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The x-Stability Limit in Spiral Sector Accelerators

L. J. Laslett and A. M. Sessler

It is the purpose of this note to exhibit a "handy formula" for the x-stability limit in spiral sector accelerators. In Part I we consider the special case of a purely sinusoidal magnetic field, deriving a formula in three distinct ways. (It might be recorded that two of these derivations were known to the first author fully ½ a year ago.) This formula is then compared with the observed values obtained with the digital computer. In Part II the methods are extended to include arbitrary median-plane magnetic fields. The resulting stability limit is seen not to be significantly modified by presently contemplated magnetic fields. It should however be emphasized that this result has not yet had the benefit of comparison with digital computer results.

Part I - Sinusoidal Fields

In the neighborhood of $\sigma_{x_0} = 2\pi/3$ it is observed that the stability boundary is characterized by three fixed points, and by a solution of fundamental frequency $N/3$. Consequently we examine the x equation for quadratic non-linearities. Taking the largest term in b_1 (notation of MURA/FTC-3) we obtain:

$$u'' + (a + b \sin N\theta)\, u = \frac{f}{2w^2} \sin N\theta \; u^2 \qquad (1)$$

where:

$$a = k + 1 - \frac{f^2}{2N^2 w^2}$$

$$b = f/w$$

For our purposes this may be replaced with the approximate equation:

$$u'' + \nu_x^2 u = \frac{f}{2w^2} \sin N\theta \; u^2 \qquad (2)$$

Derivation I:

We proceed by the method of harmonic balance, assuming $u = A \sin \frac{N\theta}{3}$:

$$-\frac{N^2}{9} A \sin \frac{N\theta}{3} + A\nu_x^2 \sin \frac{N\theta}{3} = \frac{f A^2}{4w^2} \sin N\theta \left[1 - \cos \frac{2N\theta}{3}\right]$$

$$= \frac{f A^2}{4w^2} \left\{ \sin N\theta - \tfrac{1}{2} \sin \frac{2N\theta}{3} - \tfrac{1}{2} \sin \frac{N\theta}{3} \right\}$$

thus obtaining:

$$\nu_x^2 - \frac{N^2}{9} = -\frac{fA}{8w^2} \qquad (3)$$

upon replacing ν with σ, and solving for A:

$$A = -\frac{2w^2 N^2}{f\pi^2} \left[\sigma_x - \frac{2\pi}{3}\right]\left[\sigma_x + \frac{2\pi}{3}\right]$$

or since σ is in the neighborhood of $\frac{2\pi}{3}$:

$$|A| = \frac{8}{3\pi} \frac{w^2 N^2}{f} \left| \sigma_x - 2\pi/3 \right|$$

$$|A| \simeq \frac{w^2 N^2}{f} \left| \sigma_x - \frac{2\pi}{3} \right| \qquad (4)$$

Derivation II:

This derivation follows the methods outlined

by P. A. Sturrock (A.E.R.E. x/R 1771 and Urbana lecture 30 November 1955), and we shall use his notation freely. W + φ represent the phase - and amplitude-functions (B.M.L. Report, EDC-15) characterizing the Floquet solutions of the linear differential equations,

$$\Delta m_3 = -\frac{f}{6W^2} \sin N\theta \, u^3$$

$$u = K \frac{W}{W_0} \sin(\varphi - \varphi_0 + K) \qquad (5)$$

This yields

$$\Delta U_3 = -K^3 \frac{f}{24W^2} \int_0^{2\pi/N} \left(\frac{W}{W_0}\right)^3 \left[3 \sin(\varphi - \varphi_0 + K) - \sin 3(\varphi - \varphi_0 + K)\right] \sin N\theta \, d\theta$$

or:

$$EK^3 = K^3 \frac{f}{24W^2} \int_0^{2\pi/N} \left(\frac{W}{W_0}\right)^3 \sin 3(\varphi - \varphi_0 + K) \sin N\theta \, d\theta$$

Maximized with respect to K.

$$E = \frac{f}{24W^2} \left\{ \left[\int_0^{2\pi/N} \left(\frac{W}{W_0}\right)^3 \sin 3(\varphi - \varphi_0) \sin N\theta \, d\theta \right]^2 + \left[\int_0^{2\pi/N} \left(\frac{W}{W_0}\right)^3 \cos 3(\varphi - \varphi_0) \sin N\theta \, d\theta \right]^2 \right\}^{1/2} \qquad (6)$$

