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CALCULATIONS CONCERNING PARTICLE MOTION IN SPIRALLY-RIDGED and SEPARATED-SECTOR FFAG ACCELERATORS

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MURA

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I. INTRODUCTION

The subject with which I shall be concerned today is the particle motion in a Fixed-Field accelerator of the spirally-ridged type, including computational preparations for examination of the separated-sector variety, but with no reference to acceleration processes.

To define the problem, the starting point in analysis of the spirally-ridged structure is the assumed median-plane field, which we prescribe. In contrast, the separated-sector machine, or even certain slightly-modified spiral sector machines, make the specification of the pole contour more natural. If the pole contour is specified, one has the preliminary problem of determining the fields (or the magnetostatic potential) which they produce, while with the median-plane field prescribed at the start one must find the fields at other points and the location of equipotential surfaces along which the poles may be located.

It may be noted that, perhaps through lack of ingenuity, we have not attempted to start with a system of orbits and then endeavored to find a Maxwell field which would give rise to the prescribed orbits. We have, however, always imposed a scaling requirement in a sufficiently strict sense that not only are the number of radial and axial betatron oscillations around the machine independent of particle energy, , but the orbits for different energy particles are themselves geometrically similar.

The basic idea in these structures is that the average field around the machine shall vary with radius

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as r^k, with k sufficiently great as to give an adequately large momentum content, and that stability will be provided by the A-G action which arises from having the field alternatively higher and lower than average along spiral curves which all particles must cross. The most general type of median-plane field which is considered is, then,

$$B_{z_0} = -\langle B\rangle\left(\frac{r}{r_0}\right)^k\left\{1+\sum_{m=1} f_m \sin m\left[\frac{\ln(r/r_0)}{w} - N\phi\right] + g_m \cos m\left[\frac{\ln(r/r_0)}{w} - N\phi\right]\right\}.$$

Powell[2] has shown how such a field may be developed for points out of the median-plane, to give the various field components, the vector-potential components, or the magnetostatic scalar potential. For these various quantities Powell's development is quite similar in form and is expressed in terms of the dimensionless parameters

$$x = \frac{r - r_1}{r_1} \qquad y = \frac{z}{r_1} \qquad N\theta = N\phi - \ln(r_1/r_0) \;;$$

where r_1 is chosen so that, for the particle momentum under consideration, x will be small. Then the scalar potential, for example, is written

$$\Psi = (1+x)^{k+1}\sum_{i=0}\left\{\tau_i + \sum_{m=1} \nu_{i,m} \sin m\left[\frac{\ln(1+x)}{w} - N\theta\right] + \mu_{i,m}\cos m\left[\frac{\ln(1+x)}{w} - N\theta\right]\right\}\frac{(y/1+x)^{2i+1}}{(2i+1)!} \;,$$

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where recursion relations are given by Powell for the coefficients. It is noted that the definition of x is based on use of a cylindrical coördinate system; it may be pointed out that Dr. Akeley has suggested that reference to a system of spherical coördinates may have certain advantages and has written a report on this topic. This series of Powell's has formed the basis of a computer program--the "Potentate"--whereby the height (y) of a specified equipotential (Φ) may be found digitally as a function of θ and x.

By a quite similar program--the so-called "MKY stormesh Leader"--values proportional to

$$\frac{\Phi/(1+x)^{k+1}}{\mathcal{J}/(1+x)}$$

may be obtained.

II. PROGRAMS FOR COMPUTING TRAJECTORIES[3]

For the computation of trajectories in the spiral ridge accelerator no direct use has been made so far of Powell's expansion, although it has assisted in providing the base for some new programs which may come into use soon. What has already been done computationally has been with two programs which I shall now discuss--

First, the "Ridge Runner", based on exact equations for motion in the median plane, and, second, what was supposed to have been an interim program for combined radial and axial motion, the so-called "Peckless Five". In these two programs the prescribed median plane field is taken to be of the form[4]

$$(1+x)^k\left[1 + f \sin\left(\frac{\ln(1+x)}{w} - N\theta\right)\right],$$

no harmonic components being admissible.

1. Ridge Runner[4a]

The differential equations for the Ridge Runner program are written quite readily, since the fields, and hence the forces, may be explicitly formulated immediately. The equations are written as first order equations, in terms of the canonically conjugate variables

$$x \quad \text{and} \quad p_x \equiv \frac{x'}{\sqrt{(1+x)^2 + x'^2}} ,$$

and are integrated by the Runge-Kutta procedure. Computation takes 0.37 sec/RK, or, with 32 RK/sector, about 12 sec/sector plus punching time.

2. Feckless Five [4b]

For inclusion of axial motion in the computations attention must first be given to the development of the field out of the median-plane. One wishes, in fact, to obtain the vector potential, in order that the equations of motion (which involve the velocity-dependent $\vec{v} \times \vec{B}$ forces) may be strictly Hamiltonian in character. The systematic development of the vector potential has been treated in reports by Dr. Akeley[5] about a year and a half ago--the process is strictly an infinite one, involving the repeated application of the ∇ operator to vector quantities, and becomes rather elaborate if carried out properly in cylindrical coordinates. In setting up the "Feckless Five" equations this type of development was kept in mind--the view was taken, however, that curvature effects of the sort which distinguish a cylindrical coordinate system from a Cartesian one could be regarded as small corrections which need not be included exactly and that the dominant y-dependence would be given by hyperbolic functions of an argument close to y/w. In this way an approximate vector potential with components A_θ and A_z was contrived, from which a set of exactly-Hamiltonian

equations was derived with the dependent variables x, y,

$$p_x = \frac{x'}{1+x}, \qquad p_y = \frac{y'}{1+x} + \frac{e}{p_1} A_z .$$

One supposes that the variables x and y themselves will be small, but that $\frac{x}{w}$ and $\frac{y}{w}$ may be comparable with unity.

This program requires 0.71 sec/RK-step, or, with 32 RK/sector, about 23 sec/sector plus punching time.

The program is to be regarded as an approximate one, whose accuracy is expected to be good for large-scale machines but not as great for models where curvature effects play a more pronounced rôle.

