Mura Notes/AMS
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On the Non-Liouvillian Character of Foils

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Comments

I was unable to see Lichtenberg on my visit of July 10. Symon says Lichtenberg's results do not agree in detail with those presented here, but it is not clear to me that we are calculating the same quantity. Simply to form a basis for discussion during future visits to Madison I have written this material up. With the technical group in two locations, preliminary drafts with a high probability of included errors, seem unfortunately to be essential.

Lichtenberg has constructed an ingenious proof (1) that thin foils are almost Liovillian in character. He has shown by general arguments that the change in total phase space on passage through a foil is negligibly small. This author felt the need for a specific calculation in order to confirm the general result; as well as to obtain explicit formulas for the change in betatron, synchrotron, and total phase space on traversal of a foil. The results of these calculations have been outlined here.

I. Derivation

The starting point is a mathematical characterization of a foil and its effect on the betatron oscillation coordinates x and p, and on the energy of a particle. The transformation is given in a previous memorandum (2), but will

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(1) Mura lecture of July 2, 1956, soon to be published.
(2) "A Proposed Digital Computer Program to Study Foils, etc." -- Memorandum to J. N. Snyder of May 16, 1956.

be repeated here for convenience.

Let: R = radius of particle

$R_e(E)$ = equilibrium orbit radius

E = energy of particle

$R = R_e(E)\ (1 + x)$

$U(R)$ = energy loss incured by particle at radius R on traversing foil. This energy loss is assumed independent of the particle momentum. Extension to more realistic foils should not be difficult.

$$p = \frac{dx}{d\theta} \left[(1 + x)^2 + \left(\frac{dx}{d\theta}\right)^2 \right]^{-\frac{1}{2}} \qquad (1)$$

Note that x and p are canonically conjugate in the absence of foils. The traversal of a foil then causes initial and final values to be related by:

$$R_f = R_i$$

$$\left.\frac{dR}{d\theta}\right|_f = \left.\frac{dR}{d\theta}\right|_i$$

$$\theta_f = \theta_i$$

$$E_f = E_i - U \qquad (2)$$

Which becomes in terms of betatron and synchrotron coordinates:

$$x = \frac{Re(E_i)\ (1 + x_i) - Re(E_i - U)}{Re(E_i - U)}$$

$$p = p_i$$

$$E = E_1 - U$$

$$\theta = \theta_1 \qquad (3)$$

The damping or undamping of phase space is readily evaluated by computing the Jacobians:

$$\frac{\partial\,(x\ p)}{\partial\,(x_1 p_1)}\ ,\quad \frac{\partial\,(E,\ \theta)}{\partial\,(E_1,\theta_1)}\ ,\quad \frac{\partial\,(x,p,E,\theta)}{\partial\,(x_1,p_1,E_1,\theta)}\ .$$

$$\frac{\partial\,(x,p)}{\partial\,(x_1,p_1)} = \left[\frac{Re(E_1)}{Re(E_1-U)}\right]\left[1 + (1+x_1)\frac{\partial U}{\partial R}\ ,\ \frac{Re(E_1)}{Re(E_1-U)}\frac{\partial Re(E_1-U)}{\partial\ (E_1-U)}\right]$$

$$\frac{\partial\,(E,\theta)}{\partial\,(E_1,\theta_1)} = 1 - (1+x_1)\frac{\partial U}{\partial R}\quad\frac{\partial R_e(E_1)}{\partial E_1}$$

$$\frac{\partial\,(x,p,E,\theta)}{\partial\,(x_1,p_1,E_1,\theta_1)} = \left[\frac{Re(E_1)}{Re(E_1-U)}\right] \qquad (4)$$

It can be seen that the total phase space is only slightly changed, and this change is independent of the taper of the foils -- a result in agreement with Lichtenberg's general theorem. The relative damping or undamping of betatron or synchrotron phase space is a sensitive function of the taper of the foil, reversing character with the sign of the taper.

II Numerical Evaluation for Typical Cases

For purposes of orientation as to the magnitude of the damping or undamping we have evaluated in Table II the Jacobians for two cases. Case I is characteristic of a full size FFAG accelerator, and Case II has parameters characteristic of a storage ring for the Princeton machine.

We have taken:

$$Re(E) = R_o \left[\frac{E^2 - E_o^2}{1 - E_o^2}\right]^{\frac{1}{2k+2}}$$

$$\frac{d\,Re(E)}{dE} = \frac{E}{(k+1)} \frac{Re(E)}{\left[E^2 - E_o^2\right]}$$

$$U(R) = U_o \left[1 + \lambda (R - R_{oo})\right]$$

Parameters are listed in Table I:

	K	R_{oo}	E_1	$Re(E_1)$	x_1	λ	U_o
Case I	83	76 M.	16 Bev.	76 M.	10^{-3}	10 M^{-1}	10 Mev.
Case II	1	9 M.	4 Bev.	9 M.	10^{-2}	10 M^{-1}	10 Mev.

Table I . Parameters

	$\frac{\partial(x,p)}{\partial(x_1,p_1)}$	$\frac{\partial(E,\theta)}{\partial(E_1,\theta_1)}$	$\frac{\partial(x,p,E,\theta)}{\partial(x_1,p_1,E_1,\theta_1)}$	Fractional energy loss per traversal
Case I	1.006268	.9943360	1.0000074	6.6×10^{-4}
Case II	1.121949	.879763	1.001325	2.5×10^{-3}

Table II Jacobians

It should be noted that the deviation from unity of the first two Jacobians is proportional to λ and U_o. The foil characteristics we have taken of a change in width of a factor of ten in one meter, and an energy loss of the order of 10 Mev, seem typical.

I am endebted to members of the Mura Technical Group for helpful discussions. The calculations were performed by Mr. G. T. Condo of the C.H.W.G.

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