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FIXED-FIELD ALTERNATING-GRADIENT ACCELERATORS

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The author is at present on leave of absence from Iowa State College, to work at the University of Illinois as a member of the Technical Group of the Midwestern Universities Research Association. Some of the material on which this article is based was discussed at the International Conference on accelerators, held in Geneva during the week of June 11, and at a meeting of the Canadian Association of Physicists on June 14, 1956.

I. INTRODUCTION

Developments in the art of designing high-energy particle accelerators may be of interest not only to nuclear physicists, but also to those working in chemical and engineering fields, to biologists, and to workers engaged in medical research. For the physicist the possibility of studying particle reactions at increasingly high energies may be the most exciting aspect of such developments, although a substantial increase of intensity, at energies presently available, would make possible definitive experiments now difficult to perform. For production of radiation effects on matter en gross, as in the production of cross-linkages in polymers or in various radiation damage investigations, intensity may be the more important characteristic of an accelerator. In the present article we attempt to outline a potential new development in the accelerator art which appears to offer not only the prospect of certain engineering advantages, but also the promise of a substantial increase of intensity or of the energy available for the study of particle reactions. Analysis of the particle orbits to be expected in the proposed structures affords a number of important and challenging mathematical problems concerning which it may be hoped an improved analytic understanding will be built up to supplement results obtained by digital computation.

The developments discussed here are the result of study by a group of Midwestern physicists (1) who were stimulated by the broad class of new accelerators apparently made possible by the use of the alternating-gradient principle, first announced from the Brookhaven National Laboratory (2). Specifically, in contrast to the present Brookhaven efforts, the Midwestern group has concentrated on a class of cyclic accelerators employing magnetic fields which are <u>constant</u> in time.

In any cyclic accelerator, as the cyclotron, betatron, or synchrotron, a charged particle makes a great number of revolutions within the structure, gaining a relatively small amount of energy on each turn, and the provision of suitable focusing forces is essential. It may be of interest to note in this connection that, in a number of typical accelerators now in use, the distance covered by the particle during the acceleration process ranges from one-third the distance across the United States to some six or eight times around the earth. Since particles with energies that are at least slightly different will be simultaneously present, a related property of an annular accelerator of importance in its effect on the cost of the structure is the ability to accomodate particles with various energies within an annular region of limited radial extent.

If, as is customary, the particles are guided by a magnetic field as they follow their orbits around the accelerator, it is particularly convenient to achieve the requisite focusing by adjustment of the spatial variation of this field. In the case that the fields show no variation with azimuth, a suitable index to characterize this spatial variation is

$$n \equiv \frac{r}{B}\frac{dB}{dr} ,$$

where r represents the distance from the central axis of the machine and B represents the strength of the (axial) field in the median plane. In the absence of an azimuthal variation, stability in both the radial and axial directions is obtained only if the condition

$$-1 < n < 0$$

is satisfied. The energy or momentum content of such a machine is expressed by the quantity

$$\alpha \equiv \frac{r}{p}\frac{dp}{dr} = n + 1 ,$$

where p denotes the particle momentum, and α is so small that an annular accelerator must then be operated in a pulsed manner to provide an increasing field adequate to hold particles of increasing energy within the machine.

In a conventional continuous-wave cyclotron, with the index n constrained to lie between 0 and -0.2 in order to avoid a coupling resonance between the radial and axial oscillations, the requirement that the frequency of revolution be independent of energy imposes a limitation on the attainable energy when the relativistic increase of mass becomes significant.

II. DESCRIPTION OF FIXED-FIELD ALTERNATING-GRADIENT ACCELERATORS

A markedly greater energy-content can be achieved in an annular accelerator if a rapid radial increase of the guide field is permitted by introduction of alternating-gradient focusing to maintain orbit stability. The field may then be capable of accomodating simultaneously particles of a wide range of energy and the field strength could be independent of time. Such a modification, although introducing complications associated with the significantly non-linear character of the differential equations governing the particle motion, evidently promises a number of significant advantages:

1. Direct-current magnet construction and excitation may be employed;

2. The magnetic field need only be adjusted for operation at a single level of excitation, thus avoiding the difficulties associated with remanence, saturation, and eddy-currents in a pulsed accelerator;

3. There is greater freedom in the choice of injection energy and the time-schedule for the acceleration process is flexible; and

4. High intensity appears possible, due to the permissible flexibility in planning the means of particle acceleration. Azimuthal variation of field in a cyclotron, with the associated alternating-gradient focusing effects, can also be advantageous in allowing higher energies to be reached than otherwise would be permitted by the relativistic increase of mass with energy.

We discuss below a number of specific types of structures in which fixed-field alternating-gradient focusing is present (3), (4), (5). The structures are of two general types, one employing radial sectors and the other a spiral sector pattern. The first-mentioned type is in some ways simpler and easier to construct, while the second appears to permit a smaller accelerator for a given energy. In all of the structures, particles with a wide range of energies can be simultaneously accommodated by virtue of a magnetic field whose average value around the machine varies with radius as r^k and focusing forces leading to stable motion are obtained by a suitable spatial variation of the field.

Reversed-field Design

In the reversed-field type of FFAG accelerator, the direction of the field is reversed from one sector to the next. The sector boundaries are usually supposed to be formed by geometrical planes which extend radially from the axis of the accelerator. The strength of the field in the reversed-field sectors, or the length of the reversed-field sectors, must of course be less than for the sectors of positive field in order that the particle orbits will ultimately be bent around through 360^o and permit a closed equilibrium orbit to be drawn (Fig. 1).

The magnitude of the field in the reversed-field accelerator varies at every azimuth as r^k, where r is the radius from the central axis of the machine. If k is positive there is axial defocusing in the positive field sectors and axial focusing in the reversed field sectors. The alternating-gradient action is found to yield reasonable stability for small-amplitude oscillations in both the radial and axial directions, provided the combined circumference of the forward and reversed field magnets is some five times that required by an azimuthally-constant magnetic field of the same maximum field strength. The ratio of the combined circumference to that required for a constant magnetic field is termed the "circumference factor", C.

Within the individual sectors the fields would normally be such that the complete equilibrium orbit would be formed from a series of circular arcs with their centers displaced from the axis of the machine. Denoting the radius of curvature of the orbit by ρ , the local focusing index is $n=k\cdot\rho/r$ and, if the same magnitude of field strength prevails in the positive and negative sectors, $\rho = r/C$. In linear approximation the radial and axial oscillations in such structures can then be expressed reasonably accurately, when the number of sectors is large, by the equations

$$\frac{d^2x}{d(s/r)^2} \pm kC\,x = 0$$

$$\frac{d^2z}{d(s/r)^2} \mp kC\,z = 0 ,$$

where s denotes arc-length along a reference circle of radius r, the upper and lower signs refer respectively to the sectors of positive and negative field, and where centrifugal effects have been neglected since we assume $kC \gg 1$. These equations may be solved by aid of the matrix methods customarily employed in analysis of alternating-gradient focusing. If the phase change per sector for the radial oscillations and the corresponding phase change for the axial oscillations are permitted to assume widely different values, lying near the upper and lower limits of the stable range, a design with C as low as 5 may be feasible. A more accurate calculation must, of course, take account of the edge effects which arise at the sector boundaries and would involve an expansion about an equilibrium orbit which accordingly must be determined first. For a complete account of the motion the effect of non-linear terms would also have to be included.

Attention is directed to the important scaling property of the orbits in this accelerator. Possible orbits of particles of different energies, or moments, are scaled replicas of each other. In consequence, the frequencies of the oscillations will be independent of energy and harmful resonances may be avoided at all energies by a consistent design. The momentum content is represented by $p \propto r^{k+1}$, so that the momentum compaction factor α is given by

$$\alpha + k + 1$$

and can be either positive or negative in a reversed-field accelerator.

