MURA-GP- 1
Internal
July 28, 1956

Non-Linear Resonances in the Radial Motion

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I Introduction

The following is a description of a method for treating the $\nu = \frac{1}{3} N$, $(\sigma = \frac{1}{3} 2\pi)$ resonance in the radial motion. Section II gives the general treatment and the general final results are given by eqs. (17) $\longrightarrow$ (19).

Section III contains examples applying the formulae and some simplified results obtained through them. No numerical results are given.

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II Description of the General Method

Write the equation for the radial motion in the form

$$\left\{ \frac{d^2}{d\theta^2} + E_0 - g(\theta) \right\} u = B(\theta)\, u^2 . \tag{1}$$

$g(\theta)$ and $B(\theta)$ are periodic in θ with period $2\pi/N$.
$g(\theta)$ is defined by requiring that its average value over a period be zero.

Before presenting our procedure for solving eq. (1), we would like to review the properties of the linear equation. In the linear theory $u(\theta)$ obeys the equation

$$\left\{ \frac{d^2}{d\theta^2} + E_0 - g(\theta) \right\} u = 0 . \tag{2}$$

For our purposes, we will discuss the solutions of the slightly more general equation

$$\left\{ \frac{d^2}{d\theta^2} + E - g(\theta) \right\} u = 0 \tag{3}$$

and consider the solutions of eq. (3) for all values of E.

It is known that eq. (3) has stable solutions only for certain values of E. For these "allowed" values of E, the solutions have the form

$$u_\nu(\theta) = e^{i\nu\theta} h_\nu(\theta) \tag{4}$$

corresponding to the value of E, E_ν. $h(\theta)$ is periodic in θ with the period $2\pi/N$.

In Fig. 1 the allowed E-values, E_ν, is plotted against ν. Gaps appear at $\nu = 0, \frac{1}{2}N, N, \frac{3}{2}N \ldots$ etc. E_0 is the operating value of E, and for the motion to be stable in the linear theory, E_0 must not fall inside the gaps. In the linear theory, the motion is given by

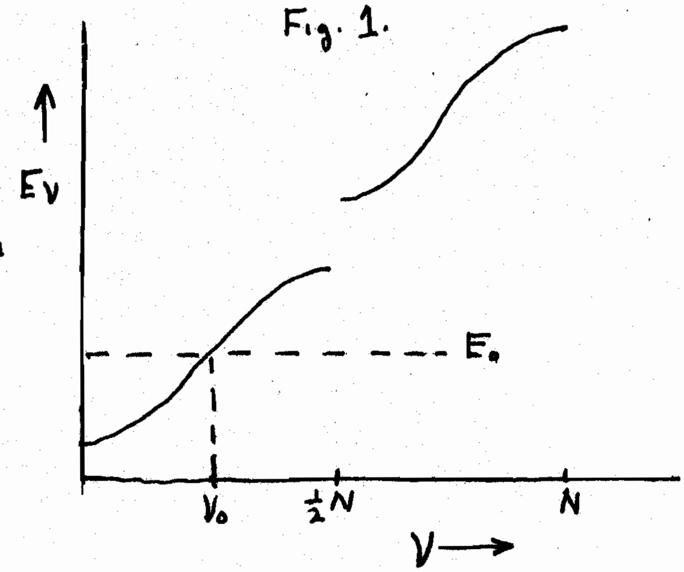

Fig. 1.

$$u = a\, u_{\nu_0}(\theta) + a^* u^*_{\nu_0}(\theta).$$

Now let us return to the non-linear eq. (1). We will start by considering a slightly more general equation

$$\left\{ \frac{d^2}{d\theta^2} + W - g(\theta) \right\} \psi = B(\theta)\, \psi^2, \tag{5}$$

and let us ask for what values of W does eq. (5) have stable solutions.