The Feckless Five program is seen to be substantially half as fast as the Ridge Runner and it would be nice if we had some equivalent transformation which could be used to carry the particle rapidly through sector after sector in studies which require continued computations through a large number of sectors.

3. <u>Overwrites:</u>

Available for use with these programs are various embellishments or "overwrites". Thus the Ridge Runner may be adapted to permit the introduction of an algebraic transformation to simulate passage of a particle through a straight section. The Feckless Five may be supplemented by an overwrite which gives once or twice a sector the square root of the quadratic forms which remain invariant in the linear approximation:

$$K_x = \sqrt{\xi_x (x - x_f)^2 + \eta_x (x - x_f)(p_x - p_{x_f}) + \zeta_x (p_x - p_{x_f})^2}$$

and K_y similarly.

In addition, as we shall illustrate later, it is possible to

introduce various kinds of "bumps" into the Feckless Five program, to simulate certain misalignments.

4. Small Five:[2]

It has been hoped that the Feckless Five will be replaced by the proposed "Small Five" program, in which a more systematic development of the magnetic field would be employed, based on Powell's series, and with which it would be possible to study cases in which a limited number of harmonic components would be present in the field. Programming of the Small Five was begun and then interrupted in the interest of other work considered to be more pressing--it is hoped, however, that work on the Small Five will be recommenced and this program completed.

5. Stormesh:[6]

Since the first of the calendar year it has become increasingly apparent that one is unnecessarily and undesirably restricting oneself by confining attention to fields which in the median-plane are strictly sinusoidal or which are even restricted to a very limited number of harmonics. This recognition was reinforced by the result of some simple field-surveys,[7] made by solving Laplace's two-dimensional equation on a "50 x 15" Cartesian not (49 units x 14 units), and by the increased interest which the separated-sector type of structure appears to warrant. It seemed important therefore to bring into operation a double program, which (i) would commence with the contour of the pole boundary, on which the magnetostatic potential would be considered as given, and solve Laplace's equation for the space between this boundary and the median-plane, and then (ii) would permit investigation of particle trajectories in this potential field. It was felt that mesh-storage would be the most practicable approach and that storage should be confined to the fast-memory of the ILLIAC, when solving the dynamical problems, in the interests of achieving reasonable speed. It was further recognized that considerable simplification would result in the potential problem and a considerable reduction of the subsequent storage requirements if advantage were taken from the start of the scaling property of the field. The desire to economize to the utmost in storage suggested that the computational programs be planned in terms of the scaler potential, despite the impossibility of strictly Hamiltonian equations of motion when using fields derived from a somewhat-in-exact scaler potential.

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The scaling character of the field can be seen by reference to Powell's expansion, cited previously. It is evident from this development that $\psi/(1+x)^{k+1}$ has the same value at all points for which both $y/(1+x)$ and $\Phi \equiv \frac{\ln(1+x)}{\nu} - N\theta$ have the same values. Also, with $y/(1+x)$ constant, $\psi/(1+x)^{k+1}$ is periodic in Φ with period 2π. The potential, with its scaling factor $(1+x)^{k+1}$ is thus coveniently expressed in terms of two independent variables which we take to be

$$\xi \equiv \frac{1}{2\pi}\left[\frac{\ln(1+x)}{\nu} - N\theta\right]$$

$$\eta \equiv \frac{\sqrt{1+(\nu N)^2}}{2\pi\nu}\,\frac{y}{1+x}\,;$$

$$\psi = \frac{1}{A}(1+x)^{k+1}\Omega(\xi,\eta),$$

with Ω periodic in ξ with period unity.

This last equation may be alternatively interpreted(i) as referring $\psi/(1+x)^{k+1}$ to the value at a suitable point in the $\theta = 0$ plane or (ii) relating it to the value at a suitable point in the cylinder $x = 0$. By virtue of this relationship, Laplace's equation for this scaling field may be reduced to a second order differential equation for Ω with only two independent variables (ξ and η). The magnetic fields, moreover, may be obtained from Ω and hence storage on a two-dimensional net will suffice. For most efficient storage it was felt appropriate to store a quantity proportional to Ω/η, since this quantity will be more nearly constant than Ω itself and a greater number of significant figures would be retained. The field-strengths which enter into the (First-order) differential equations of motion are then to be obtained by First-order differentiation of Ω/η, interpolation and interpolation-differentiation being necessary because these quantities are stored on a net. To insure continuity in the differential equations it was then felt desirable to use an interpolation formula which would exhibit a continuous derivative upon crossing from the region covered by one cell to that covered by an adjacent cell. Such an interpolation Formula, being the only reasonable

one of its type extending through u^3 and based on four values of the function, is used throughout in preference to Bessel's more conventional but only slightly-different form.

This formula is:

$$\Lambda(t_0 + u h) = u\Lambda_1 + (1-u)\Lambda_0 + \frac{1}{4}u(1-u)(-\Lambda_2 + \Lambda_1 + \Lambda_0 - \Lambda_{-1}) + \frac{1}{2}u(\tfrac{1}{2}-u)(1-u)(\Lambda_2 - 3\Lambda_1 + 3\Lambda_0 - \Lambda_{-1}),$$

in contrast to Bessel's form:

$$\Lambda(t_0 + u h) = u\Lambda_1 + (1-u)\Lambda_0 + \frac{1}{4}u(1-u)(-\Lambda_2 + \Lambda_1 + \Lambda_0 - \Lambda_{-1}) + \frac{1}{6}u(\tfrac{1}{2}-u)(1-u)(\Lambda_2 - 3\Lambda_1 + 3\Lambda_0 - \Lambda_{-1}).$$

The quantity $\dfrac{\Omega/\eta}{\langle \Omega/\eta \rangle_0}$ is stored for each of as many as some 2000 mesh points as 13 binary bits. The differential equations are written in terms of coordinates

$$S = \ln(1+x) \quad \text{and} \quad T = \frac{y}{1+x}$$

to avoid the complications of a logarithm routine in the program, although print-out is performed in terms of the more-familiar variables x, y, px, and py.

The part of this program which seeks a solution to the p. d. equ. for Ω is termed the "SCAPOCYL" and the dynamics portion, the "Stormesh". Trouble-shooting of the first portion and testing of the second is currently in progress. The speed of the Stormesh program has been found to be intermediate between that of the Ridge Runner and that of the Feckless Five.

