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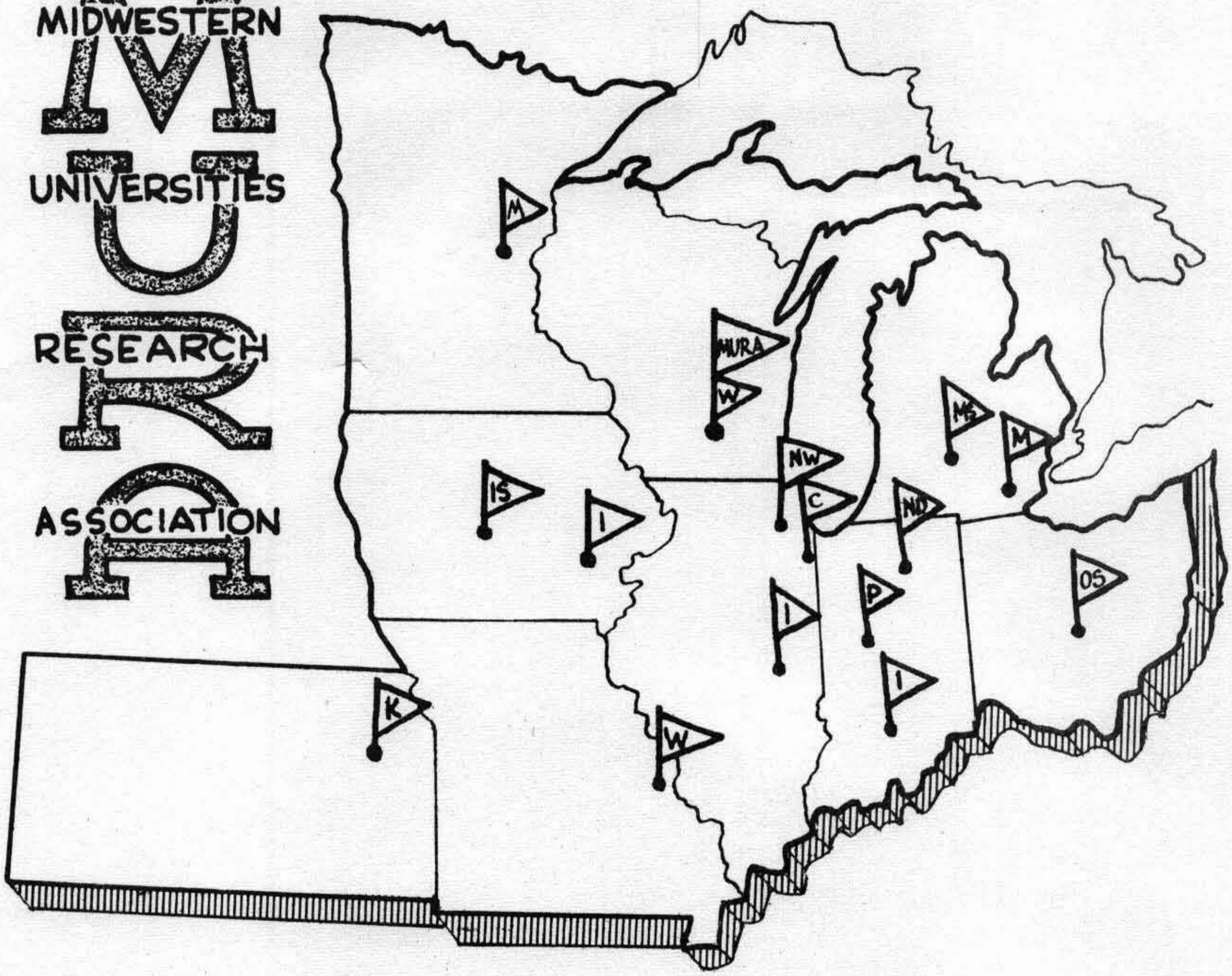

The x-Stability Limit in Large f Structures

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The x-Stability Limit in Large f Structures