We will solve eq. (5) by using a perturbation procedure similar to that used in quantum mechanics. Every solution $u_\nu(\theta)$ of the linear equation (3) corresponding to the E-value E_ν will, because of the perturbation $B(\theta)\psi^2$, go over into the solution $\psi_\nu(\theta)$ of eq. (5) corresponding to the W — value, W_ν. We will show how a little later. We can then draw a curve of the allowed W-values as in Fig. 2.

We will show that a new gap appears at $\nu = \frac{1}{3}N$. However, the size of this gap will depend on the amplitude of the motion; and it will be larger for larger amplitudes.

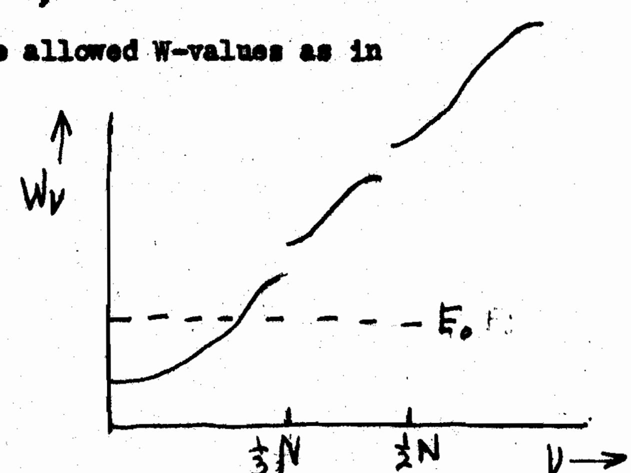

Fig. 2

The question of stability is answered by where E_0 falls in Fig. 2, since our particle motion is given by solving eq. (5) with $W = E_0$. If E_0 does not fall in the gap at $\nu = \frac{1}{3} N$, the motion is stable. However, as the amplitude of the motion increases, the gap becomes larger and E_0 will eventually lie inside the gap and the motion becomes unstable. This will determine the stability limit amplitude.

To solve eq. (5) for Ψ we will expand Ψ in terms of the solutions $\mathcal{U}_\nu(\theta)$ of eq. (3). The solutions $\mathcal{U}_\nu(\theta)$ are a complete orthogonal set. They are a continuous complete set. It is more convenient to make them a discrete set, which we can do by imposing the boundary condition on $\mathcal{U}_\nu(\theta)$ that it be periodic between $\theta = 0$ and $\theta = T$ where T is some arbitrary large angle which we will eventually let become infinite.

The solutions of the linear equation (3) can then be written as

$$\mathcal{U}_\nu(\theta) = \frac{1}{\sqrt{T}} e^{i\nu\theta} h_\nu(\theta) \tag{6}$$

where $\nu = \frac{2\pi}{T} q$, $q = 0, \pm 1, \pm 2, \pm 3$ etc............

The values of ν are now discrete and the $\mathcal{U}_\nu(\theta)$ form an orthonormal set,

$$\int_0^T d\theta \; \mathcal{U}_{\nu'}^*(\theta)\, \mathcal{U}_\nu(\theta) = \delta_{\nu\nu'} \, , \tag{7}$$

if $h_\nu(\theta)$ is so normalized that its average value over the period $2\pi/N$ is one,

$$\frac{N}{2\pi} \int_0^{\frac{2\pi}{N}} d\theta \; |h_\nu|^2 = 1 \tag{8}$$

Now expand Ψ in $\mathcal{U}_\nu(\theta)$,

$$\Psi = \sum_{\nu_i} a_{\nu_i} \mathcal{U}_{\nu_i}(\theta) \, , \tag{9}$$

and by putting this expansion in eq. (5), we get an equation for the a_{ν_i}

$$(W-E_{\nu_i})\,a_{\nu_i} = \sum_{jk} B_{\nu_i,\nu_j\nu_k}\, a_{\nu_j}\, a_{\nu_k}\,, \tag{10}$$

where

$$B_{\nu_i,\nu_j\nu_k} = \int d\theta\; u_{\nu_i}^{*}\, B\, u_{\nu_j}\, u_{\nu_k}\,. \tag{11}$$

Now we will solve eq. (10) for the a_ν using a perturbation procedure which is very similar to the "weak binding approximation" in solid state theory.