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III. COMPUTATIONAL RESULTS

By use of the Ridge Runner and Feckless Five programs, surveys have been made of the particle motion in spirally-ridged structures. Although the larger portion of this work was with parameters characteristic of models, the general features of the results no doubt apply also to large-scale machines.

1. Radial Motion [8]

The results of computation pertaining to motion with one degree of freedom are appropriately and conviently represented by means of phase plots, depicting on invariant curves the position and associated momentum of a particle as it progresses through successive "sectors" (periods of the structure) from one homologous point to another. Such studies provide information concerning the location of "fixed-points", corresponding to an equilibrium orbit; the phase-change of the betatron oscillation per sector (σ); the displacement associated with trajectory directions different from that of the equilibrium orbit; and the extent of the region within which stable motion is possible. The characteristics of small-amplitude motion found in this way agree well, for the sinusoidal fields, with the analytic work to be discussed later. At large amplitudes, unstable fixed points--representing an unstable equilibrium orbit--make their appearance. Associated with the unstable fixed-points one finds a separatrix, constituting an effective stability limit, which in the majority of cases the ILLIAC results depict as a sharp boundary and outside of which it is frequently possible to draw the initial portions of what appear to be invariant curves for the unstable motion.

With regard to stability the requirement for a strictly linear system is that the condition

$$0 < \sigma < \pi$$

be satisfied for operation in the first stability zone. Due to the non-linear character of the oscillations in a spirally-ridge FFAG, however, it is not surprising that for such structures the permissible amplitude of oscillation is much curtailed if σ lies near $\frac{2\pi}{3}$ or $\frac{2\pi}{4}$. In fact

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Dr. Christian at Los Alamos has made computations which show the amplitude limit to be reduced, although not to zero, for $\sigma = \frac{2\pi}{5}$. If the small-amplitude σ is in the neighborhood of $2\pi/3$, σ will at first change only slowly but then quite near the stability boundary will rapidly approach $2\pi/3$ and three unstable fixed-points will appear. These correspond to an unstable equilibrium orbit which repeats after progress through three sectors. Similarly, near $2\pi/4$, four unstable fixed-points may be expected to develop. When the machine parameters are such that σ is essentially midway between the values $2\pi/3$ and $2\pi/4$, a comparatively large stable region is found and the apparent limit of stability is defined by a separatrix which may be associated with a larger number of unstable Fixed-points, 7 such points being found in one example. In special cases rather elaborate island-structure is seen to develop within the main stability region. In some cases the phase curves near the stability boundary do not appear well defined and the location of the stability boundary can not be fixed with high precision.

A case with one of the largest radial-amplitude limits for machines with model-like parameters has a σ_x near $4\pi/7$ In this case $k=0.8$ $1/w=23.0$ $f=1/4$ $N=5$ and the "ears" of the phase-plot (at $N\theta = 0$, mod. 2π) extend to $x = \pm 0.09$; similarly for a case currently of considerable interest in connection with freezing the parameters of the Illinois spiral-sector model $k = 0.74$, $1/w = 23.7$, $f = 1.4$, $N = 5$, $(\sigma_{x_0} = .563\pi)$, and the ears extend beyond $x = \pm 0.06$. It has been noted that, for reasons which will be suggested later, if f is increased and $1/w$ concurrently decreased to maintain a similar σ_{x_0}, the amplitude limit may be made substantially greater. For comparison with this last example, the case $k = 0.74$ $1/w = 5.925$ $f = 1$ $N = 5$ (with $\sigma_{x_0} = .5896\pi$) led to a material change in the shape of the phase-plot but to a radial amplitude limit some 2.5 times that found in the earlier case. A similar result for the limit of stable axial motion would not be unexpected.

2. Axial Motion: 9

Introduction of axial motion into a study of spiral-sector accelerators produces complications for all but the smallest-amplitude oscillations, since in general there is coupling

between this motion and that occurring in the radial direction. For small-amplitude axial motion one can find the σ_{y_o} and the various matrix elements which characterize linear oscillations. For large-amplitude axial motion one can undertake an experimental survey to determine how large an initial y-amplitude can tolerated if the motion is to "hold on" (i.e., not exceed the limits of the computer, which are normally given by $y < 3.1\ w$) for some arbitrary number of sectors (e.g., 80 sectors). By way of example, one finds in this manner for the first of the model-like structures cited earlier ($k = 0.8$ $1/w = 23.0$ $f = 1/4$ $N = 5$) an axial amplitude limit close to $y = 0.014$; this limit applies to locations such that $N\theta = 0$ (mod. 2π), near the center of an axially defocusing region, and has associated with it amplitude limits which become almost twice as large at intermediate points. As with the radial motion, the limiting amplitude is curtailed if the operating point approaches search "resonant" values as $2\pi/3$ of such resonances has been given by Moser,[10] Hagedorn,[11] Sturrock,[12] and others.

3. Motion in Two Degrees of Freedom:

With motion in two degrees of freedom one can make searches with a wide variety of initial conditions to determine emperical stability limits. Beyond this, however, it is difficult to proceed systematically. As a result of a suggestion by Sturrock,[13a,b] it was hoped that investigation of motion in two degrees of freedom could be systemitized by use of the quadratic forms K_x^2 and K_y^2, mentioned earlier, which remain invarient on the basis of linear theory. We were led to expect that plots of K_y vs K_x would depict the point which represents a single trajectory moving on a portion of a conic curve and that regions of stability or instability could be distinquished. From a limited number of results obtained to-date, it appears that the expectation is an over-simplification--the values of K_x and K_y scatter sufficiently that a true curve is not defined, the nature of a curve near which the points lie appears sometimes to be elliptical and sometimes hyperbolic, and the regions of stability or instability are not readily appearant. It may be that further work along these lines is merited, however, A possible refinement of this technique would involve the plotting of running averages of K_x and of K_y, averaged over possibly 20 values in the interests of smoothness; in addition, one could consider use of more elaborate algebraic forms,[13a] in place of K_x and K_y themselves, between which simple relationships may be expected to apply.

