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A SHORT SURVEY OF DIGITAL COMPUTER RESULTS FOR RADIAL MOTION IN F.F.A.G. MARK V SPIRAL RIDGE ACCELERATORS

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ABSTRACT

General remarks are made pertaining to the graphical presentation and interpretation of numerical computations on non-linear one dimensional motion. A survey of numerical results is then given for the radial motion on the median plane in a set of large scale F.F.A.G. Mark V Spiral Ridge accelerators.

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INTRODUCTION

During the last 3/4 year an investigation of the radial motion of charged particles in the median plane of large scale F.F.A.G. Mark V Spiral Ridge accelerators has been performed using the fast digital computer (Illiac) at the University of Illinois. The Illiac - program used is the so-called "Ridge Runner"[1] which solves the exact equations of motion on the median plane in a magnetostatic field characterized by the median plane vertical component of magnetic induction

$$B_z = B_0\left(\frac{r}{r_0}\right)^k \left[1 + f \sin\left(\frac{1}{w} \ln \frac{r}{r_0} - N\vartheta\right)\right] \tag{1}$$

r, ϑ = radial and angular coordinates in median plane

r_0 = radius of reference circle

k = field index.

f = flutter factor

$\frac{1}{w}$ = spiral index

N = number of magnetic field periods per revolution.

B_0 = proportionality constant.

The computations are exact in the sense that they contain no approximations other than those inherent in the Illiac method of solution. The investigation is by no means claimed to be

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1 See: J. N. Snyder: The MURA programs for Illiac, p. 48.

very extensive; nor can it be said that all interesting features of the motion has been systematically looked into.

In the "Ridge Runner" program the quantities computed are the canonical conjugate variables (x, p) defined by

$$x = \frac{r}{r_0} - 1$$

(2) $$p = \frac{\frac{dx}{d\vartheta}}{\sqrt{(1+x)^2 + \left(\frac{dx}{d\vartheta}\right)^2}} \approx \frac{dx}{d\vartheta}$$

The numerical results may thus be presented graphically on a (x, p) - phase plane.

General Remarks

The manner of graphical presentation now generally adopted for non-linear motion of the kind here studied, is to plot the positions of the phase point (x, p) at angular intervals of $\frac{2\pi}{N}$, i.e. at homologous points in each period of the magnetic field structure. As discovered by J.L. Powell[2], each set of such phase points seems to define uniquely a continuous closed invariant curve in the phase plane; provided that the motion under investigation is a stable one. In fact, for the interpretation of numerically obtained results for non-linear motion, this property of the homologous phase points is the only known criterium for stability. If the motion is unstable (in this sense) the phase points may also be ascribed

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2 MAC - JLP -2, p.7.

to a curve, but curves of this category will not be closed, and the variables (x, p) will quickly assume values too large to be handled by the Illiac-program used. Also, since the set of homologous phase points in an unstable case do not densely cover an unstable curve, these curves will not be uniquely defined. One may in fact draw an infinite number of invariant curves through an unstable set of homologous phase points. The term "invariant" occuring above refers to the transformation taking the phase point through one angular period $\frac{2\pi}{N}$. On figs. 2, 3, and 4 are shown examples of stable and unstable invariant curves. The homologous phase points through which the curves are drawn are not shown. The unstable curves shown are the ones that smoothly cover the initial set of phase points. All graphs are plotted at the angles $\theta = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots\ldots$.

When the motion under consideration happens to be a periodic one with a period of some multiple $q \cdot \frac{2\pi}{N}$ (q = integer) of the angular field period, the stable invariant curve degenerates into a set of q points. The equilibrium orbit in the accelerator has this property with q = 1, and is thus represented by a single point (the "fixed point" marked F_1 on figs. 2, 3, and 4) in the phase plane. An invariant set of q points is in the usual MURA terminology termed a set of "q^{th} order fixed points"[3].

The motion of a point with initial conditions

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[3] The expression: "Periodic points of order q" introduced by J.M. Jauch (MURA/YMY/1, p.7) seems more precise.

arbitrarily close to a fixed point may be stable or unstable. If it is possible to select a neighbourhood of the fixed point such that all initial points in this neighbourhood lead to motions arbitrarily close to that of the fixed point itself, the fixed point is called "stable". If this condition is not fulfilled we talk about an "unstable fixed point"[4]. Stable fixed points occur occasionally on the graphs as centres for an invariant set of small closed curves ("pearls"). On Fig. 4 a set of 19 such "pearls" are shown. Each of these enclose a 19th order stable fixed point.