Andrew M. Sessler

Preliminary digital computer studies on large f structures seemed to indicate a disagreement with the "handy formula"[1] for the x stability limit. In view of this the present study was undertaken, although before its completion the digital computer studies were shown to be good agreement[2] with the handy formula. The slightly different results presented here are also in agreement with the relatively inaccurate data. It is nevertheless instructive to see that a careful derivation can be easily carried through yielding a more exact formula. [2] The methods used previously [1] could be employed, but would be more tedious than the more powerful technique adopted here. This technique has been developed by Dr. G. Parzen, [3] to whom I am indebted for many hours of careful explanation and advise.

I. Differential Equations

The "handy formula" for the x stability limit is derived from the differential equation:

$$u'' + \left[(K+1) - \frac{1}{2n^2}\left(\frac{f}{w}\right)^2 + \left(\frac{f}{w}\right)\cos N\theta\right]u = \frac{1}{2w}\left(\frac{f}{w}\right)\sin N\theta\, u^2 \qquad (1)$$

This equation has the property that if f is changed, but $\lambda = f/w$ is held constant, then the stability limit increased proportionally to 1/f, or w. This may be readily seen by an appropriate choice of scale, fa

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(1) Mura Notes 6/1/56 L. J. Laslett & A. M. Sessler

(2) L. J. Laslett Madison Lecture 7/17/56.

(3) G. Parzen Madison Lectures, July 1956.

let $u = w\bar{u}$. Then Eq. 1 becomes:

$$\bar{u}'' + \left[(k+1) - \frac{1}{2N^2}\lambda^2 + \lambda \cos N\theta\right]\bar{u} = \lambda \sin N\theta\, \bar{u}^2 \qquad (2)$$

which is an equation independent of w. Whatever stability limit Eq. 2 has for a particular value of λ ; as 1/w is varied, the stability limit changes proportionally to w.

Any difference between the stability limit in structures with a large f, and structures with a small f must arise from terms not included in Eq. 1. For model size parameters Cole has repeatedly emphased (4) the presence of additional large terms which are neglected in Eq. 1. If in the face of this we concentrate our attention on large scale machines, we observe that for large f there are other terms which should be included in Eq. 1. An examination of Cole's work indicates that for parameters $k \approx 100$, $f \approx 1$, $N \approx 40$, $1/w \approx 512$ the largest terms in the radial motion are:

$$u'' - a_1 u = \frac{1}{2} b_1 u^2 \qquad (3)$$

where: $$a_1 = -(k+1) + \frac{f^2}{2w^2N^2} - \frac{f}{w}\cos N\theta - f(k+1)\sin N\theta - \frac{f^2}{2w^2N^2}\cos 2N\theta$$

$$b_1 = \frac{f}{w^2}\sin N\theta + \frac{2kf}{w}\cos N\theta u^2 \qquad (4)$$

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(4) F. T. Cole Public communication

We may to first approximation ignore the cos 2 N θ in a_1, but must take the phase shifted term into account correctly.

Thus if we let;

$$\bar{a} = K - \frac{f^2}{2 w^2 N^2}$$

$$\bar{b} = f \left[1/w^2 \quad K^2 \right]^{\frac{1}{2}}$$

$$\delta = \tan^{-1} K w$$

(5)

then Eq. 3 becomes:

$$u'' + \left[\bar{a} + \bar{b} \cos (N\theta + \delta) \right] u = \frac{1}{2} \frac{f}{w^2} \left[\sin N\theta + 2 K w \cos N\theta \right] u^2$$

(6)

If we now change variables to:

$$\varphi = \theta + \delta / N$$

(7)

and ignore terms in $(kw)^2$ compared to 1, we obtain:

$$u'' + \left[\bar{a} + \bar{b} \cos N\varphi \right] u = \frac{1}{2} \frac{f}{w^2} \left[\sin N\varphi + K w \cos N\varphi \right] u^2$$

(8)

It should be noticed that the non-scaling term in Eq. 8 is of the order of magnitude of 1/5 of the leading non-linear term. It is furthermore worth noting that neglect of the phase shifted term in a_1 would have over-emphasized this term by just a factor or two.