4. Coupling Resonances:

Evidence of apparant instability appears for operating points in the neighborhood of certain "coupling resonances",

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notably:

$$\sigma_x = 2\sigma_y$$
$$\sigma_x + 2\sigma_y = 2\pi$$
$$2\sigma_x + 2\sigma_y = 2\pi .$$

In such cases one finds a exponential growth of y-amplitude, conveniently represented by semi-logarithmic plot of K_y vs the number of sectors traversed, which may begin with very small initial y-amplitudes (or even in the "noise" of the computer) and extends over many orders of magnitude. The y-growth appears to be the more rapid the greater the amplitude of the radial motion, above a certain threshold, and the more pronounced the closer one is to the resonance in question. This phenomenon has so far been studied in greatest detail for the $\sigma_x = 2\sigma_y$ resonance. If the initial radial amplitude is not too great, the exponential growth may be seen eventually to terminate rather suddenly (For example, near y = 0.01 or y = 0.02 in some typical cases involving model-like parameters) and to change to an exponential decrease for a time. This decrease may then be followed by an interval characterized by an exponential growth. When one is very close to the $\sigma_x = 2\sigma_y$ resonance, so that the y-growth occurs for even a rather small amplitude of the radial oscillation, it is possible to see from the computed values of K_x and K_y that as the axial increases there is some decrease of the amplitude of the radial oscillation. Becuase of the cessation of growth at axial amplitudes not far from those at which violent instability might be expected to occur, it is questionable whether machine operation would be satisfactory such under conditions--in such cases the majority of the particles would be expected to find themselves near the outer limits of the beam from time-to-time and misalignments may be expected to reap a heavy toll in such cases. These matters are being further explored computationally at the present time and theoretical progress has been made with respect to these aspects of the phenomena which concern the initial exponential growth.

5. Studies of Misalignments:

Some computational investigations of the effects of various "bumps" have been made before making a final commitment

concerning the parameters of the Illinois spiral sector model. The results obtained will be reported in a factual way, little theory being available for organization of the results, and the work will be seen to represent no more than a coarse survey of the effects which certain misalignments can cause in a specific case. The slowness of such computational work is a real handicap, which arises in part from the misalignments being a property of the machine-as-a-whole and the consequent necessity of going through a number of sectors to traverse one period of the perturbed structure. It may, moreover, be noted that, as Symon and Christian have emphasized, certain types of bumps may excite certain potentially dangerous resonances only indirectly; hence, unless a suitable, and perhaps unrealistic, perturbation is selected, a resultant instability may develop so slowly as to pass unnoticed in a run of reasonable duration. It would be of considerable convenience in such work to have at hand the most general transformation required to represent the effects of harmful misalignments and a knowledge of the manner in which the parameters of such transformation are related to the magnitudes of the constructional misalignments which the transformation represents.

The work reported here pertained primarily to an operating point chosen to be clear of the $\sigma_x = 2\sigma_y$ resonance, to lie between $\sigma_x = 2\pi/4$ (For which $\nu_x = 5/4$) and the half-integer resonance $\nu_x = 3/2$ and to fall below $\nu_y = 1$. The nearby inherent resonance $3\sigma_x + \sigma_y = 2\pi$ had not been found harmful in a machine free of imperfections. The nearest other imperfection resonances, aside from difference resonances, were those for which $3\nu_y + \nu_x = 4$ and $3\nu_x = 4$. The parameters of the machine selected for most of the studies (denoted "d") were:

$$k = 0.74 \qquad \frac{1}{w} = 23.7 \qquad f = \frac{1}{4} \qquad N = 5 ,$$

for which $\nu_x = 1.40\dot{8}$ and $\nu_y = 0.873$.

In studying certain bumps, neighboring operating points were also included. The computations were performed by aid of various overwrites, applicable to the Feckless Five master program.

Results

(i) Bumps Absent: -

In the absence of bumps, radial motion was stable in

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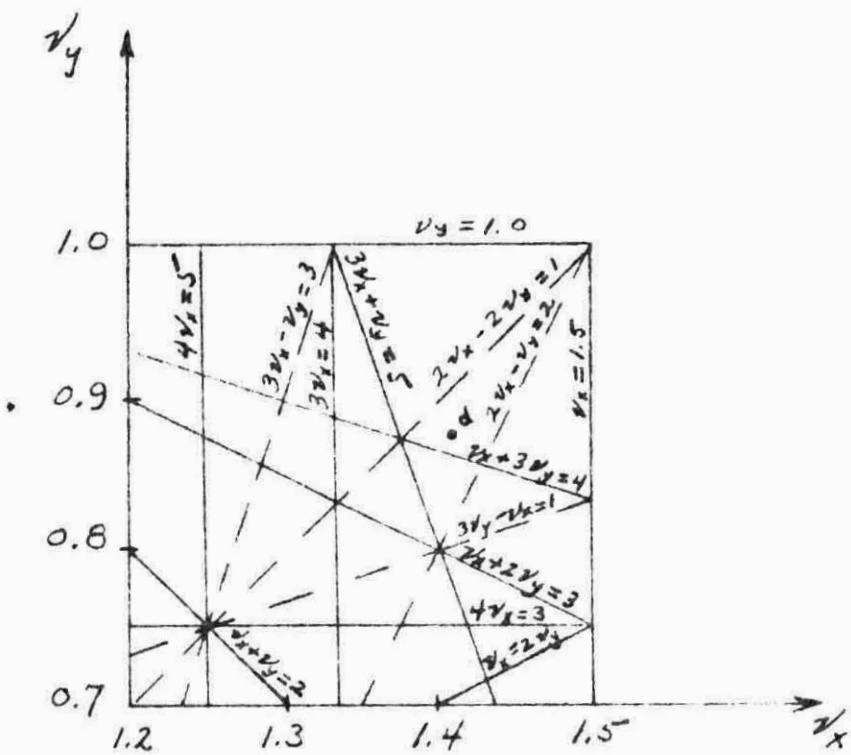
machine "d" for an x-displacement $\begin{cases} -.033 \\ +.040 \end{cases}$ from the stable fixed point, and the ears of the phase plot extended beyond $\pm$ .06. Similarly an initial y-displacement of $\pm$ .011 appeared to be stable. These displacements refer to points for which $N\theta = 0$, Mod. 2π .