On the basis of the obtained phase plots described above it is possible to extend the definition of the constant σ, known from the linear Floquet theory, also to cover the case of non-linear stable motion. σ may in both cases be defined as the phase advance of the oscillation per period of the magnetic field structure. To obtain σ for a particular stable motion one has to count the number n of field periods (homologous phase points) the motion must go through to complete one revolution on the closed phase plane curve. Then

$$\sigma = \frac{2\pi}{n} \tag{3}$$

If n is non-integer one will have to estimate the fraction it contains; or better: count phase points not only through one revolution of the closed curve, but through a large number of

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4 "Labile fixed point" might perhaps be a better term, since these points may perfectly well be stable in the sense that any arbitrarily close point may never proceed to infinity.

revolutions, say S, and then (only roughly) estimate the fraction. In this case

$$\sigma = \frac{2\pi S}{n} \tag{4}$$

Clearly σ is obtained with any desired degree of accuracy by this method. In the case of a linear motion σ will be the same for all curves and coincide with the σ of Floquet's theory[5]. If the motion is non-linear, σ will vary from one closed curve to the next, and hence be a function of the amplitude of oscillation.

It follows from the fact that the motion is Hamiltonian, that a transformation of the invariant phase plane curves through any interval in the angular coordinate ϑ will be a topological one. Hence, σ as defined above is a true constant of the motion. It will be the same for any set of homologous phase points belonging to the same particle. Also, and for the same reason as above, the area enclosed by a closed phase plane curve will be constant against any transformation in ϑ (Liouville's theorem), while of course the curves themselves will change both in shape and position.

It appears from the foregoing that any motion with a rational value of the constant σ will be periodic; its period being some multiple of the field period $\frac{2\pi}{N}$. The corresponding

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5 For calculation of linear σ see: MURA/NVN/3.

phase plot consists as we have seen of a set of fixed points of some order. This fact supplies the necessary information to locate the positions of expected fixed points on the graphs. However, in most cases the accompanying "pearls" simply fail to show up; the indication being that they are either non-existing or too small to be detected with the x, p - scales applied.

The graphs indicate the existence of an important relationship between the amplitude dependent constant σ and the limit of stability. The latter is then defined as the closed invariant curve with the maximum enclosed area on the graph; the underlying idea being that the set of phase points homologous to any initial point inside this area will also lie inside the area. In practically all cases so far investigated at the University of Illinois the limit of stability runs through 3 or 4 unstable fixed points corresponding to $\sigma_{limit} = \frac{2\pi}{3}$ or $\frac{2\pi}{4}$. The points F_3 on fig. 2 are 3rd order points of this category. However, R. Christian[6] at Los Alamos Scientific Laboratory has observed cases with 5 unstable fixed points at the stability limit, and L. Jackson Laslett and F. T. Cole[7] has discovered a case showing indications of 7 fixed points at the stability limit.

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6 Private communication

7 See: L. Jackson Laslett: MURA NOTES, 7 Dec., 1955.

If one follows the variation of σ versus amplitude of oscillation in a certain machine having a small amplitude σ (the σ of the corresponding linearized problem) either a little above or a little below one of the values $\frac{2\pi}{3}$ or $\frac{2\pi}{4}$, the normal effect of increasing amplitudes has been to push σ towards the respective value $\frac{2\pi}{3}$ or $\frac{2\pi}{4}$. This value is reached at the stability limit. Increasing the amplitude further will of course give unstable motion. The effect is shown on fig. 5.

As will be seen from the foregoing, the concept of invariant phase plane curves is purely based on experimental information obtained by digital computations. As opposed to this it should be mentioned that from an analytical point of view[8] these invariant curves do not seem to exist in a strict mathematical sense - at least not as stationary phenomena. In view of this the obtained phase plane graphs should be regarded only as first approximations too coarse to show the finer details in the actual motion. Furthermore the graphs are only valid over a certain (so far unknown) length of time, after which one must expect the particle to move gradually away from the closed "invariant" phase curve and eventually reach the unstable region of the phase plane. As a consequence the

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8 Lectures by J. Moser (Princeton University) and R. d'Vogelaere (Notre Dame University) at the University of Illinois, Fall, 1955.

definition of stability based on the invariant curves on the graphs should perhaps more precisely be termed "quasi-stability" or "short time stability".