II Approximate Solutions

Parzen has observed that the solution to a non-linear equation such as Eq. 8, can at a stability limit be expressed in terms of the solution to the linearized equation. As he has developed the method one can use accurate numerical solutions to the linearized equations. In order to compare this method with the method of harmonic balance, and also in order to obtain analytic expressions for the stability limit, we will forego accuracy in the solution to the linear problem and use instead an approximate analytic solution. This solution may be most easily obtained by a variational principle, (2) and is approximately, for $\sigma = 2\pi/3$:

$$u = A u_1 + B u_2 \tag{9}$$

where:

$$u_1 = \sin \frac{N\varphi}{3} - \frac{5}{6}\frac{\bar{b}}{N^2} \sin \frac{2N\varphi}{3} + \frac{1}{6}\frac{\bar{b}}{N^2} \sin \frac{4N\varphi}{3}$$

$$u_2 = \cos \frac{N\varphi}{3} + \frac{5}{6}\frac{\bar{b}}{N^2} \cos \frac{2N\varphi}{3} + \frac{1}{6}\frac{\bar{b}}{N^2} \cos \frac{4N\varphi}{3}$$

A, B arbitrary (10)

Following Parzen, we assume the solution of Eq. 8 to be of the form of Eq. 9. Inserting this into Eq. 8, one obtains:

$$A u_1 \left[\bar{a} - a\frac{2\pi}{3}\right] + B u_2 \left[\bar{a} - a\frac{2\pi}{3}\right] =$$
$$\frac{1}{2}\frac{f}{\omega^2}\left[\sin N\varphi + K\omega \cos N\varphi\right]\left[A^2 u_1^2 + B^2 u_2^2 + 2AB u_1 u_2\right] \tag{11}$$

where $\bar{a} \simeq V_x{}^2 - \dfrac{f^2}{2\omega^2 N^2}$

$$a\,\frac{2\pi}{3} \simeq \left(\frac{N}{3}\right)^2 - \frac{f^2}{2\omega^2 N^2}$$

so that: $\bar{a} - a\,\dfrac{2\pi}{3} = V_x{}^2 - \dfrac{N^2}{9}$ (12)

Observing that:

$$\int_0^{3\left(\frac{2\pi}{N}\right)} u_1^2(\varphi)\,d\varphi = \int_0^{3\left(\frac{2\pi}{N}\right)} u_2^2(\varphi)\,d\varphi = 3\pi\left[1 + \frac{3}{2}\,\frac{\bar{b}^2}{N^4}\right]$$

$$\int_0^{3\left(\frac{2\pi}{N}\right)} u_1(\varphi)\,u_2(\varphi)\,d\varphi = 0$$

(13)

we may from Eq. 11 obtain two algebraic equations for A and B:

$$A\,3\pi\left[1 + \frac{3}{2}\,\frac{\bar{b}^2}{N^4}\right]\left(V_x{}^2 - \frac{N^2}{9}\right) = \frac{f}{2\omega^2}\left[I_3 A^2 + I_1 B^2 + 2 I_2 A B\right]$$

$$B\,3\pi\left[1 + \frac{3}{2}\,\frac{\bar{b}^2}{N^4}\right]\left(V_x{}^2 - \frac{N^2}{9}\right) = \frac{f}{2\omega^2}\left[I_2 A^2 + I_0 B^2 + 2 I_1 A B\right]$$

(14)

where:

$$I_n = \int_0^{3\left(\frac{2\pi}{N}\right)} \left[\sin N\varphi + K\omega \cos N\varphi\right] \mu_1^{\,n}(\varphi)\, \mu_2^{\,3-n}(\varphi)\, d\varphi \tag{15}$$