(ii) Momentem Bumps:

As a first attempt at the introduction of bumps, the combination $\Delta p_x = -.005$, $\Delta p_y = +.0005$ was introduced once a revolution. . The permissible amplitude of the radial phase plot appeared to be reduced by a factor of about 2.



RESONANCES IN NEIGHBORHOOD OF "d".

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(iii) Coordinate-Dependent Momentum Bumps:

In this series of runs a few different combinations of momentum bumps (Δp_x , Δp_y) were abruptly introduced into the computations once every 5 sectors--i.e., once per revolution. In one case the combination

$$\Delta p_x = -.005 - .2x - 2x^2 - 10x^3$$
$$\Delta p_y = +.0005 - .2y \quad ,$$

for which the Jacobian of the transformation is unity, was employed. In this case the stable region of the radial phase plot was very materially reduced, each dimension of the plot being reduced by a factor estimated as close to 3. With the signs of the x-dependent terms reversed, the decrease of each dimension was similarly by a factor close to two.

(iv) Radially-Displaced Sector:

In this series a displacement, Δx , was introduced for an interval $2\pi/N$ to simulate a radially-displaced sector. Various phases for introduction of the bump were investigated, as well as various magnitudes of Δx . In this case a reduction of the stable region, of the radial phase plot, by a factor of 2 seemed to result from a displacement, Δx , lying between 0.0021 and 0.0063. Thus, with the smaller bump, motion with an initial x lying .0250 to the left of the fixed point was stable regardless of the phase of the bump while motion with an itial x lying .0375 to the left was stable in none of the cases studied; with the larger bump, an initial x lying .0250 to the left of the fixed point led to instability in most cases.

(v) Axially-Displaced Sector:

With an axially-displaced sector there resulted a very noticable increase in the frequency of axial betatron oscillations, an increase which varied predominently as the square of the sector displacement Δy .
Because of the proximity of the integer resonance $\nu_y = 1$,

it was felt appropriate to suppose that in actual practice suitable tuning controls would be employed to restore the operating point to its desired location despite the presence of unavoidable misalignments. For this reason the work to be reported here is concerned with a structure (denoted "e") for which

$$k = 0.7464 \qquad \frac{1}{w} = 23.252 \qquad f = \frac{1}{4} \qquad N = 5 ;$$

in the absence of any sector-displacement

$$\nu_x = 1.405 \qquad \nu_y = 0.834 ,$$

but with a displacement $|\Delta y| = .00350$ introduction for one sector

$$\nu_y = 0.8747 .$$

The axially displaced sector, $\Delta y = -.00350$, was found to effect a reduction of the stable x-amplitude by a factor of 2 and the y-amplitude by a factor of 3.

(vi) Tilted Sector:-

In this series a tilted sector was simulated by introducing, once per revolution, two bumps, $(\Delta y_1, \Delta p_{y_1})$ and $(\Delta y_2, \Delta p_{y_2})$, [redacted] at points one sector apart. Specifically, $\Delta y_1 = \Delta y_2$, $\Delta p_{y_1} = -\frac{5}{\pi}\Delta y_1$, and $\Delta p_{y_2} = +\frac{5}{\pi}\Delta y_1$. A reduction of the stable amplitudes of radial and axial motion by a factor of nearly 2 was found to occur when $\Delta y_1 = -.00350$.

(vii) Parameter-Shift:

In this series the parameter 1/w was changed for an interval corresponding to one sector and a con-current change was made in F in an effort to allow for the increased spatial modulation of the field which would be expected to result if the ridges of a spiral sector accelerator were separated. Work has been confined to the case in which one sector of accelerator "d" was modified as follows:

Unperturbed Sectors: $1/w = 23.7$, $f = 0.25$;
Perturbed Sectors: $1/w_s = 23.07423$, $f_s = 0.2533996$.

In this case little reduction of the stable region appeared to result, although a radial phase plot of a nearly-limiting amplitude run appeared to be a bit more ragged than for the unperturbed case.

In summary, it is seen that displacements which correspond to about 1mm, when $r_1 = 300$ mm, under a number of circumstances can cause a serious reduction of the stability region.

IV. ANALYTIC WORK PERTAINING TO UNPERTURBED STRUCTURES

1. The Equilibrium Orbit:

One of the distinctive features of the spirally-ridged accelerator is that the equilibrium orbit is not circular.[4] If one expands the equations which govern the motion in the median plane about a reference circle, a forcing term makes its appearance and leads to a scalloped equilibrium orbit. The departure from a circle is, in fact, close to a sinusoid, given by[4]

$$x_f \doteq -\frac{f}{N^2 - (k+1)} \sin N\theta$$

and has been determined with greater accuracy by Judd[14] and by Cole[15].

2. The Small-Amplitude Betatron Oscillations:

The character of small-amplitude betatron oscillations must be obtained by expansion of the equations of motion about the equilibrium orbit and leads to frequencies materially different[4,16] from those which would be obtained by ignoring the effect of the forcing term. Qualitatively this is to be expected, since the field gradient is in a sense to favor radial focusing over a smaller interval of θ if one examines the gradient in the neighborhood of the scalloped orbit instead of along a circular path.

We will not undertake here to discuss development of the equations for betatron oscillations on the basis of Symon's unified theory of FFAG machines,[17] but shall outline a more specific approach developed with increasing degrees of completeness by myself,[4] by Judd,[14,18] and by Cole.[15] From the prescribed median plane field, vector potential components are developed and employed in a space-like Lagrangian from which, by the principle of least action, the differential equations for the trajectories may be derived directly:[19]

$$\begin{aligned}\mathcal{L}(x,y;x',y';\theta) &= p\, ds/d\theta + e\,\bar{A}\cdot \overline{ds}/d\theta \\ &= pr_1\sqrt{(1+x)^2 + x'^2 + y'^2} \\ &\quad + er_1\left[(1+x)A_\theta + x'A_r + y'A_z\right].\end{aligned}$$

A change of variable is then made ($u \equiv x - x_f$) to modify the Lagrangian so as to eliminate the forced motion, and the differential equations which result from the modified Lagrangian are then taken as the equations governing the betatron oscillations. In this way the coefficients of the linearized equations, applying to small-amplitude motion, are obtained and the major non-linear terms also may be noted.