Numerical Results

Twelve different sets (Table 1) of the machine parameters k, $\frac{1}{w}$, f and N in formular (1) were chosen for the survey. The corresponding twelve operating points on the (σ_ζ, σ_y) necktie diagram[9] are shown on fig. 1. Of course

Table 1

Point #	k	$\frac{1}{w}$	f	N
1	160	2432	.25	40
2	123	2368	.25	40
3	96	2304	.25	40
4	64	2240	.25	40
5	160	2302	.25	40
6	128	2240	.25	40
7	96	2176	.25	40
8	64	2112	.25	40
9	160	2176	.25	40
10	128	2112	.25	40
11	96	2048	.25	40
12	64	1984	.25	40

9 By L. Jackson Laslett, July 30, 1955.

only the value of σ_x, pertaining to the radial motion, is of interest here.

Some of the most conspicuous features of the motion are compared for the twelve sets of parameters in table 2. Here

(x_o, p_o) are the phase plane coordinates of the 1st order stable fixed point F_1 representing the equilibrium orbit at the angles $\theta = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \cdots$.

σ_o is the σ -constant for a particle performing a small-amplitude oscillation about the equilibrium orbit.

x_m, X_m are the minimum and maximum values of the displacement x at the limit of stability measured at the angles $\theta = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \cdots$.

σ_{lim} is the σ -constant at the limit of stability.

A_{lim} is the area enclosed by the limit of stability (information on this quantity is only included for about half of the necktie points).

Table 2

Point #	$\frac{\sigma_o}{\pi}$	x_o p_o	x_m x_m	$\frac{\sigma_{lim}}{\pi}$	A_{lim} $\cdot 10^8$	Remarks
1	.79	-.000032 -.007242	-.0004 +.00031	2/3	$\sim$ 87	Case of 19 pearls at $\frac{\sigma}{\pi} = \frac{14}{19}$. Points scatter at stability limit. Graph on fig.4.
2	.67	-.0000298 -.007107	-.000043 -.0000175	2/3	0.14	
3	.56	-.0000287 -.006970	-.00075 +.00071			
4	.45	-.0000276 -.0068378	-.0007 +.00068	2/4		
5	.77	-.0000308 -.0072452	-.0004066 +.0003220	2/3	89	Location of 3rd order unstable fixed points at stability limit (see footnote 7): x = -.0004066 -.0000121 +.0003220 p = -.0056114 -.0058299 -.0073597
6	.66	-.0000295 -.007104	-.00007 +.00001	2/3		
7	.53	-.00003 -.00695	-.001 +.001	2/4		Points scatter at stability limit

Table 2 cont'd.

Point #	$\frac{\sigma_0}{\pi}$	X_0 / P_0	X_m / X_M	$\frac{\sigma_{lim}}{\pi}$	$A_{lim} \cdot 10^8$	Remarks
8	.44	-.000026 -.00684	-.0009 +.0009	2/4		
9	.75	-.0000304 -.007242	-.00040 +.00031	2/3	87	Graph on fig. 2
10	.645	-.0000292 -.007099	-.00014 +.00009	2/3	0.3	Graph on fig. 3
11	.548	-.000033 -.006960	-.0012 +.00098	2/4	810	Points scatter at the stability limit.
12	.44	-.000027 -.00682	-.00123 +.00123	2/4		

Notice that in some cases (Points #1, 7, 11) the homologous phase points scatter at the limit of stability. In these cases the points seem to define a closed band rather than a curve; but there is no clear evidence of a real instability (in the sense that the phase point moves rapid' outwards) during the relatively short lengths of time (usually 80 sectors) used in the survey. Also in these cases it has not been possible to draw smooth invariant curves in the unstable region of the phase plane.