A small working model of a reversed-field FFAG accelerator has been put into operation (6). This model, shown in Fig. 2, employs eight sectors of positive field and eight shorter sectors of negative field. Electrons are accelerated, at present by betatron action, from 25 Kev to 400 Kev. Tuning controls have been provided for the model, so that various oscillation frequencies can be produced. These frequencies can be measured accurately by a radio-frequency knock-out technique (7) and the effect of certain resonances on the beam noted. The model affords an opportunity to study operation with a high duty-factor, as is possible in FFAG accelerators employing betatron acceleration. Radio-frequency acceleration methods will also be investigated.

Possible parameters for a large-scale reversed-field FFAG accelerator for the production of 10 Bev protons have been examined. Although such a machine would be expected to have many desirable characteristics, the large magnet mass and power requirements direct interest to other FFAG designs of smaller circumference factor. By virtue of its essential simplicity, however, the reversed-field type may remain of interest for accelerators of low or intermediate energy, especially if a high duty-factor can be efficiently realized with betatron acceleration.

Spiral-Sector Design

To avoid the considerable circumference required for a reversed-field FFAG accelerator, an alternative arrangement has been suggested by Dr. Kerst and others of the MURA group in which the alternating-gradient action is provided by a smaller but more rapid spatial variation of the field, the field being alternatively high and low along spiral curves which all particles must cross. Illustrative of the type of field present in the median plane of such a structure, one may take

$$B_{z_0} = \langle B \rangle \, (r/r_0)^k \left\{ 1 + f \sin \left[\frac{\ln(r/r_0)}{w} - N\phi \right] \right\} .$$

From this expression it is seen that N is the number of spiralling ridges passed over by a particle in going around the machine once. The coefficient f is the fractional flutter in the magnetic field due to the ridges. Finally, if the radial width of the annulus is small in comparison to the outer radius, r_0, $\lambda = 2\pi\, r_0 w$ is substantially the radial separation of the ridges. The exponent k is taken to be positive.

In the spiral-sector design, as in the radial-sector case, the fields and the orbits satisfy the scaling condition. In passing from one energy to another there is, however, a rotation of the geometrically similar orbits, which presents complications if one wishes to introduce straight-sections (field-free regions) whose boundaries extend radially from the central axis of the machine.

The equilibrium orbit in the spiral-sector machine departs from a circle by an amount which affects significantly the character of the small-amplitude oscillations. For analytic work (8a,b,c,d) it is most appropriate to expand the equations of motion about the scalloped equilibrium orbit. In terms of cylindrical coordinates (r, z, ϕ) we introduce the notation

$$x \equiv \frac{r - r_1}{r_1} \qquad y \equiv \frac{z}{r_1} \qquad N\Theta \equiv N\phi - \frac{1}{w} \ln(r_1/r_o)$$

and choose r_1 so that the dimensionless variable x will be small. The forced motion which produces the non-circular equilibrium orbit is found to be quite well represented by

$$x_f = - \frac{f}{N^2 - (k + 1)} \sin N\Theta$$

and the linearized equations describing small-amplitude oscillations are represented by Hill equations of substantially the following form:

$$u'' + (a_u + b_u \cos N\Theta + c_u \cos 2N\Theta)\, u = 0$$

$$y'' + (a_y + b_y \cos N\Theta + c_y \cos 2N\Theta)\, y = 0,$$

where

$$u \equiv x - x_f$$

$$a_u \cong k + 1 - \tfrac{1}{2} \frac{(f/w)^2}{N^2 - (k + 1)} \qquad b_u \cong \frac{f}{w} \qquad c_u \cong \tfrac{1}{2} \left(\frac{f}{wN}\right)^2$$

$$a_y \cong - k + \tfrac{1}{2} \frac{(f/w)^2}{N^2 - (k + 1)} \qquad b_y \cong \frac{f}{w} \qquad c_y \cong - \tfrac{1}{2} \left(\frac{f}{wN}\right)^2 .$$

Non-linear terms in the equations of motion can also be obtained.

The frequencies and other characteristics of the oscillations characterized by the foregoing linear equations can be obtained by use of tables prepared with the aid of the electronic digital computer of the Graduate College of the University of Illinois (ILLIAC). Useful orientation is provided, however, by writing the frequencies which are given by a simple approximate solution (9), ignoring the relatively small effect of the terms involving cos 2NΘ and taking $N^2 \gg k + 1$:

$$\nu_x = \left[k + 1\right]^{1/2}$$

$$\nu_y = \left[\left(\frac{f}{wN}\right)^2 - k\right]^{1/2} .$$

It is thus seen that the frequency of the free radial oscillations is substantially determined by the exponent k characterizing the radial increase of average field strength, so that $k + 1$ must be positive, and that axial stability may be obtained if the term $\left(\frac{f}{wN}\right)^2$ is sufficiently large to dominate $-k$. The stability region for the small-amplitude oscillations represented by the Hill equations cited above has been mapped by aid of the ILLIAC tables and is depicted in Fig. 3.

The non-linearities associated with large-amplitude motion in the spiral-sector accelerator make the use of automatic digital computation particularly helpful in trajectory studies. Results pertaining to motion with one degree of freedom are appropriately and conveniently represented by phase plots which depict the position and associated momentum of a particle as it progresses through successive "sectors" (periods of the structure) from one homologous point to another (Fig. 4). For small-amplitude motion the particle is represented by a point which moves around an elliptical curve in phase space, while with larger amplitudes curves departing from elliptical shape may be followed. At still larger amplitudes, unstable fixed points -- representing an unstable equilibrium orbit -- make their appearance. Associated with the unstable fixed points one finds a separatrix, constituting an effective stability limit to the motion, which in the majority of cases the ILLIAC results depict as a sharp boundary and outside of which it is frequently possible to draw the initial portion of unstable phase curves.

Due to the non-linear character of the oscillations, it is not surprising (10) (11) (12) that the permissible amplitude of oscillation is much curtailed if σ, the phase change per sector, lies near $2\pi/3$ or $2\pi/4$. It has, in fact, also been found (13) that the amplitude limit is reduced, although not to zero, for $\sigma = 2\pi/5$. For cases in which σ_x is near $2\pi/3$, the limit of radial stability is characterized by the appearance of three unstable fixed points. In this case an examination of the non-linear differential equation for the trajectory permits a rough estimate to be made of the limiting amplitude (14):

$$A_x \cong 2(w^2N^2/f) \left| (\sigma_x/\pi)^2 - (2/3)^2 \right| .$$

It may be noted that, since the oscillation frequencies are esentially determined by k and f/wN, this formula suggests a desirable increase of stable amplitude might be expected if f and w were each increased by the same factor.

Introduction of axial motion into a study of spiral-sector accelerators produces complications for all but the smallest amplitude oscillations, since there is coupling between this motion and that occurring in the radial direction. Surveys can be made, however, to determine initial conditions which appear to exhibit short-time stability. In typical cases the permissible amplitude for axial motion appears to be materially smaller, possibly by a factor of five, than is allowable for the radial motion. When oscillations in two degrees of freedom are treated, the characteristics of the axial motion and inferences concerning stability limits are materially affected by proximity to certain coupling resonances, notably those for which

$$\sigma_x = 2\sigma_y ,$$

$$\sigma_x + 2\,\sigma_y = 2\pi , \quad \text{or}$$

$$2\sigma_x + 2\,\sigma_y = 2\pi \quad .$$

Near such resonances the amplitude of axial motion exhibits an exponential increase, over a considerable amplitude range, the rate of growth being the greater the more the radial amplitude exceeds a certain threshold value and the closer one is to the resonance in question. Some quantitative success in accounting for the growth of axial amplitude can be obtained by treating the differential equation for the axial motion as linear and inserting a prescribed expression for the radial oscillations into certain coupling terms which are linear in the axial coordinate.