Assuming, for simplicity, the motion to be simple harmonic, we have:

$$\frac{W}{W_0} = 1 \quad , \quad \varphi - \varphi_0 = \frac{N\sigma}{2\pi}\theta \quad , \quad \sigma_{\text{Sturrock}} = \nu$$

So that:

$$E = \frac{f}{2\pi N \omega^2}\left\{\left[\int_0^{2\pi} \sin\left(\frac{3\sigma t}{2\pi}\right)\sin t\, dt\right]^2 + \left[\int_0^{2\pi} \cos\left(\frac{3\sigma t}{2\pi}\right)\sin t\, dt\right]^2\right\}^{\frac{1}{2}}$$

which for σ near $\frac{2\pi}{3}$ becomes:

$$E \simeq \frac{f\pi}{2\pi N \omega^2} \qquad (7)$$

Now Sturrock has shown that

$$K_{max} = \frac{1}{3}\left|\frac{\sigma_{Sturrock}\, E}{E}\right| \qquad (8)$$

Yielding finally,

$$K_{max} = \frac{8 N^2 \omega^2}{3\pi f}\left|\sigma - \frac{2\pi}{3}\right| \qquad (9)$$

<u>Derivation III</u>:

This derivation employs a variational method and proceeds by replacing Eq. 2 with the following:

$$I = \int_0^{2\pi/N}\left\{-\frac{1}{2}u'^2 + \frac{1}{2}\nu_x^2 u^2 - \frac{f}{6\omega^2}\sin N\theta\, u^3\right\} d\theta$$

$$\delta I = 0 \qquad (10)$$

Take as a trial function the simplest possible form:

$$u = A \sin\frac{N\theta}{3} \qquad (11)$$

Then:

$$u'^2 = \frac{N^2}{9} A^2 \sin^2 \frac{N\theta}{3}$$

$$u^2 = A^2 \sin^2 \frac{N\theta}{3}$$

$$u^3 = \frac{A^3}{4}\left[3 \sin \frac{N\theta}{3} - \sin N\theta\right]$$

Thus:

$$\langle 2I \rangle = -\frac{N^2}{9}\frac{A^2}{2} + V_x^2 \frac{A^2}{2} + \frac{f A^3}{24 r^2} \qquad (12)$$

Which varying with respect to A, yields:

$$\left(V_x^2 - \frac{N^2}{9}\right) A + \frac{f}{8 r^2} A^2 = 0$$

or, in agreement with our other derivations:

$$|A| = \frac{8 r^2}{f}\left|V_x^2 - \frac{N^2}{9}\right| \qquad (13)$$

It is evident that the derivation is particularly adaptable to generalization to more complicated trial functions, and hence to more accurate expressions for the stability limit.

In Table I, we have compared the predicted values of A, as given by Eq. 4, with two sample digital computer runs, employing both model and full-size machine parameters.

Table I Predicted and Observed Stability Limits

k	1/w	f	N	$\lvert\sigma_0 - 2\pi/3\rvert$	A calculated	A observed (Nθ=0, mod.2)
1.1	26.4	1/4	40	$\lvert 1.647-.667\rvert\ \pi = .033\pi$	.015	.017
128	2112	1/4	40	$\lvert 1.647-.667\rvert\ \pi = .020\pi$	.00009	.00010

On Graph I we have compared the calculated limits with observed values, in the neighborhood of $\sigma_x = 2\pi/3$ for various machines characterized by model size parameters.

On Graph II we have plotted predicted values of x-stability limit as a function of x-tune, for fixed y-tune.

Part II - Arbitrary Median Plane Fields

We wish to examine the x-stability limit in an accelerator with median-plane magnetic fields involving a number of harmonics. The dominant non-linear term in the x equation will be of the form[1]

$$\frac{1}{2} u^2 \sum_n \sin n N\theta \; \frac{f_n n^2}{n^2} \qquad (14)$$

where the magnetic field has been specified by Fourier components with coefficients f_n. This leads to the approximate x equation:

$$u'' + \nu_x^2 u = \frac{1}{2w^2} u^2 \sum_n f_n n^2 \sin n N\theta \qquad (15)$$

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[1] We are indebted to Dr. F. T. Cole for assurance that this is in fact the dominant non-linear term.

