frac{r}{r_o}\right)^K, \quad a_x = \frac{r}{k}\frac{\Delta p}{p}$$

where Δp is the azimuthal momentum gained in crossing the cavity.

$$\Delta p = \frac{ev}{\beta c}$$

V = cavity voltage

$\beta = \frac{v}{c}$

$$\frac{\Delta p}{p} = \frac{eV}{\beta^2 E_{Total}}$$

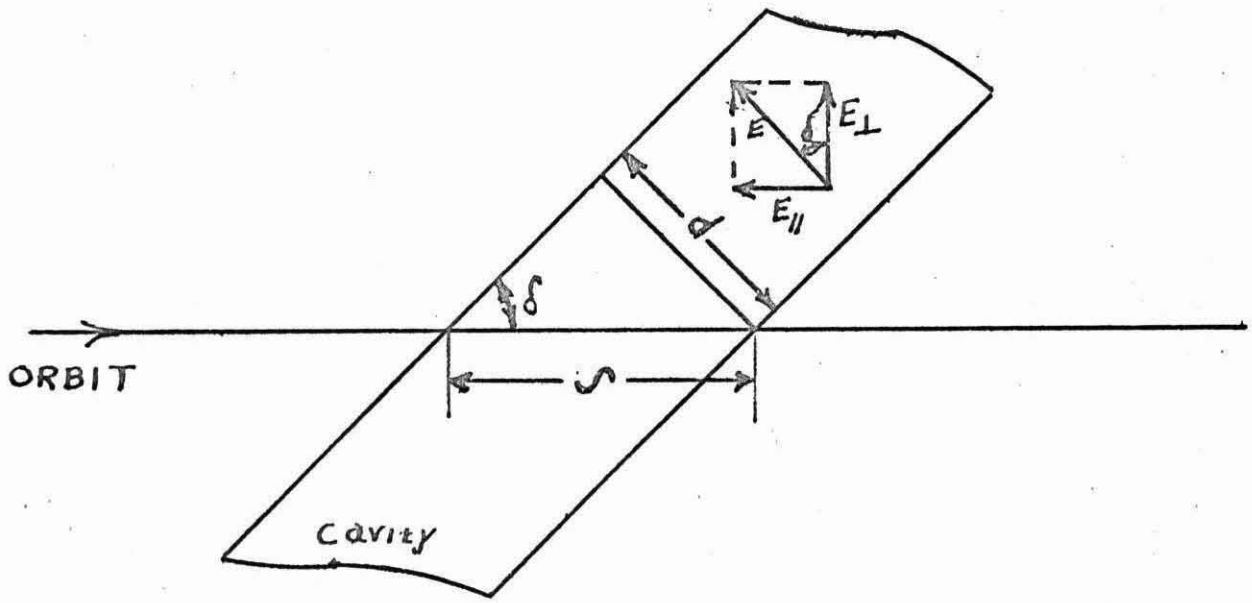
So $$a_x = \frac{r}{K}\frac{V}{\beta^2 E_{Total}}$$

Where V and E_{Total} are in the same energy units.

b) Non radial cavity.

In this case, in addition to the amplitude generated by a change in the equilibrium orbit, there is a transverse kick resulting in an angle change

$$\Delta\theta = \frac{\Delta P_\perp}{P}$$ Where $\Delta P_\perp$ is the transverse momentum gained.



$$\Delta P_\perp = F \cdot t = e E_\perp \frac{s}{\beta c} = \frac{e E \cos\delta}{\beta c} \cdot \frac{d}{\sin\delta}$$

$$= \frac{e V \cot\delta}{\beta c}$$

$$\Delta\theta = \frac{\Delta P_\perp}{P} = \frac{e V \cot\delta}{\beta^2 E_{Total}}$$

$$a_x \cong \frac{2r}{v_x} \Delta\theta = \frac{2r \cot\delta \, V}{v_x \beta^2 E_{Total}}$$

V and E_{Total} again in same units.

5. Numerical examples.

To plug in some numbers, consider a system like that of the proposal in which beam is stacked at ~ 1 Bev. Going around frequency for the stacked beam is 500 KC and $\frac{d\nu_{RF}}{dt} \simeq 30 \times 10^6$ cycles/sec, 2 going from 5.5 $\longrightarrow$ 11 mc three times/sec. $\nu_x = 10.8$. Resonances are crossed when

$$\frac{\omega_{RF}}{\omega_0} = \pm \nu_x \pm n, \qquad \text{assuming one cavity, no gradients}$$

$$11 \rightarrow 22 = \pm 10.8 \pm n$$

Resonances occur at n = 1 $\rightarrow$ 11, 22 $\longrightarrow$ 32 = 22 resonances/sweep = 66 resonances/sec ~ 8 in phase resonances/sec.

The equivalent number of in phase kicks/resonance = $\frac{f}{\sqrt{d\nu_{RF}}} = \frac{.5 \times 10^6}{\sqrt{30 \times 10^6}} \simeq 100$

So the number of in phase kicks/sec = $8 \times 100 = 800$.

If the cavity is radial

$$a_{peak} = \frac{\pi}{K} \frac{V}{\beta^2 E_{Total}} = \frac{7.5 \times 10^3 \times .05 \text{ mev}}{80 \times .75 \times 2 \times 10^3 \text{ mev}}$$

$$= .003 \text{ cm/Kick}$$

$$\bar{a} = \frac{a_{peak}}{2} = .0015 \text{ cm/kick}$$

Total amplitude = $\bar{a} \times 800 = 1.2$ cm for a one second stack. Seems just tolerable for a beam to be $\pm \frac{1}{2}$ cm at 15 Bev.

If the cavity is slanted at $\delta = 8.5^\circ$

$$a_{peak} = \frac{2\pi}{\nu_x} \frac{\cot \delta \, V}{\beta^2 E_{Total}} = \frac{2 \times 7.5 \times 10^3 \times 7.3 \times .05}{10.8 \times .75 \times 2 \times 10^3}$$

$$= .34 \text{ cm} \qquad \bar{a} = .17 \text{ cm}$$

Total amplitude = $800\ \bar{a} = 1.3$ meters!

If one is going to keep stacked particles around, the system must be designed so that betatron oscillations are not excited by the RF field of a slanted cavity.