In an actual accelerator, the N individual sectors will not be exactly identical, due to the presence of unavoidable small differences in construction, excitation, or alignment. The basic period of the structure will thus be strictly N sectors, representing the machine as a whole, and additional resonances based on values of $N\sigma$ may be of importance. Computational study of the effect of realistic misalignments can be very informative prior to fixing the specifications of a proposed machine. By way of example, studies of a proposed five-sector model ($\nu_x = 1.41$, $\nu_y = 0.87$) indicated that an axial displacement of one sector by 1/300 of the radius effected a reduction of the stable radial and axial amplitudes by factors of about 2 and 3, respectively.

Separated-Sector Modification

In the spiral-sector accelerator discussed in the foregoing paragraphs, an unnecessary and probably undesirable limitation was introduced by requiring that the field in the median plane have a precisely sinusoidal variation. The aperture which is magnetostatically possible is severely limited (15), especially if f differs markedly from the value ¼. In addition, the angle $\tan^{-1} Nw$ of the ridges (measured with respect to a reference circle) may be inconveniently small in a large machine and a convenient construction may be difficult to realize. Attention is accordingly directed to structures involving separated poles (Fig. 5), a design which affords improved accessability to the vacuum chamber and beam, easy realization of a more generous magnet gap, a considerably higher value for the root mean square field flutter, and a corresponding increase of the spiral angle. In this design it would be important to retain the scaling feature of the field and to take note of the high-order Fourier components which some pole configurations may introduce into the field. Retention of the scaling requirement makes it possible to solve the magnetostatic problem, which is defined by a specified pole contour, by relaxation methods on a <u>two-dimensional</u> grid which represents variables conveniently taken as

$$\xi = \frac{1}{2\pi}\left[\frac{\ln(1+x)}{w} - N\theta\right],$$

$$\eta = \frac{\sqrt{1+(wN)^2}}{2\pi w}\,\frac{y}{1+x}.$$

The result of such computations may then be stored, again on a two-dimensional grid, for use in trajectory computations (16a,b).

Plans are being completed for construction, at the University of Illinois, of electron models which will provide experience pertaining to spiral-sector and separated-sector FFAG accelerators. These models will be similar in size to the reversed-field model mentioned earlier and likewise will employ betatron acceleration in the initial tests. Provisional designs of a large-scale machine have been attempted. It has been estimated that a separated-sector FFAG magnet for the production of 15 Bev protons would weigh about 12000 tons and consume some 5 MW of electrical power. This estimated magnet weight is intermediate between estimates which one would make for reversed-field and spiral-sector magnets, for which the estimated weights would be roughly 3 times greater or 3 times smaller, respectively. Although such a separated-sector structure may be some six times as massive as a pulsed accelerator of the same design-energy, it may be felt that this feature is compensated to a considerable degree by the many simplifications which a direct-current design affords and that, as will be emphasised later, the increased freedom in detailed acceleration methods may permit a very isgnificant increase of intensity.

Cyclotrons

It is attractive to consider the possible applicability of a spiral field-variation to continuous-wave cyclotrons, as a generalization of the early suggestions of Thomas (4), in the interests of increasing the attainable energy. If, to permit continuous-wave operation, the frequency of revolution is to be independent of particle energy, the field index k which characterized (differentially) the radial increase of the average field must satisfy the relation

$$k + 1 = (E/E_0)^2 ,$$

where E and E_0 are respectively the total and rest energies of the particle. In a cyclotron, therefore, k must increase with energy, the oscillations will not satisfy the scaling requirement, and the possibility of encountering dangerous resonances during the acceleration process must be carefully considered. If we regard the relation $\nu_x = [k + 1]^{1/2}$ as sufficiently accurate for the present purpose, then $_{x} = E/E_o$,

the first half-integral and integral machine resonances for the radial motion ($\nu_x = 3/2$ and $\nu_x = 2$) would be encountered at kinetic energies of $\frac{1}{2}E_o$ and E_o respectively (17), and the $\sigma_x = 2/3$ inherent resonance at $(\frac{N}{3} - 1)E_o$. The design of FFAG cyclotrons is currently being pursued by a number of groups and design modifications which hold the promise of ameliorating the foregoing difficulties are being explored.

III. ACCELERATION METHODS

In small size annular accelerators employing the FFAG principle the use of betatron acceleration is highly attractive from the standpoint of intensity. If charged particles are injected into the gap of the fixed-field magnet during a substantial portion of the time the central flux is rising, they may be accelerated and arrive at the target with full energy so long as the flux continues to rise (Fig.6). If the total change of flux within the core is twice that required to accelerate the beam from the low to the high magnetic field region, the duty cycle would approach 25%.

For larger machines radio-frequency acceleration methods would appear to be more practicable. The lack of dependence on a fixed magnet excitation cycle may permit in the FFAG accelerators a more rapid recycling of the radio-frequency program and a desirable flexibility in the design of this program. In analyzing the synchrotron motion it is noteworthy that, in distinction to pulsed machines, the orbit radius and revolution frequency are a function only of the particle energy, rather than of energy and time. To study in detail the effects of radio-frequency handling systems it is helpful to employ a Hamiltonian theory for the synchrotron oscillations, in order that general theorems such as that of Liouville may be brought to bear on the problem. With $\omega(E)$ denoting 2π times the revolution frequency of the particle and E the energy, suitable canonical coordinates are the electrical phase-angle ϕ with which the particle crosses the acceleration gap and the quantity w, related to energy, defined as

$$w \equiv \int^{E} \frac{dE}{\omega(E)} .$$

For a single cavity of peak voltage V, frequency $\nu/2\pi$, and operating at the h^{th} harmonic of the nominal particle frequency, the equations characterizing the synchrotron motion can then be derived from the Hamiltonian

$$\mathcal{H} = V \cos\phi + 2\pi\left[\nu w - hE(w)\right],$$

in which V and ν are specified functions of time.

To avoid the large frequency swing -- perhaps as great as a factor of 11 -- which would be required to carry a proton from its initial to its final energy in a single modulation cycle, it is attractive to think of raising the particle energy in a series of steps each involving a comparatively small amount of frequency modulation. Such an arrangement provides a sort of "bucket-lift" process whereby groups of particles are simultaneously and progressively accelerated by means of a single radio-frequency source whose frequency is successively a smaller multiple of the increasing revolution frequency of the particle. If one commences with an oscillator frequency which is $s.p^M$ times the rotation frequency of the injected particle and modulates by a factor p/q, the particle frequency is raised by this factor and the particle may be further accelerated in the $s.q.p^{M-1}$ harmonic during the next frequency-modulation cycle.

The modulation cycle may thus be employed by the particle some $M + 1$ times, as it progresses to higher energies, before synchronism is lost. The modulation factor, p/q, could be 3/2 for example and a factor 2/1 might be particularly suitable.

If one thinks of using a bucket-lift process to stack particles at some intermediate energy prior to a final acceleration of the accumulated group by a second radio-frequency system, conservation of area in (ϕ,w) phase-space tells us that the particles in successive buckets cannot be superposed exactly. Physically speaking, one group is slightly disturbed and displaced by the oscillator when it brings up a later group. This displacement has been studied computationally and is not sufficient to preclude the practicality of stacking a number of groups in a region of synchrotron phase-space sufficiently limited that a second radio-frequency system could then accomodate them all.

For efficient stacking, it is of interest to ascertain the number of buckets which may be brought up empty at the end of the process. If $q = 1$ and $p = 2$ and if particles are injected only once per frequency-modulation cycle, the number of such empty buckets may readily be shown to be s, but these extra buckets can presumably be used with a consequent increase of intensity by more frequent injection.

There are several variants of this bucket-lift arrangement, which may present advantages chiefly of convenience. With an <u>unscheduled</u> bucket lift, particles not caught in a bucket at the onset of a particular frequency-modulation cycle will usually be displaced downward in energy by a passing bucket, but will be caught on occasional frequency-modulation cycles and in the end may be carried up in energy. The use of a completely stochastic acceleration method has been discussed in a Soviet paper (18) and shown to lead to acceleration of some particles by a sort of random-walk process.