On fig. 5 one may study the amplitude dependence of the constant σ . Shown are curves of σ versus the area of the corresponding closed invariant curve for values of σ in the neighbourhood of $\frac{2\pi}{3}$. One notices the effect mentioned earlier, that the σ in these cases tend towards $\frac{2\pi}{3}$, and also that the value of $A_{lim.}$ is smaller the smaller the difference $\left|\sigma_0 - \frac{2\pi}{3}\right|$. An analoguous effect is noticed in the neighbourhood of $\sigma_0 = \frac{2\pi}{4}$, but here no calculations of area have been performed. However, from the table one may conclude that the amplitudes of oscillation at the stability limit are 3 or 4 times larger in the neighbourhood of $\sigma_0 = \frac{2\pi}{4}$ than in the neighbourhood of $\sigma_0 = \frac{2\pi}{3}$.

NVN:tr

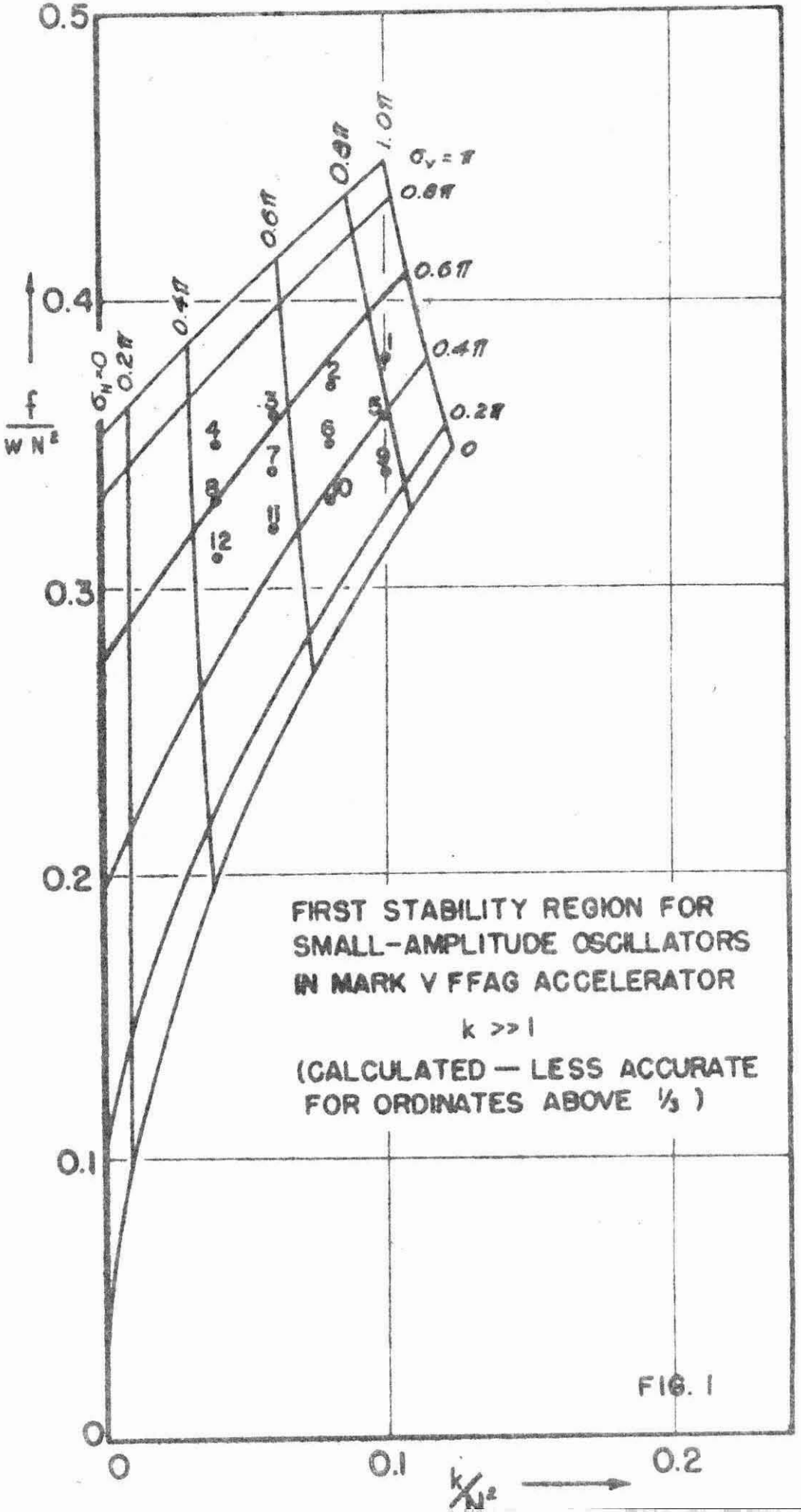
0.5
0.4
0.3
0.2
0.1
0
$\frac{f}{wN^2}$
0
0.1
0.2
$\frac{k}{N^2}$
$\sigma_H = 0$
0.2π
0.4π
0.6π
0.8π
1.0π
$\sigma_V = \pi$
0.8π
0.6π
0.4π
0.2π
0
1
2
3
4
5
6
7
8
9
10
11
12
FIRST STABILITY REGION FOR
SMALL-AMPLITUDE OSCILLATORS
IN MARK V FFAG ACCELERATOR
k >> 1
(CALCULATED — LESS ACCURATE
FOR ORDINATES ABOVE ⅓)