The linear equations are of the Hill form and, if relatively small terms are ignored, are substantially of the form

$$u'' + (a_x + b_x \cos N\theta + c_x \cos 2N\theta)u = 0 ,$$

and similarly for the y-equation. For orientation, it is helpful to note the frequencies which the smooth approximation[20] gives for the solutions to these equations. Ignoring the relatively small contribution from the term involving $c_x \cos 2N\theta$, one obtains

$$\nu_x^2 \cong a_x + \frac{1}{2}\frac{b_x^2}{N^2}$$

$$\doteq \left[(k+1) - \frac{1}{2}\frac{(f/w)^2}{N^2 - (k+1)}\right] + \frac{1}{2}\frac{(f/w)^2}{N^2}$$

$$\doteq k + 1$$

$$\nu_y^2 \cong \left[-k + \frac{1}{2}\frac{(f/w)^2}{N^2 - (k+1)}\right] + \frac{1}{2}\frac{(f/w)^2}{N^2}$$

$$\doteq -k + \left(\frac{f}{wN}\right)^2 .$$

It is thus seen that the frequency of the free radial oscillations is substantially determined by the exponent k characterizing the radial increase of average field strength, so that $k + 1$ must be positive, and that axial stability may simultaneously be obtained if the enhanced A-G term, $(f/wN)^2$, is sufficiently large to dominate $-k$.

More exact information concerning the solutions of the Hill equation, with the term $c \cos 2N\theta$ retained, may be obtained by use of tables pertaining to this equation and which were calculated by aid of the ILLIAC digital computer. The first set of tables was prepared by a variational method which is believed to be quite accurate at the stability limits, $\sigma = 0$ or π, and also for the smaller values of σ in general. The most satisfactory form in which to use these results is by recourse to the graphs which accompany the tables. A second set of finer-mesh tables has been subsequently prepared for the Hill equation by direct integration of the differential equation. These tables, which have been duplicated and are about to be distributed, give $\cos \sigma$, σ/π, and selected values of a quantity $\left(\frac{\sin\sigma}{\sigma}\right)$ proportional to the square of the amplitude function and from which the Floquet solutions

in the phase-amplitude form can be obtained. In each of these tables an independent variable is used such that the argument of the cosine functions is 2t, and multiples thereof.

By use of the foregoing theory, and by aid of the available tables or graphs, the first stability region may be plotted in terms of machine parameters. The basic variables, when $k \gg 1$, are k/N^2 and $f/(wN^2)$. The result of direct integration of the equations of motion, by use of the Ridge Runner or Feckless Five programs, yield results which for small-amplitude motion are consistent with the predictions of the analytic theory.

3. The Limiting Stable Amplitudes:

(i) The stability-limit for radial oscillations:-

In a large fraction of cases the limit of stability for the radial motion is characterized by the appearance of three fixed points. For such cases a convenient approximate formula may be developed by recourse to a differential equation in which the important non-linear term is taken from Cole's report:[15]

$$u'' + (a + b \cos N\theta)u = \frac{f}{2w^2} (\sin N\theta)\, u^2 .$$

One may attempt the solution of this equation by substitution of the trial solution $u = A \sin(N\theta/3)$ and application of harmonic balance. One may alternatively replace the differential equation by an equivalent variational statement and then employ the same type of trial solution as before. Finally one may employ a variational procedure of the type outlined by Sturrock.[13] These various methods appear to agree in giving for the limiting amplitude the expression

$$|A| \cong 8(w^2/f) \left|\nu_x^2 - \frac{N^2}{9}\right| = 2(w^2N^2/f) \left|(\sigma_x/\pi)^2 - (2/3)^2\right|$$

$$\doteq (8/3)(w^2N^2/f) \left|\frac{\sigma_x}{\pi} - \frac{2}{3}\right| .$$

It is noted that the character of the trial function taken in this work was extremely simple; the formula appears, however, to give estimates in good accord with the Ridge Runner stability limits in both model-like and full-scale machines for which the nearest resonance is that for which $\sigma_x = 2\pi/3$. It may be noted that, since the betatron frequencies are essentially determined by k/N^2 and $f/(wN^2)$, a desirable increase of stable amplitude might be expected if f and w were each increased by the same factor. Dr. Sessler has extended this formula in an attempt to take account of fields containing higher-order Fourier components.

(ii) The Stability Limit for Axial Oscillations:-

In considering the stability limit for axial motion, it has been pointed out that larger amplitudes of axial oscillation cause the particle to sample fields of a necessarily greater flutter-factor. The flutter-factor of a simple sinusoidal variation increases approximately by the factor $\cosh(y/w)$ for points out of the median plane. The suggestion has then been advanced that

the effect of this increased flutter in the field is to "tune" the oscillation frequency towards the next higher resonance and that instability will result when this resonant value is reached. On the basis of this simple, and perhaps not entirely true, idea, one may proceed to write

$$\nu_y^2 = \frac{f^2}{N^2 w^2} - k$$

$$\Delta\nu_y \doteq \frac{f\,\Delta f}{\nu_y\, N^2 w^2} .$$

With $$f = f_o \operatorname{Cosh}\frac{\langle y\rangle_{eff}}{w} ,$$

$$\Delta f \doteq \frac{f\langle y^2\rangle}{2\,w^2} ,$$

and $$\Delta\nu_y = \frac{f^2\,\langle y^2\rangle}{2\,\nu_y\, N^2\, w^4} .$$

If we write $\langle y^2\rangle = A^2/2$ and consider σ_y to be below but fairly close to the value $2\pi/3$,

$$A^2 \doteq \frac{4\nu_y\, N^2\, w^4}{f^2}\left[\frac{N}{3} - \nu_y\right]$$

$$\doteq \frac{2\nu_y\, N^3\, w^4}{f^2}\left[\frac{2}{3} - \frac{\sigma_y}{\pi}\right] .$$

Since ν_y is presumed close to $N/3$ (i.e., σ_y close to $2\pi/3$), this result may be expressed in the simpler form

$$A \cong \sqrt{\frac{2}{3}}\,\frac{N^2 w^2}{f}\sqrt{\frac{2}{3} - \frac{\sigma_y}{\pi}}$$

$$\cong 0.8\,\frac{N^2 w^2}{f}\sqrt{\frac{2}{3} - \frac{\sigma_y}{\pi}} .$$

Comparison of this equation with Feckless Five results suggests that the formula may over-estimate the permissible amplitude but that within a factor of about 4 it gives a correct estimate in a variety of cases. Again, the desirability of increasing f and w together by the same factor is suggested. Sessler has undertaken to extend this formula, by the same type of reasoning, to cases in which higher-order Fourier components are present in the field.