It appears clear that the flexibility which fixed-field accelerators permit in regard to design of particle-handling methods offers many promising possibilities. These possibilities are being further studied within the MURA group, chiefly by Drs. A. M. Sessler and K. R. Symon, both analytically and with the aid of digital computation. As a related endeavor, the characteristics of mechanically-modulated radio-frequency cavities are being studied by Dr. Zaffarano and his associates at Iowa State College. The accumulation of intense beams, within an accelerator or in adjacent storage rings (19)(20), by a suitable stacking process may open the door to study of a new field of high-energy physics.

IV. INTERSECTING-BEAM ACCELERATORS

With the possibility in sight of attaining beam intensities higher than have been possible heretofore, the opportunity arises (21) of studying high-energy particle interactions by directing one beam against another (Fig. 7). The outstanding advantage of such a system would be the large increase of effective center-of-mass energy which could be reached in this way. If two beams, each of energy E_1, are directed against each other, the total energy is, of course, $E_{CM} = 2E_1$. In contrast, a single beam of energy E_1' (measured in units of the rest energy) directed against a stationary target makes available a center-of-mass energy which is approximately $E_{CM} = (2E_1')^{1/2}$ for $E_1' \gg 1$. Thus two 15 Bev proton beams, oppositely directed, are equivalent to a single beam of 500 Bev directed against a stationary target and two 21.6 Bev accelerators would be equivalent to one machine of 1 Tev (10^{12}ev)!

In estimating the practicality of intersecting-beam accelerators one must, of course, judge whether it is feasible to produce beam intensities which will result in a sufficiently large reaction rate. The interactions of interest must, moreover, be studied in the presence of background radiation produced by the individual beams and will bear a more favorable ratio to the background the greater the density of intersecting particles. In this regard, however, it may be noted that the background radiations will be confined to directions differing little from the beam direction, while the reactions of interest will be essentially isotropic in the laboratory system. The background and beam survival will be directly dependent on the degree of vacuum which can be maintained in the system so that recent developments for the realization of high pumping speeds (22) and the measurement of high vacuua (23) will be of importance. The additional focusing or defocusing effects which arise from space-charge forces, possibly modified by the effect of any electrons which may be captured by the beam, and the difficulties of handling safely a concentrated beam which may possess an energy of 1 megajoule will also require careful attention.

The intensities which one may be able to build up will certainly depend on the efficiency of stacking and on the ingenuity employed in the injection process. Although these techniques may be developed and improved as experience is gained with completed FFAG accelerators, an upper limit to the particle density in a stacked beam is imposed by Liouville's theorem. In regard to this limitation we may estimate the number of injected pulses which theoretically could be assembled, after acceleration, in a region of reasonably small cross-sectional area. With respect to the energy-spread associated with the motion in synchrotron phase-space, we may consider the fate of particles injected with an energy spread ΔE_1, assuming for simplicity that synchrotron and betatron phase-space are separately conserved. If the most efficient particle handling system is used, the number of pulses which can be contained with a region ΔE_2 in energy at the completion of the acceleration process is

$$n_p = (\Delta E_2/\Delta E_1)/(\omega_2/\omega_1),$$

for $\Delta\phi$ constant, since the area in phase-space is $\Delta\phi_2\, \Delta E_2/\omega_2 = n_p \Delta\phi_1 \Delta E_1/\omega_1$.

ΔE_2 in turn may be expressed conveniently in terms of the associated radial spread of the beam

$$E_2 = (k + 1)(p_2^2 c^2/E_2)(\Delta r_2/r_2)$$

$$\cong (k + 1)\, E_2\, (\Delta r_2/r_2)\ , \text{ ultra-relativistically.}$$

Thus, if $k + 1 = 100$, $E_2 = 15 \times 10^9$ ev, $\Delta r_2 = 0.5$ cm, $r_2 = 10^4$ cm, $\omega_2/\omega_1 = 11$, and $\Delta E_1 = 4 \times 10^3$ ev, we find

$$\Delta E_2 = 7.5 \times 10^7 \text{ ev and}$$

$$n_p = 1700 \text{ particle pulses.}$$

Similarly, in regard to the phase-space for betatron oscillations, if the injector is imagined to scan the aperture, the number of horizontal and vertical scans which theoretically could be accommodiated can be written

$$n_x = \frac{p_2}{p_1}\ \frac{(\Delta r_2)^2}{r_2 \beta_x \Psi_x \Delta r_1}$$

$$n_y = \frac{p_2}{p_1}\ \frac{(\Delta z_2)^2}{r_2 \beta_y \Psi_y \Delta z_1}$$

where Ψ_x, Ψ_y denote the angular spread of the injected beam. β_x, β_y relate the angular and linear displacements experienced during the course of a betatron oscillation ($\Delta r_2 = r_2 \beta_x \Psi_{x2}$), and the momentum ratio p_2/p_1 accounts for the adiabatic damping of the oscillations. Accordingly, approximating $\beta_{x,y}$ by $2/\nu_{x,y}$,

$$n_x n_y = \left(\frac{p_2}{p_1}\right)^2 \frac{\nu_x \nu_y (\Delta r_2)^2 (\Delta z_2)^2}{4 r_2^2 \Psi_x \Psi_y \Delta r_1 \Delta r_2}\ .$$

If we now substitute $\rho_2/\rho_1 = 100$, $\nu_x = 10$, $\nu_y = 5$, $r_2 = 10^4$ cm, $\Delta r_2 = \Delta z_2 = 0.5$ cm, and $\Psi_x \Delta r_1 = \Psi_y \Delta z_1 = 0.5 \times 10^{-3}$ radian cm, we find

$$n_x n_y = 1250.$$

This large value for the theoretically admissible number of scans implies a very complex scanning procedure and suggests that an injector with a much larger beam spread and correspondingly higher current would be desirable (24).

Based upon the considerations of the preceding paragraphs, one would estimate that a 1 ma. injector would permit the accumulation of

$$N_p = \frac{10^{-3}}{1.6 \times 10^{-19}} \times \frac{2\pi \times 10^4}{3 \times 10^{10}/11} \times 1700 \times 1250$$
$$\cong 3 \times 10^{17} \text{ particles}$$

within a tube of about 1 cm^2 cross-sectional area. If we estimate that we actually may have 1/600 as large a beam as this, or 5×10^{14} particles circulating in each machine, some 10^7 interactions per second (proportional to N_p^2) may be expected to be produced in an interaction region which is one meter in length (21). With a vacuum of the order of 10^{-6} mm Hg of nitrogen gas, the background produced in this target volume may be expected to be larger by about one order of magnitude, but, as pointed out previously, the background radiations will be confined primarily to the median plane. Interaction with the residual gas also has the effect of limiting the beam life, possibly to a time not much longer than 1000 seconds in the present example, so that groups of particles must be injected to replenish the beam at a rate not less than the reasonable value of one group per second.

It is the hope of the MURA group that further theoretical and experimental work will lead to the design and construction of models which will permit testing means for efficient particle acceleration, the investigation of high-current beams, and permit the eventual realization of a research machine which will take full advantage of the benefits to be derived from the FFAG principle. It is impossible here to give explicit credit to the many physicists who have contributed to the development of these ideas, but it is fitting to indicate our special appreciation of the courtesy which the University of Illinois has extended to the MURA group in making the ILLIAC available for numerous computational studies and our indebtedness to Dr. J. N. Snyder for directing this phase of the program. The writer wishes also to express his appreciation to Dr. D. W. Kerst, Dr. K. R. Symon, and Dr. A. M. Sessler for assistance in the preparation of this article and to Dr. K. Lark-Horovitz for his courtesy in reading the manuscript in draft form.