FIG. 1

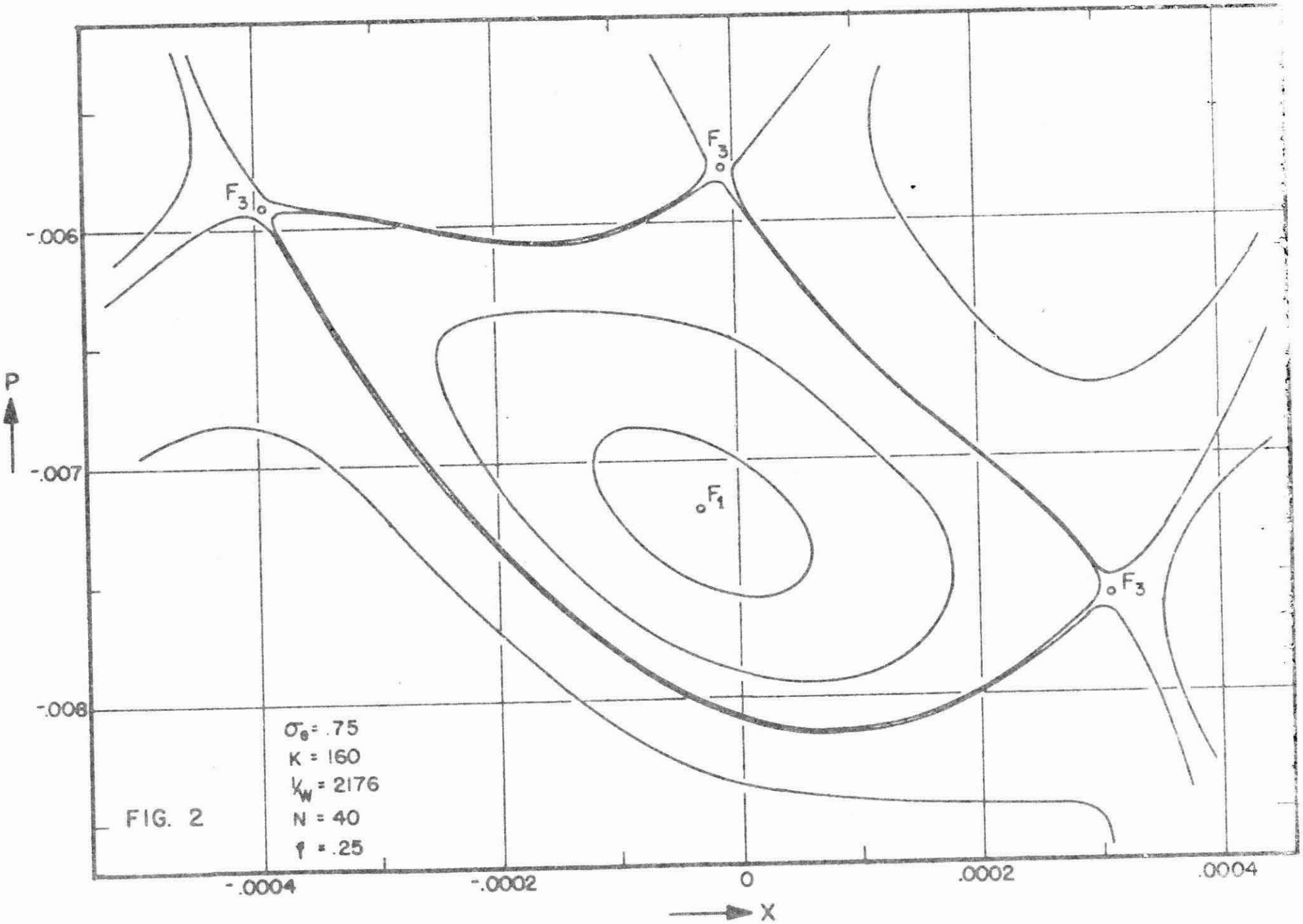
F3
F3
F3
F1
-.006
-.007
-.008
P
-.0004
-.0002
0
.0002
.0004
X
σ0 = .75
K = 160
1/W = 2176
N = 40
f = .25