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4. y-Growth:

We shall discuss here analytical work relevant to the exponential growth of axial amplitude observed in the neighborhood of certain resonances, with particular emphasis on the $\sigma_x = 2\sigma_y$ resonance.

(i) Walkinshaw's approach:-

In a recent memorandum, Walkinshaw[21] has pointed out that the differential equation for axial motion, although properly treated as linear in y for small amplitudes, contains a coefficient which involves the radial motion. Just as the forced radial motion is known, as we have seen, to affect the axial focusing, so the presence of an appreciable amplitude of radial oscillation may be expected to affect the axial motion under suitable circumstances. The view is taken that the superposition of the comparitively-long-wavelength radial oscillations on the forced motion in effect modulates the smooth-approximation coefficient in the axial equation, to yield a Mathieu equation with a coefficient having the period of the radial motion. Under "resonant" conditions, which appear to include the case $\sigma_x = 2\sigma_y$, the equation may have unstable solutions.

Walkinshaw commences with the basic differential equations (in our notation):

$$x'' + (k+1)x = -f\sin(\tfrac{x}{w} - N\theta) ,$$

$$y'' + \left[-k - \tfrac{f}{w}\cos(\tfrac{x}{w} - N\theta)\right] y = 0 .$$

A solution for the radial motion, representing a free oscillation of amplitude A superposed on the forced motion, is taken of the form:

$$x = x_\beta + x_f ,$$

with

$$x_\beta = A\sin(\nu_x\theta + \alpha) ,$$

$$x_f = -(f/N^2)(1 - \tfrac{x_\beta'}{wN})^{-2}\sin(N\theta - \tfrac{x_\beta}{w}) ,$$

and

$$\nu_x = \sqrt{k+1} << N .$$

This solution is then introduced into the axial equation and, after some approximation, gives

$$y'' + \left[-k + \tfrac{f^2}{w^2N^2}(1 - \tfrac{x_\beta'}{wN})^{-2} - \tfrac{f}{w}\cos(N\theta - \tfrac{x_\beta}{w})\right] y = 0 .$$

At this point the attempt is made to eliminate the cosine term by application of the "smooth approximation", in effect replacing the N which usually appears by $N - x_\beta'/w$:

$$y'' + \left[-k + \tfrac{f^2}{w^2N^2}(1 - \tfrac{x_\beta'}{wN})^{-2}\right] y = 0 ,$$

or

$$y'' + \left[-k + \tfrac{f^2}{w^2N^2} + \tfrac{2f^2 x_\beta'}{w^3N^3}\right] y = 0 .$$

It is noted that the terms within the square bracket and which do not contain x_β' are just those which normally give $\nu_{y_0}^2$ by the smooth approximation. Hence, with this substitution and replacement of x_β' by $A\,\nu_x \cos(\nu_x\theta + \alpha)$, one obtains

$$y'' + \left[\nu_{y_0}^2 + \frac{2A f^2 \nu_x}{w^3 N^3}\cos(\nu_x\theta + \alpha)\right] y = 0 .$$

This Mathieu equation may be put into standard form by the change of independent variable

$$\nu_x\,\theta + \alpha = 2\tau ,$$

to obtain

$$\frac{d^2y}{d\tau^2} + \left[(2\nu_{y_0}/\nu_x)^2 + \frac{8A f^2}{w^3 N^3 \nu_x}\cos 2\tau\right] y = 0 ,$$

with a coefficient whose period is π in the independent variable τ. Such a Mathieu equation will exhibit instability when the constant term in the coefficient is equal or close to the square of an integer -- in particular, there is a fairly broad band of instability near

$$(2\nu_{y_0}/\nu_x)^2 = 1 ,$$

corresponding to $\nu_x = 2\nu_{y_0}$. This instability will be expected to extend over a wider range of values of ν_{y_0} the greater is A, the amplitude of the radial betatron oscillations; similarly, for a fixed value of $2\nu_{y_0}/\nu_x$ and within the unstable zone, the lapse rate characterizing the growth of the axial amplitude will be the greater the larger the radial oscillations. The predictions of this theory, both with respect to the threshold at which instability sets in and with respect to the lapse rate in the unstable region, appear to be in good accord with the results of the ILLIAC computations.

(ii) An Alternative Approach:-

Despite the success of Walkinshaw's ingeneous and successful account of the $\sigma_x = 2\sigma_y$ resonance, it was felt that the method involved some uncertainties, especially in the first application of the "smooth approximation", which were difficult to rationalize. It was thought desirable to develop an alternative, and perhaps more general, method which would be applicable to other resonances and which would be based in a straight forward way on the differential equations developed by Cole.[15]

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If we regard the amplitude of the betatron oscillations themselves, taken with respect to the closed equilibrium orbit, as small, they may be supposed adequately represented by the linear differential equation

$$u'' + \left[a_x + b_x \cos N\theta\right] u = 0 ,$$

with

$$a_x = k + 1 - \frac{f^2}{2w^2\left[N^2 - (k+1)\right]} \cong k + 1 - \frac{f^2}{2w^2N^2} \quad \text{and}$$

$$b_x = f/w .$$

A suitable solution to this equation may be sought conveniently by a variational method in a variety of ways. A method which we shall employ again imagines that the frequency of the oscillation and the basic frequency of the structure are commensurate in a sufficiently large interval and, hence, that the solution may be regarded as "periodic" in such an interval. Such a periodic solution might normally be thought to correspond to a stability boundary, but in the present instance we find that there are two periodic solutions and the zone of instability which one might imagine to be present is of zero width.