V. REFERENCES AND NOTES

1. The technical group has been under the direction of Dr. D. W. Kerst. As the interest in the work grew, a number of Midwestern Institutions formalized this cooperative effort by forming the Midwestern Universities Research Association (MURA). The work of the technical group has been assisted by the National Science Foundation, the Office of Naval Research, and the Atomic Energy Commission.

2. E. D. Courant, M. S. Livingston, and H. S. Snyder, Physical Review **88**, 1190 (1952).

3. The interest of the Midwest group in Fixed-Field Alternating Gradient (FFAG) accelerators arose from the original suggestions of Dr. K. R. Symon, made during meetings of the technical group in the summer of 1954. The idea of accelerators employing annular D.C. magnets was also proposed earlier, in at least one form, by T. Ohkawa at a meeting of the Physical Society of Japan and appears to have received brief consideration by others working in the accelerator field. A special form of cyclotron employing an azimuthal variation of field is the design proposed by Thomas (4).

4. L. H. Thomas, Physical Review **54**, 580, 588 (1938).

5. Discussion of FFAG accelerators has been given in papers presented before the American Physical Society and in the following paper, to be published in The Physical Review: K. R. Symon, D. W. Kerst, L. W. Jones, L. J. Laslett, and K. M. Terwilliger, Fixed Field Alternating Gradient Particle Accelerators. In this article a general theory of orbits, applicable to a variety of types of cyclic accelerators, is derived in linear approximation.

6. The reversed-field model was constructed at the University of Michigan under the direction of K. M. Terwilliger and L. W. Jones. The magnet design was carried to completion by R. Haxby, of Purdue University, and injectors were supplied from Dr. Kerst's laboratory

at the University of Illinois. A substantial portion of the theoretical work was contributed by F. T. Cole, of the State University of Iowa.

7. C. L. Hammer, R. W. Pidd, and K. M. Terwilliger, Rev. Sci. Instr. **26**, 555 (1955).

8. Details of some of the analytic work pertaining to spiral-sector accelerators are given in a series of MURA reports:

(a) L. J. Laslett, Character of Particle Motion in the Mark V FFAG Accelerator, LJL(MURA)-5, et seq. (July 30 and August 1, 1955).

(b) D. L. Judd, Analytical Approximation in Mark V Scalloped Orbits and to Radial Betatron Oscillations about them, MURA DLJ-1 (August 10, 1955).

(c) _______, Non-Linear Terms in Mark V Radial Betatron Equation, MURA DLJ-2 (August 12, 1955).

(d) F. T. Cole, Mark V Expanded Equations of Motion, MURA/FTC-3 (January 19, 1956).

9. A useful method for obtaining approximate characteristics of solutions to linear or non-linear differential equations with periodic coefficients is the "smooth approximation" developed by K. R. Symon and summarized in two MURA reports:

(a) K. R. Symon, A Smooth Approximation to the Alternating Gradient Orbit Equations, KRS(MURA)-1 (July 1, 1954).

(b) _______, An Alternative Derivation of the Formulas for the Smooth Approximation, KRS(MURA)-4 (August 10, 1954).

10. J. Moser, Nachr. Akad. Wiss. Göttingen, Math.-Physik Kl. **II-a**, No. 6, 87 (1955).

11. _______, Communications on Pure and Appl. Math., **VIII**, 409 (1955).

12. P. A. Sturrock, Static and Dynamic Electron Optics (Cambridge University Press, Cambridge, 1955), Chapt. 7. In place of the quantity which we have denoted σ, Moser (10)(11) employs $2\pi w$ or $2\pi\alpha$ and Sturrock employs θ. Our quantity σ may be related to the number

of betatron oscillations, ν , executed by the particle as it passes through N sectors to make a complete circuit of the accelerator, by the relation $N\sigma = 2\pi\nu$.

13. R. Christian (unpublished).

14. This relation was derived originally by Dr. A. M. Sessler and the present author and has recently been treated more carefully by Dr. G. Parzen (unpublished).

15. In the absence of back-wound currents on the pole surface and with f assuming its optimum value 0.24, the available magnet gap is limited to $G = 0.28\,(2\pi\omega)r = 0.28\,\lambda$, where λ is the radial wavelength of the magnet structure.

16. This computational method is outlined and certain useful general features of the fields are treated, respectively, in the following MURA reports:

(a) L. J. Laslett, *Proposed Method for Determining Mark V Trajectories by Aid of Grid Storage*, MURA-LJL-8 Revised (February 21, 1956);

(b) J. L. Powell, *Mark V FFAG Equations of Motion for Illiac Computation*, MURA-JLP-6 (August 16, 1955).

17. *Cf.* D. S. Falk and T. A. Welton, *Bull. Amer. Physical Soc.* II, **1**, 60, Paper U4 (1956).

18. E. L. Burshtein, V. I. Veksler, and A. A. Kolomensky, *A Stochastic Method of Particle Acceleration*, translated by V. N. Rimsky-Korsakoff from *Certain Problems of the Theory of Cyclic Accelerators* (USSR Akad. Sci., Moscow, 1955), pp 3-6.

19. D. Lichtenberg, R. Newton, and M. Ross, *Intersecting Beam Accelerator with Storage Ring*, MURA Report MURA-DBL/RGN/MHR-1 (April 26, 1956).

20. G. K. O'Neill, *The Storage-Ring Synchrotron -- A Device for High-Energy Physics Research*, Princeton University Report (April 11, 1956); *Physical Review* **102**, 1418 (June 1, 1956).

21. D. W. Kerst, et al., Physical Review **102**, 590 (April 15, 1956).

22. R. H. Davis and A. S. Divatia, Rev. Sci. Instr. **25**, 1193 (1954).

23. D. Alpert, Science **122**, 729 (1955)

24. Interesting information concerning working designs of high current linear accelerators is given by E. O. Lawrence, Science **122**, 1127 (1955).

TITLES FOR FIGURES

Fig. 1. Orbits in a reversed-field FFAG accelerator.

Fig. 2. An operating electron model of a reversed-field FFAG accelerator. Eight sectors of positive field and eight narrower sectors of negative field are employed. The betatron core is seen linking the region occupied by the particle orbits.

Legend:

- F Magnet sector with forward or positive field
- R Magnet sector with reversed or negative field
- C Betatron core
- I Injector
- M Pump manifold.