FIG. 2

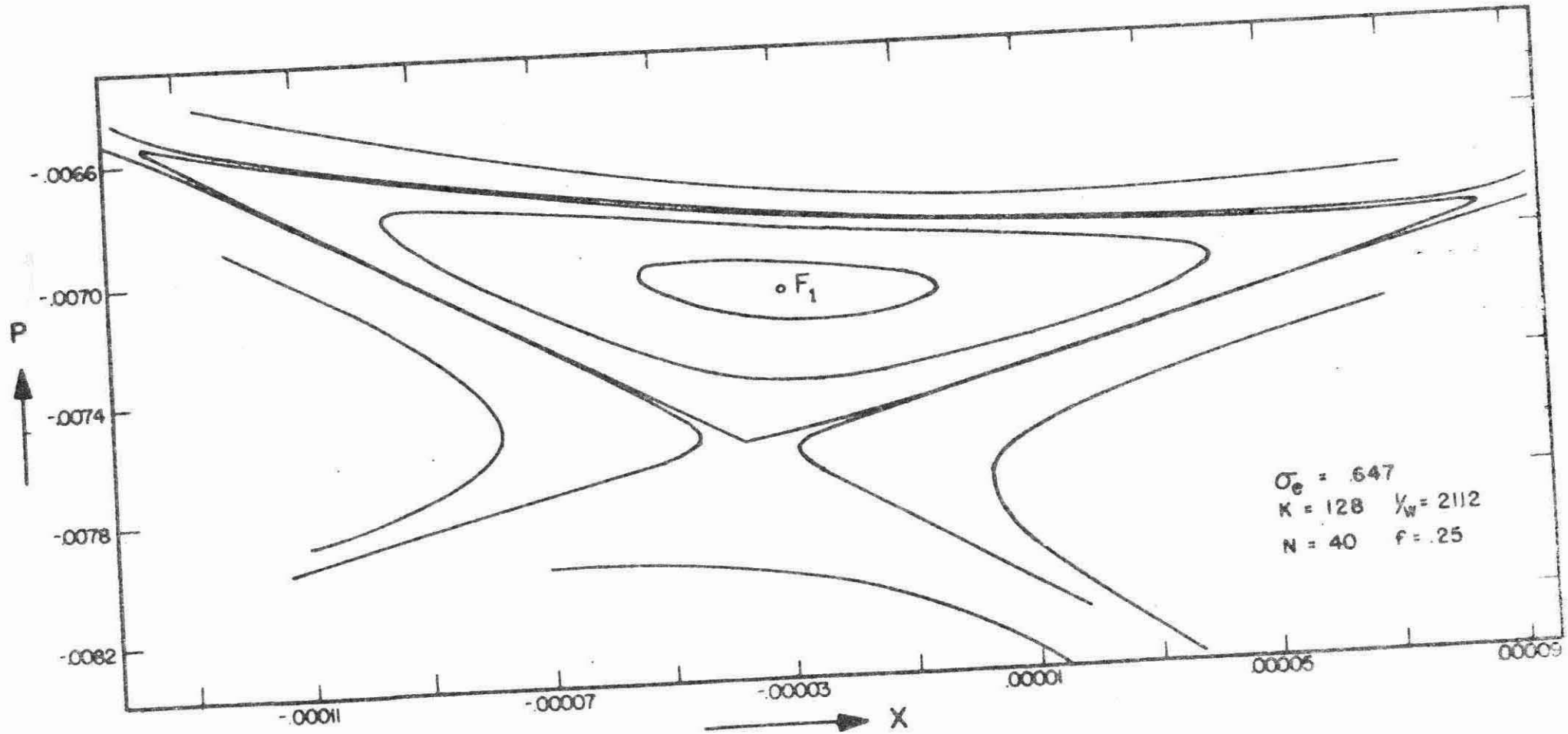
-.0066
-.0070
-.0074
-.0078
-.0082
P
-.00011
-.00007
-.00003
.00001
.00005
.00009
X
F1
σe = .647
K = 128
1/w = 2112
N = 40
f = .25