Consider the particular real solution of the linear equation $a\,u_{\nu_s} + a^* u_{-\nu_s}$, which corresponds to the E-value E_{ν_s}. Through the perturbation $B(\theta)$, this solution will go over into a solution of eq. (5) with a value of W which differs slightly from E_{ν_s}. Thus to find the solution of the non-linear eq. (5) which corresponds to the solution $a\,u_{\nu_s} + a^* u_{-\nu_s}$ of the linear equation, we will first assume that in the expansion eq. (9) for Ψ, only a_{ν_s}, $a_{-\nu_s}$ are large and all the other a_ν are small. We will see later that this is not always true.

Thus to the first approximation, eq. (10) becomes

$$(W-E_i)\,a_i = B_{i,ss}\,a_s^2 + 2B_{i,s\bar{s}}\,a_s a_{\bar{s}} + B_{i,\bar{s}\bar{s}}\,a_{\bar{s}}^2 \tag{12}$$

where we have written $E_{\nu_s} = E_s$, $u_{\nu_s} = u_s$ and $-\nu_s = \nu_{\bar{s}}$ for the sake of brevity.

Let us notice that the matrix element $B_{i,jk}$ will vanish unless

$$\nu_i = \nu_j + \nu_k + \omega_n\,, \tag{13}$$

where $\omega_n = n\omega$, $\omega = N$, and n is any positive or negative integer. This follows if one expands $\beta(\theta)$, which is periodic, as $\beta(\theta) = \sum_n B_n e^{i\omega_n\theta}$ and puts the expansion into eq. (11).

Thus the matrix elements $B_{i,ss}, B_{i,s\bar{s}}, B_{i,\bar{s}\bar{s}}$, will vanish unless $\nu_i = \pm 2\nu_s + \omega_n$ or $\nu_i = \omega_n$

We will see now that the levels E_s do not get shifted unless $\nu_s \simeq \pm \frac{1}{3}\omega_n$. The two levels corresponding to $\nu_s = \pm\frac{1}{3}\omega_n$ will get split apart and a gap appears in the allowed E values of the non-linear eq. (5) around $\nu_s = \pm\frac{1}{3}\omega_n$ as shown in Fig. 2.

The shift in the level E_s is given by eq. (12) by putting $i = s$.

$$(W - E_s)\, a_s = B_{s,\bar{s}\bar{s}}\, a_{\bar{s}}^2 \qquad (14)$$

since $B_{s,s\bar{s}} = B_{s,ss} = 0$ by eq. (13) unless $\nu_s = \omega_n$ which we assume is not so.

Also for $i = \bar{s}$ we get

$$(W - E_s)\, a_{\bar{s}} = B_{\bar{s},ss}\, a_s^2 \qquad (15)$$

Now $B_{s,\bar{s}\bar{s}} = 0$ unless $\nu_s = \frac{1}{3}\omega_n$ by eq. (13). So if $\nu_s \neq \frac{1}{3}\omega_n$, $W = E_s$; that is, there is no level shift.

For $\nu_s = \pm\frac{1}{3}\omega_n$, $B_{s,\bar{s}\bar{s}} \neq 0$, and we get a splitting of these levels which we can find by multiplying eq. (14) by eq. (15) and find that

$$(W - E_s) = \pm\, |B_{s,\bar{s}\bar{s}}|\,|a_s| \qquad (16)$$

Note the splitting depends on $|a_s|$ the amplitude of the motion.

For $\nu_s \sim \frac{1}{3}\omega_n$ but $\nu_s \neq \frac{1}{3}\omega_n$ we also get a shift in the levels which is not shown by eq. (12) since it turns out that for $\nu_s \sim \frac{1}{3}\omega_n$ but $\nu_s \neq \frac{1}{3}\omega_n$ eq. (12) is not valid as some other a_i besides a_s and $a_{\bar{s}}$ can get large as assumed. We will not demonstrate this point here.