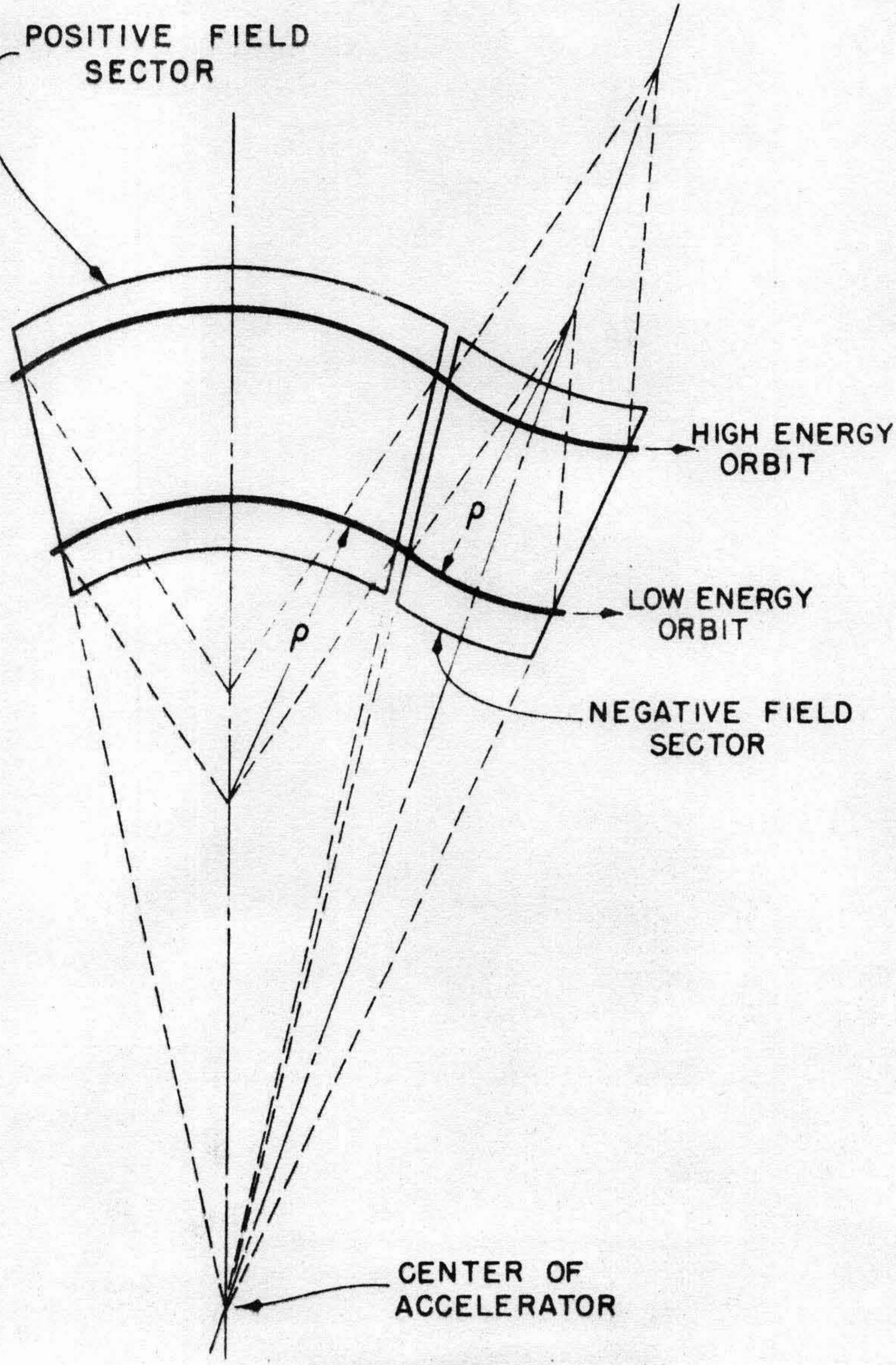
Fig. 3. First stability region ($0 < \sigma \equiv 2\pi\nu/N < \pi$) for small-amplitude oscillations in spiral-sector FFAG accelerators. The curves are calculated for the case $k \gg 1$ and are believed to be most accurate for ordinates less than 1/3. When the condition $k \gg 1$ is not satisfied the diagram can best be used by entering at the point $(k/N^2, f/wN^2)$ and proceding up a curve of constant σ_v until an abscissa of $(k + 1)/N^2$ is reached.

Fig. 4. Phase plot representing radial motion, at $N\theta = 0$ mod. 2π, in a spiral-sector FFAG accelerator. The machine parameters are those of a proposed model, for which $k = 0.8$, $1/w = 23.0$, $f = ¼$, and $N = 5$. In this case σ_x is close to 0.571π for small-amplitude motion. σ_x does not change greatly with increasing amplitude and it is noteworthy that ultimately seven unstable fixed points make their appearance in this particular example.

Fig. 5. Pole configuration illustrative of the separated-sector modification of a spiral-sector magnet. The currents carried by the pole-face windings are instrumental in achieving the r^k dependence of the magnetic field.

Fig. 6. Operation cycle of a FFAG betatron with a high duty factor.

Fig. 7. Schematic method of effecting the intersection of high-energy beams. In the case illustrated the individual accelerators are considered to be of the separated-sector type.


POSITIVE FIELD SECTOR
HIGH ENERGY ORBIT
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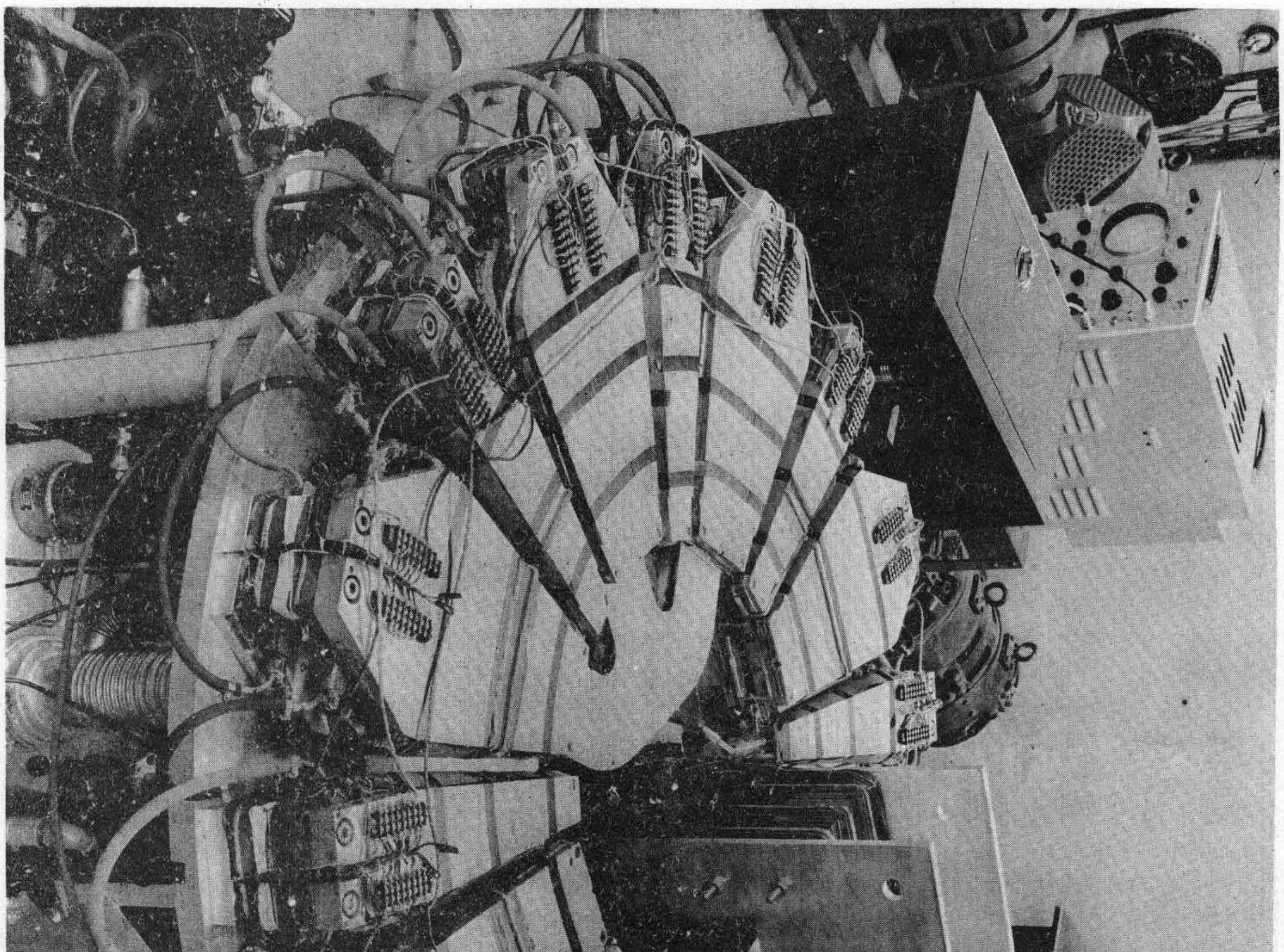
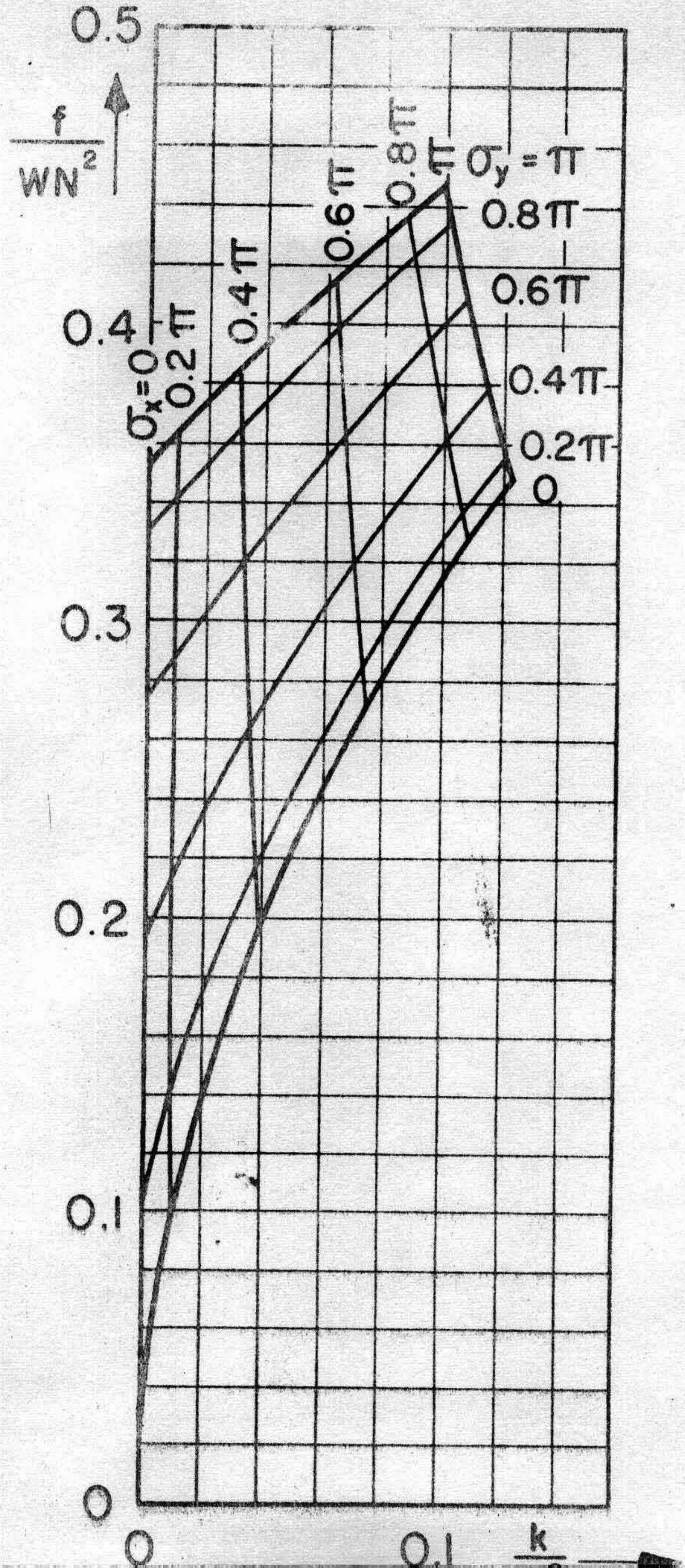
ρ
LOW ENERGY ORBIT
NEGATIVE FIELD SECTOR
CENTER OF ACCELERATOR