FIG. 3

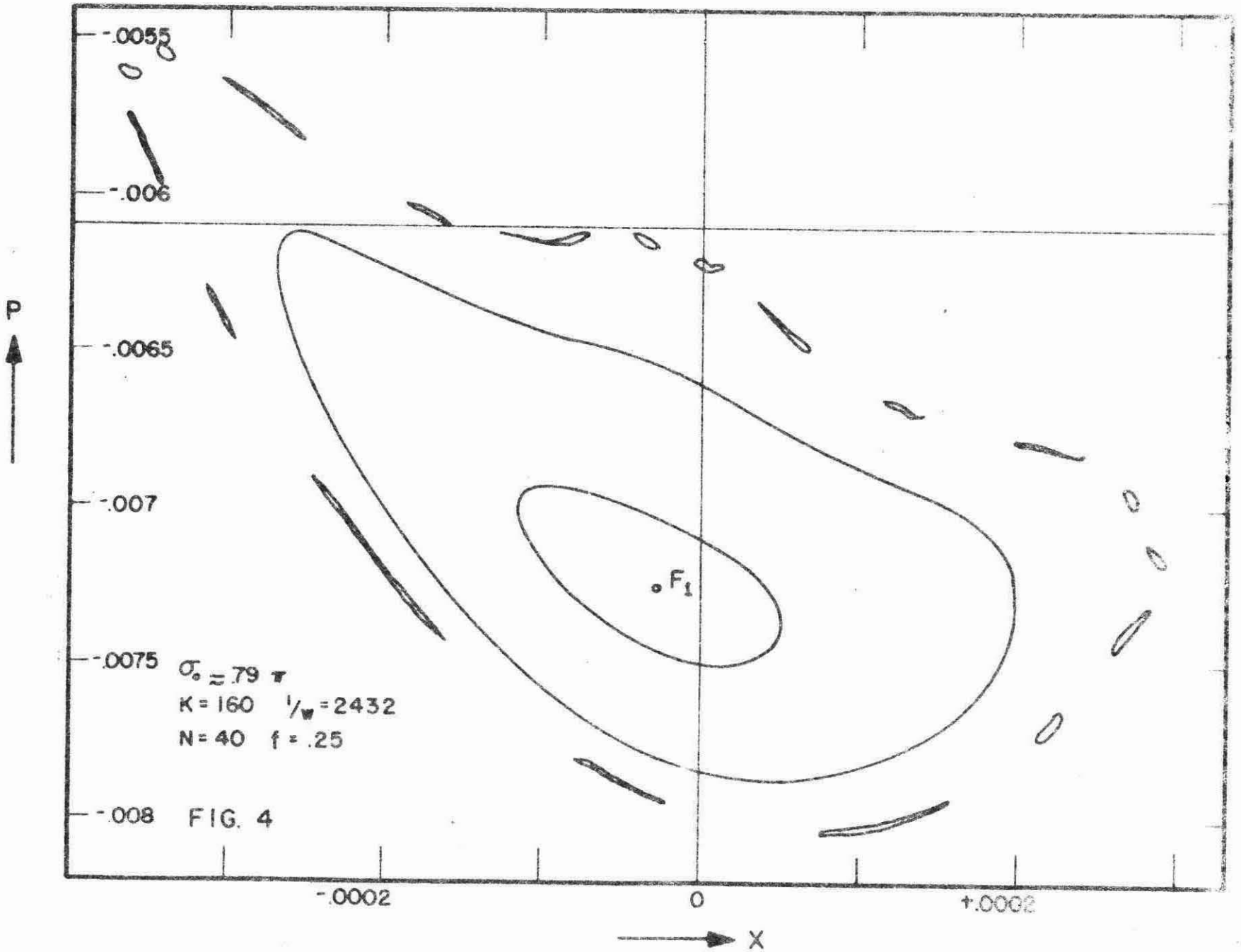
-.0055
-.006
-.0065
-.007
-.0075
-.008
P
F_1
$\sigma_0 \simeq .79\,\pi$
K = 160 $1/w$ = 2432
N = 40 f = .25
-.0002
0
+.0002
X