By eq. (16), the gap shown in Fig. 2 has the width $2|B_{s,\bar{s}\bar{s}}||a_s|$, $\nu_s = \frac{1}{3}\omega_n$. If E_0, the operating point, falls outside the gap, the motion is stable. Suppose E_0 lies below the gap. Then by increasing a_s we can bring the gap down until E_0 lies inside the gap and the motion becomes unstable. This will happen when $|a_s|$ becomes so large that

$$E_s - |B_{s,\bar{s}\bar{s}}||a_s| = E_0 \quad ,$$

$$\text{or} \quad |a_s| = \frac{|E_s - E_0|}{|B_{s,\bar{s}\bar{s}}|} \tag{17}$$

when $\nu_s = \frac{1}{3}\omega_n$.

Eq. (17) gives the stability limit amplitude; that is the largest permissible amplitude $|a_s|$ before the motion becomes unstable.

The phase of a_s is also determined by eq. (14),

$$\text{phase } a_s = \frac{1}{3}\left\{ \text{phase } B_{s,\bar{s}\bar{s}} - \text{phase } (W - E_s) \right\} \tag{18}$$

It might be noticed that one can add $\pm 2\pi/3$ to the above phase. This is because there are three stability limit orbits which differ only by a shift along θ by $\pm 2\pi/3$.

Finally, let us notice that the equation of the stability limit orbit is given by

$$\psi = a_s u_s + a_{\bar{s}} u_{\bar{s}} \tag{19}$$

where $\nu_s = \frac{1}{3}\omega_n$.

Equations (17), (18) and (19) give all the general results about the stability limit orbit. In the next section we will give applications of the above general results.

III Specialized Formulae and Examples

For a sine wave spiral sector machine, $B(\theta)$ to a fair approximation is given by[1]

$$B(\theta) = \frac{f}{2w^2} \sin N\theta \tag{20}$$

To determine the stability limit orbit, we use equations (17) - (19).

We calculate $B_{s,\bar{s}\bar{s}}$,

$$B_{s,\bar{s}\bar{s}} = \int_0^T d\theta \, e^{-i\omega_1\theta} h_s^* h_{\bar{s}} h_{\bar{s}} \, B(\theta) \tag{21}$$

In eq. (21) we have specialized to $\nu_s = \frac{1}{3}\omega_1 = \frac{1}{3}N$ resonance.

We get a very simple and fairly accurate result by assuming $h_s \simeq h_{\bar{s}} \simeq 1$ in (21).

So that

$$B_{s,\bar{s}\bar{s}} = B_1 \tag{22}$$

Where B_1 is defined by $B(\theta) = \sum_n B_n e^{i\omega_n\theta}$

For $B(\theta) = (f/2w^2)\sin N\theta$, $B_1 = -i f/4w^2$

The amplitude of the stability limit orbit is given by eq. (17), thus,

$$|a_s| \simeq \frac{4w^2}{f} \left| E_s - E_0 \right| \tag{23}$$

The simplest possible answer is gotten by assuming that for the solutions of the linear equation $E_\nu = \nu^2$. Thus $E_s = \nu_s^2 = \frac{1}{9}N^2$; and the operating point $E_0 = \nu_0^2$.

Then $$|a_s| \simeq \frac{4w^2N^2}{f} \left| \frac{1}{9} - \frac{\nu_0^2}{N^2} \right| . \tag{24}$$

Eq. (24) is the Laslett-Sessler result.

Finally the phase a_s is given by eq. (18) and phase $a_s = 30^\circ$ in this case.

The actual stability limit orbit can now be calculated using eq. (19) and the a_s calculated above. One must know the Flocquet functions $\mu_s(\theta)$ to use eq. (19).

I am indebted to Drs. Sessler, Laslett, Cole and Ohkawa for much discussion of their results for the radial motion case.

References

1. MURA/FTC - 3