0.5
0.4
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0
$\frac{f}{WN^2}$
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0.8π
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0
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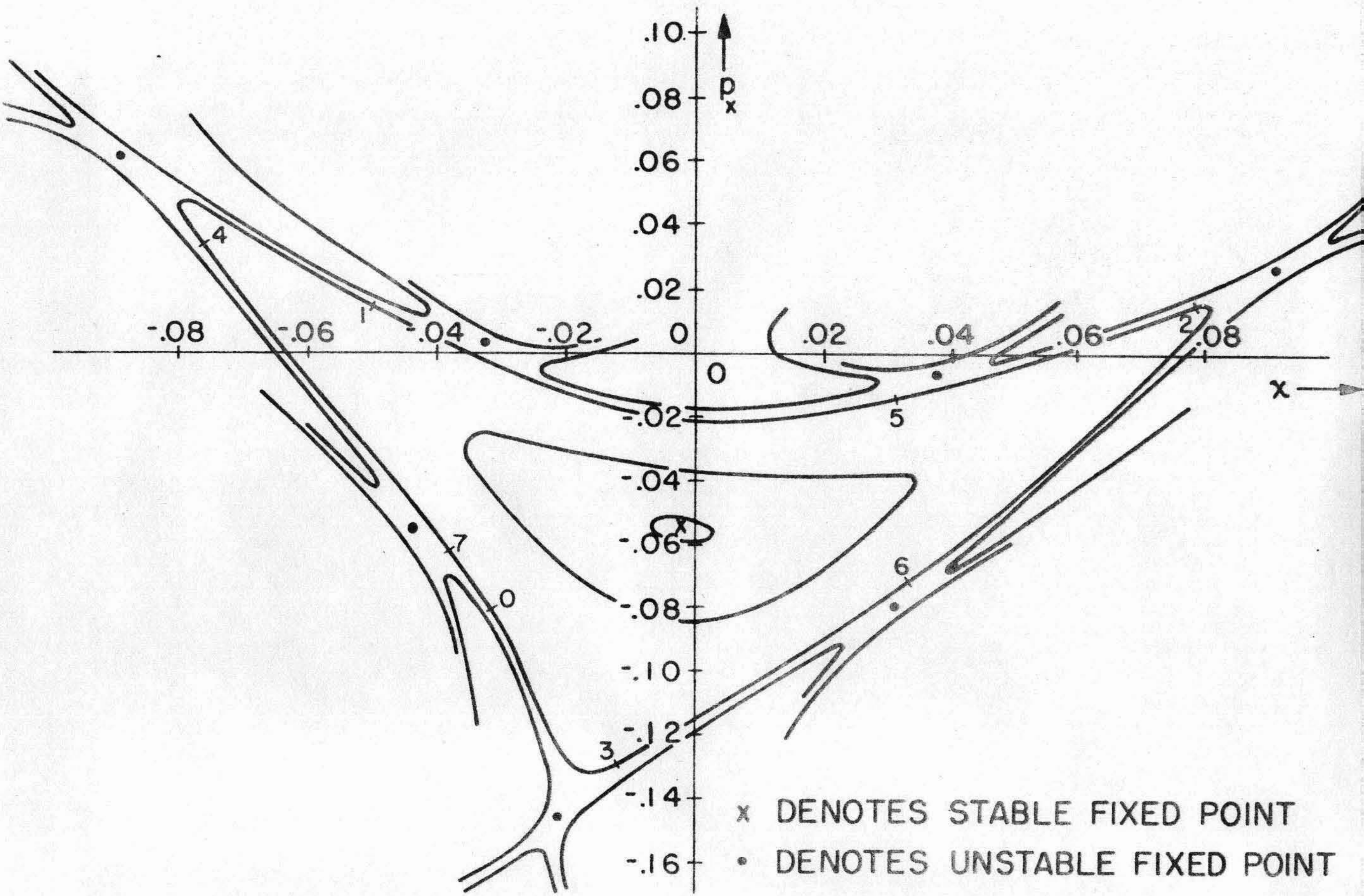