FIG. 4

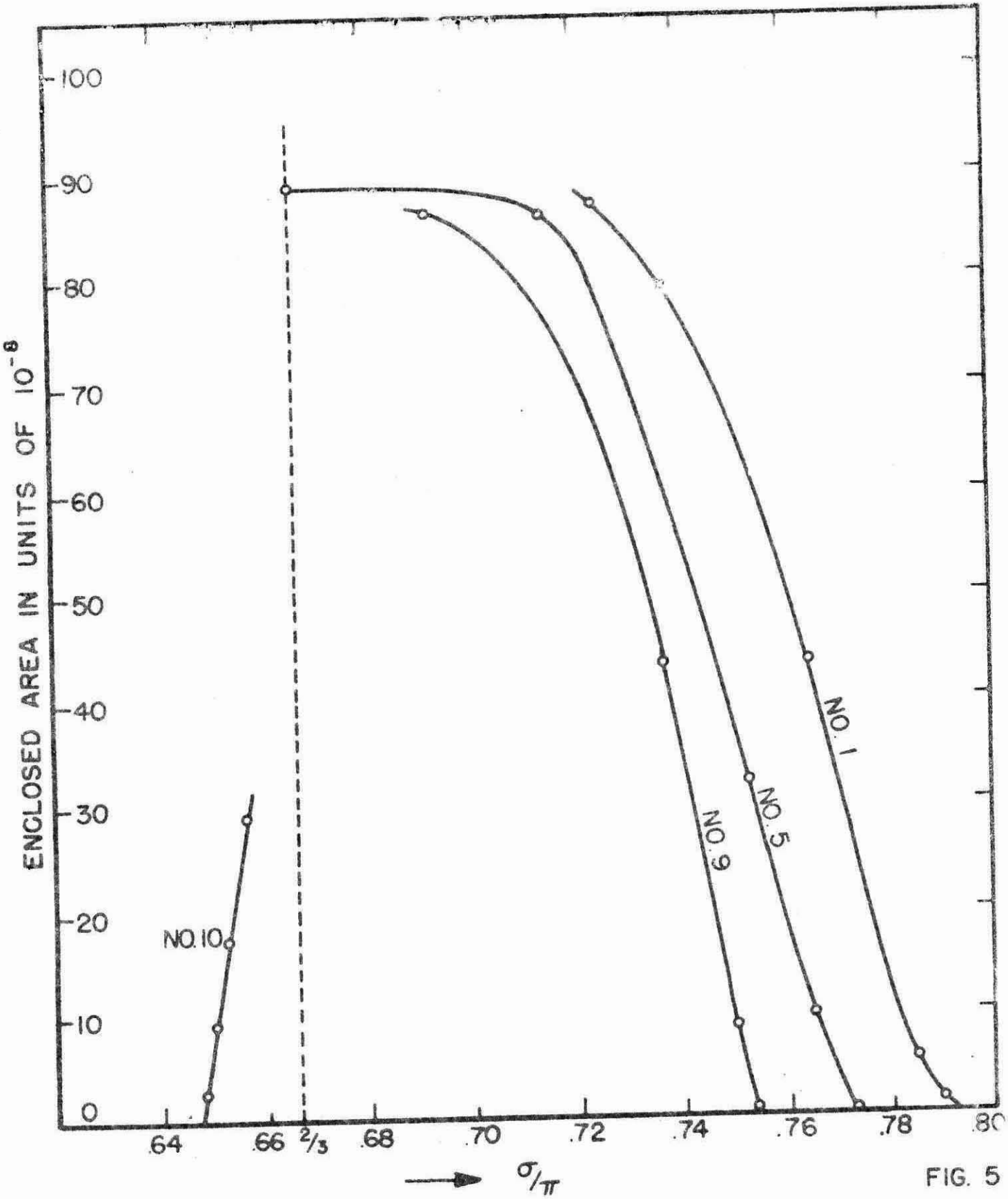
ENCLOSED AREA IN UNITS OF 10^{-8}
100
90
80
70
60
50
40
30
20
10
0
.64
.66
2/3
.68
.70
.72
.74
.76
.78
.80
σ/π
NO. 1
NO. 5
NO. 9
NO. 10

FIG. 5