Although it is simple enough to exactly evaluate I_n, we shall only use the first term in μ_1 and μ_2. This yields:

$$I_0 = +\, 3\pi/4\; K\omega$$

$$I_1 = 3\pi/4$$

$$I_2 = -\, 3\pi/4\; K\omega$$

$$I_3 = -\, 3\pi/4 \tag{16}$$

The equations for A & B (Eq. 14) becomes:

$$A\left(V_x^2 - \frac{N^2}{9}\right) = \frac{f}{8\omega^2}\left[(B^2 - A^2) - 2K\omega AB\right]$$

$$B\left(V_x^2 - \frac{N^2}{9}\right) = \frac{f}{8\omega^2}\left[(B^2 - A^2)K\omega + 2AB\right] \tag{17}$$

These equations are exactly the equations one would obtain from Eq. 8 by harmonic balance assuming $u = A \sin N\theta/3 + B \cos N\theta/3$. The advantage of Parzen's method is that the coefficients A & B may be used with the functions u_1 & u_2 of Eq. 10, to a first approximation. To obtain a solution of this accuracy by means of harmonic balance one would have to use a trial function with six terms. Thus in this context, the advantage of Parzen's method is it relates these coefficients initially (by the linear solution), reducing a six parameter problem to a two parameter problem. Of course, if one used numerical functions for u_1 and u_2 Parzen's technique would yield a very accurate result, although it will always yield the same result as obtained by harmonic balance, if enough terms are kept in the latter method.

Equations 17 can be readily solved in powers of $kw \approx .2$. To first approximation one obtains:

$$A = \frac{4w^2}{f}\left(V_x^2 - \frac{N^2}{9}\right)\left[1 - \frac{4}{\sqrt{3}} Kw + \dots\right]$$

$$B = \frac{4\sqrt{3}\, w^2}{f}\left(V_x^2 - \frac{N^2}{9}\right)\left[1 + \frac{4}{3\sqrt{3}} Kw + \dots\right] \qquad (18)$$

where terms of order $(kw)^2$ are ignored compared to one. These coefficients may be validly used with Eqs. 9 & 10, even though I_n was evaluated using only the first terms in u_1 & u_2. If we keep only the first terms in u_1 & u_2 we obtain the result (identical with what one obtains by harmonic balance with two terms)

$$u = \frac{4w^2}{f}\left(V_x^2 - \frac{N^2}{9}\right)\left(1 - \frac{4Kw}{\sqrt{3}}\right)\left[\sin\frac{N}{3} + \sqrt{3}\,\mu \cos\frac{N\varphi}{3}\right]$$

$$u' = \frac{4w^2}{f}\left(V_x^2 - \frac{N^2}{9}\right)\left(1 - \frac{4Kw}{\sqrt{3}}\right)\frac{N}{3}\left[\cos\frac{N\varphi}{3} - \sqrt{3}\,\mu \sin\frac{N\varphi}{3}\right]$$

where: $\mu = 1 + \dfrac{16Kw}{3\sqrt{3}}$

(19)

Neglecting the small difference between φ and θ, one obtains for the unstable periodic points at $N\theta = 0$:

$$u(0) = A\sqrt{3}\left(1 + \frac{11}{3}\delta\right) \qquad u'(0) = \frac{AN}{3}\left(1 - \frac{\delta}{3}\right)$$

$$u(2\pi) = -2\sqrt{3}\,A\delta \qquad u'(2\pi) = -\frac{2}{3}AN$$

$$u(4\pi) = -\sqrt{3}\,A\left(1 + \frac{5}{3}\delta\right) \qquad u'(4\pi) = \frac{AN}{3}\left(1 + \frac{\delta}{3}\right)$$

where:

$$A = \frac{4\omega^2}{f}\left(\nu_x^2 - \frac{N^2}{9}\right)$$

$$\delta = \frac{4K\omega}{3\sqrt{3}} \qquad (20)$$

The effect of the δ term is indicated in Fig. 1, where the periodic points at $N\theta$ are plotted, as well as arrows indicating the direction & relative amount by which they move as δ increases.