We write, then, the variational statement

$$\delta \int \tfrac{1}{2}\left[u'^2 - (a_x + b_x \cos N\theta)\, u^2\right] d\theta = 0$$

as equivalent to the differential equation.

A trial solution of the form

$$u = A \cos \nu\theta + B \cos(\nu + N)\theta + C \cos(\nu - N)\theta$$

may be introduced into the integral, the integration performed, and the resultant algebraic expression adjusted to be stationary by proper selection of the frequency ν and proper proportioning of the coefficients A, B, C. One finds in this way

$$\nu^2 \cong a_x - \frac{b_x^2}{2N^2}$$

$$B \cong \frac{b_x}{2N^2}\,(1 - 2\nu/N)\, A$$

$$C \cong \frac{b_x}{2N^2}\,(1 + 2\nu/N)\, A .$$

Thus the value found for ν in this approximation is concordant with the result of the smooth approximation and we have the approximate solution for u:

$$u = A\left[\cos \nu\theta + \frac{b_x}{2N^2}\,(1 - 2\nu/N)\cos(\nu + N)\theta + \frac{b_x}{2N^2}\left(1 + \frac{2\nu}{N}\right)\cos(\nu - N)\theta\right]$$

$$= A\left[\cos\nu\theta + \frac{b_x}{N^2}\cos\nu\theta\cos N\theta + \frac{2b_x\nu}{N^3}\sin\nu\theta\sin N\theta\right].$$

Likewise, if a trial function employing sine functions had been employed, a similar result would have been obtained:

$$u = A\left[\sin\nu\theta + \frac{b_x}{N^2}\sin\nu\theta\cos N\theta - \frac{2b_x\nu}{N^3}\cos\nu\theta\sin N\theta\right].$$

We accordingly take the general solution to be:

$$u = A\left[\sin(\nu_x\theta+\epsilon) + \frac{b_x}{N^2}\sin(\nu_x\theta+\epsilon)\cos N\theta - \frac{2b_x\nu_x}{N^3}\cos(\nu_x\theta+\epsilon)\sin N\theta\right]$$

$$= A\left[\sin(\nu_x\theta+\epsilon) + \frac{f}{wN^2}\sin(\nu_x\theta+\epsilon)\cos N\theta - \frac{2f\nu_x}{wN^3}\cos(\nu_x\theta+\epsilon)\sin N\theta\right].$$

The complete radial motion is $x = -\frac{f}{N^2}\sin N\theta + u$ and is found to agree with Walkinshaw's form when the latter is expanded.

For study of the y-motion near the $\sigma_x = 2\sigma_y$ resonance we again refer to Cole's report[15] to write the linear equation in y:

$$y'' + \left[a_y + b_y\cos N\theta - b_5 u\right]y = 0,$$

where $a_y = -k + \frac{f^2}{2w^2[N^2-(k+1)]} \cong -k + \frac{f^2}{2w^2N^2}$,

$b_y = -f/w$,

and $b_5 \cong (f/w^2)\sin N\theta$ to sufficient accuracy for the present purpose.

We then substitute our solution u into this equation, ignoring terms in $2N\theta$ and dropping the phase-shift ϵ as a matter of convenience, to obtain

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$$y'' + \left[a_y + b_y \cos N\theta - \frac{Af}{w^2} \sin N\theta \sin \nu_x\theta + \frac{Af^2\nu_x}{w^3 N^3} \cos \nu_x\theta\right] y = 0 .$$

This equation is of the form

$$y'' + \left[a_y + b_y \cos N\theta + \tfrac{1}{2}c_y \cos(N - \nu_x)\theta - \tfrac{1}{2}c_y \cos(N + \nu_x)\theta + d_y \cos \nu_x\theta\right] y = 0 ,$$

with a_y and b_y as before,

with $c_y = - Af/w^2$, and

with $d_y = + Af^2\nu_x/(w^3N^3)$.

The equation may be cast in the form of a variational statement and stability boundaries sought by the use of trial functions

$$y_1 = B \cos \frac{\nu_x}{2} + P_1 \cos \frac{2N - \nu_x}{2}\theta + P_2 \cos \frac{2N + \nu_x}{2}\theta ,$$

$$y_2 = C \sin \frac{\nu_x}{2} + Q_1 \sin \frac{2N - \nu_x}{2}\theta + Q_2 \sin \frac{2N + \nu_x}{2}\theta .$$

One finds in this way that the stability boundaries in the neighborhood of $\nu_x = 2\nu_{y_e}$ (where $\nu_{y_e}^2 \equiv a_y + b_y^2/(2N^2)$ corresponds to solutions of the y-equation when $A = 0$) are given by

$$\left| \nu_x^2 - (2\nu_{y_e})^2 \right| = 2 \left| \frac{\nu_x b_y c_y}{N^3} + d_y \right|$$

$$= 4 \frac{\nu_x f^2}{w^3 N^3} |A| .$$

This result is in agreement with the location of the stability boundaries of the "equivalent" Mathieu equation originally suggested by Walkinshaw.

Continuation of the analysis of our equation, along lines indicated for the Mathieu equation by McLachlan,[22] moreover leads to lapse rates in the unstable zone which agree with the values implied by Walkinshaw's equation and which appear to be in reasonable accord with the ILLIAC results.

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(iii) Other Resonances:-

We have applied our methods to the examination of other resonances where y-growth may occur. It appears possible in this way to account for the behavior at the resonance $\sigma_x + 2\sigma_y = 2\pi$ and at $2\sigma_x + 2\sigma_y = 2\pi$. In this latter case one should consider not only the term $\frac{1}{2}c_{10}\, u^2 y$ in the y-equation but also the double frequency $(2\nu_x)$ terms which can enter the term $b_5 uy$ by use of supplementary terms in u obtained by a perturbation solution of the non-linear u-equation. It appears, however, that the direct contribution from $\frac{1}{2}c_{10}\, u^2 y$ definitely dominates.

In the neighborhood of the possible $\sigma_x = \sigma_y$ resonance, the ILLIAC results have revealed no y-growth. Our analysis, differing in detail from Walkinshaw's, indicates that instability leading to y-growth would occur over a quite restricted range of radial amplitudes and that the lapse rate within this narrow zone of instability would be so small as to be far beneath notice.

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