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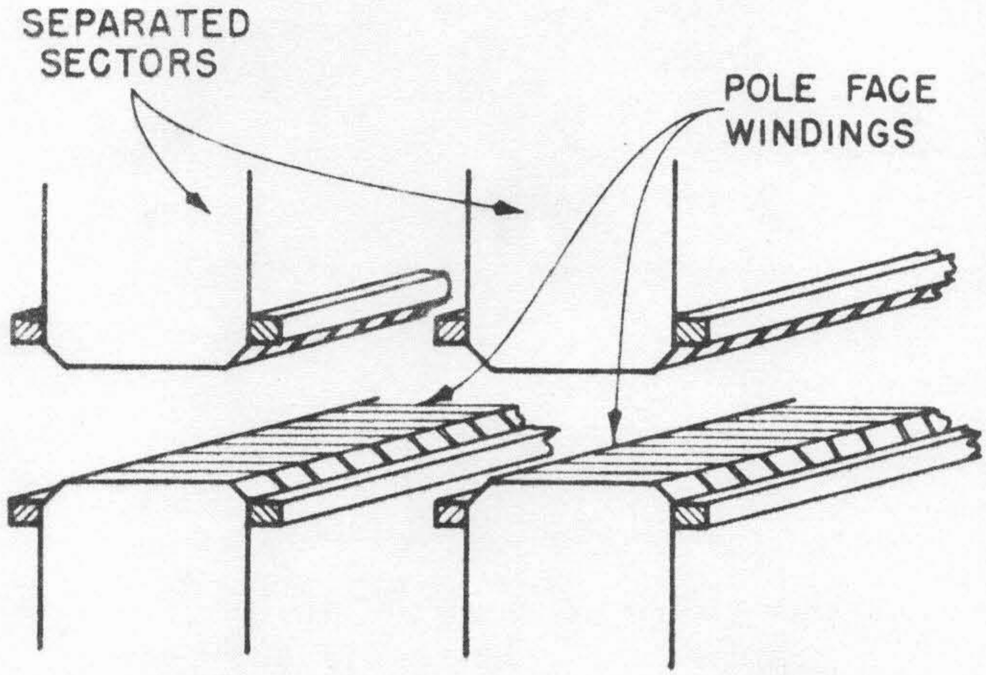
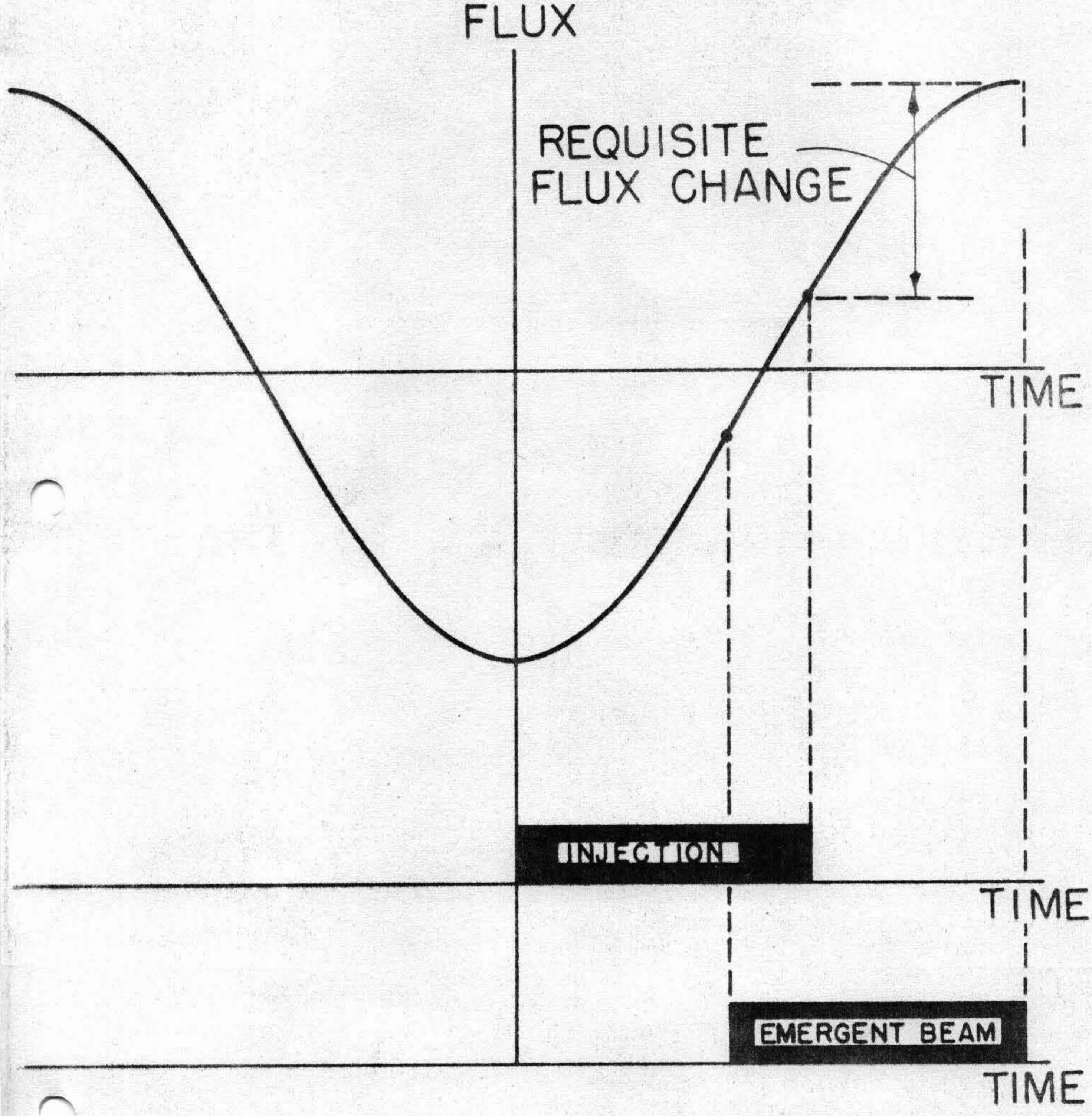
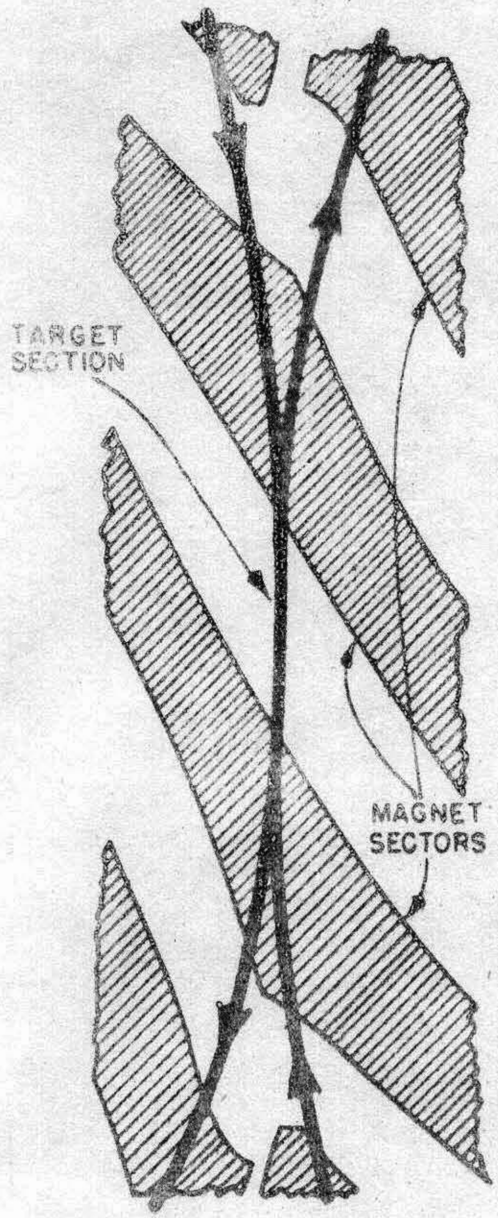

SEPARATED
SECTORS
POLE FACE
WINDINGS



FLUX
REQUISITE
FLUX CHANGE
TIME
INJECTION
TIME
EMERGENT BEAM
TIME


TARGET
SECTION
MAGNET
SECTORS