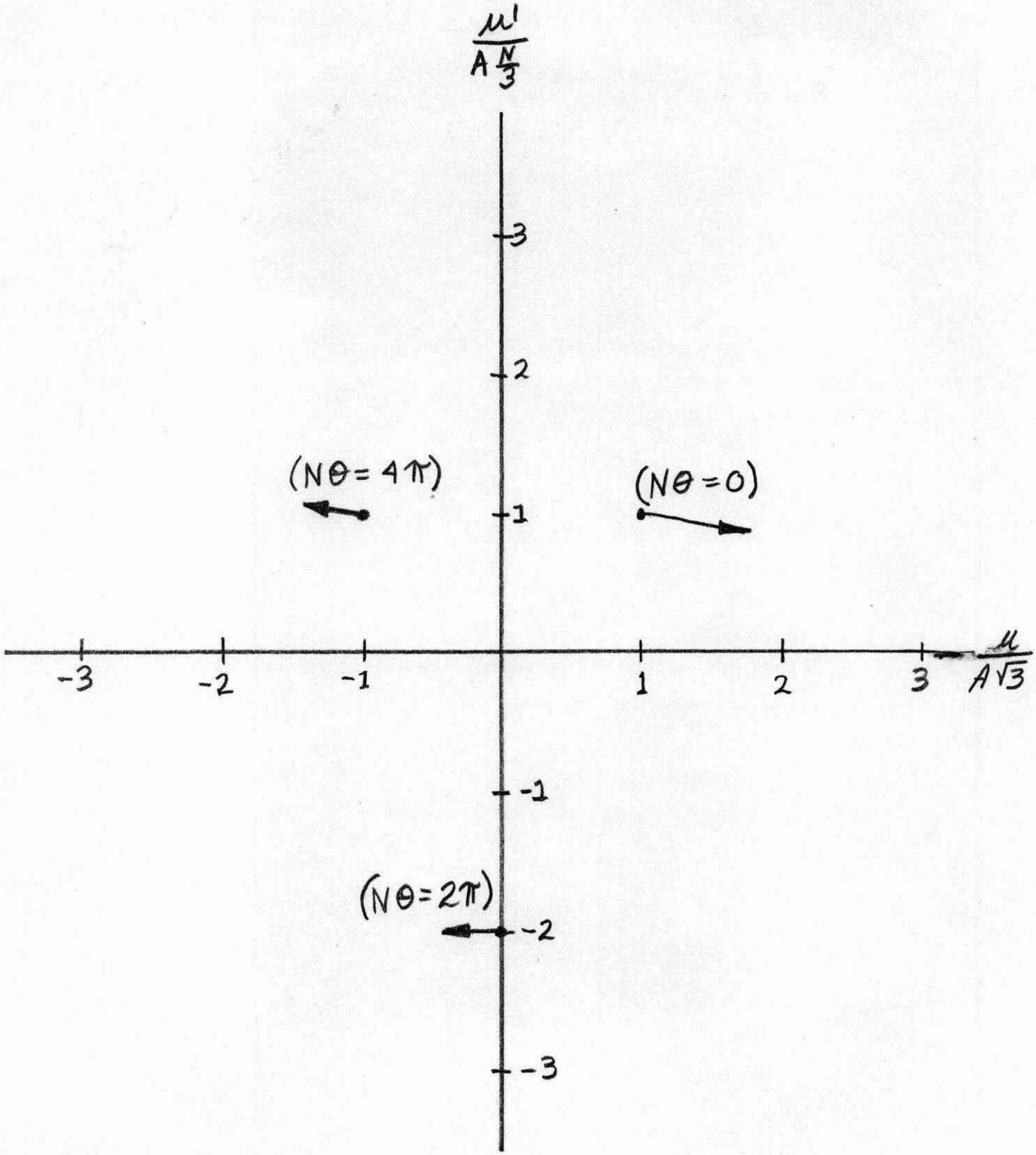

Figure 1 Periodic Points at $N\theta = 0$.