Guided by our work of Part I, we assume a trial function of the form:

$$u = B_1 \sin\frac{N\theta}{3} + \sum_{n=2}^{\infty} B_n \sin\frac{nN\theta}{3} + A_n \cos\frac{nN\theta}{3} \qquad (16)$$

We assume that A_n, $B_n \ll B_1$ and proceed to evaluate the stability limit by the method of harmonic balance, keeping only linear terms in A_n, B_n:

$$\left(\nu_x^2 - \frac{N^2}{9}\right) B_1 \sin\frac{N\theta}{3} + \sum_{n=2}^{\infty} B_n \left(\nu_x^2 - \frac{n^2N^2}{9}\right) \sin\frac{nN\theta}{3} +$$

$$+ \sum_{n=2}^{\infty} A_n \left(\nu_x^2 - \frac{n^2N^2}{9}\right) \cos\frac{nN\theta}{3} =$$

$$= \frac{1}{2w^2} \sum_{n=1}^{\infty} f_n\, n^2 \sin nN\theta \Big\{ B_1^2 \sin^2\frac{N\theta}{3} +$$

$$+ 2B_1 \sum_{n'=2}^{\infty} B_{n'} \sin\frac{N\theta}{3} \sin\frac{n'N\theta}{3} +$$

$$+ 2B_1 \sum_{n'=2}^{\infty} A_{n'} \sin\frac{N\theta}{3} \cos\frac{n'N\theta}{3} \Big\} \qquad (17)$$

$$R.H.S. = \frac{1}{2\omega^2} \sum_{n=1}^{\infty} f_n n^2 \left\{ \frac{B_1^2}{2} \left[\sin N\theta - \tfrac{1}{2} \sin\left(\frac{3n+2}{3}\right) N\theta - \tfrac{1}{2} \sin\left(\frac{3n-2}{3}\right) N\theta \right] \right.$$

$$+ \frac{B_1}{2} \sum_{n'=2}^{\infty} B_{n'} \left\{ \sin(3n+n'-1)\frac{N\theta}{3} + \sin(3n-n'+1)\frac{N\theta}{3} + \right.$$

$$\left. - \sin(3n+n'+1)\frac{N\theta}{3} - \sin(3n-n'-1)\frac{N\theta}{3} \right\}$$

$$+ \frac{B_1}{2} \sum_{n'=2}^{\infty} A_{n'} \left\{ \cos(n'-3n+1)\frac{N\theta}{3} + \cos(n'-3n-1)\frac{N\theta}{3} + \right.$$

$$\left. - \cos(n'+3n+1)\frac{N\theta}{3} - \cos(n'+3n-1)\frac{N\theta}{3} \right\} \Bigg\}$$

(18)

Thus equating terms of the same frequency:

$$A_n \left(\nu_n^2 - \frac{n^2 N^2}{9} \right) = 0 \qquad n \neq 1$$

(provided $\frac{n-2}{3}$, $\frac{n+2}{3}$ are integers)

$$B_n \left(\nu_n^2 - \frac{n^2 N^2}{9} \right) = -\frac{B_1^2}{8\omega^2} \left\{ f_{\frac{n-2}{3}} \left(\frac{n-2}{3}\right)^2 + f_{\frac{n+2}{3}} \left(\frac{n+2}{3}\right)^2 \right.$$

$n \neq 1$

$$\left(\nu_x^2 - \frac{N^2}{9}\right) B_1 = \frac{1}{2w^2}\left(-\tfrac{1}{2} f_1 \frac{B_1^2}{2}\right) +$$

$$+ \frac{B_1}{4w^2} \sum_{n=1}^{\infty} f_n\, n^2 \left\{ B_{2-3n} + B_{3n} - B_{-3n} - B_{3n-2} \right\}$$

(19)

where in the last two equations we have ignored terms of order $(B_1 A_n f_n)$. and $(B_1 A_n f_n)$. Rewritting;

$$\left(\nu_x^2 - \frac{N^2}{9}\right) = -\frac{f_1}{8w^2} B_1 + \frac{1}{4w^2} \sum_{n=1}^{\infty} f_n\, n^2 \left\{ B_{3n} - B_{3n-2} \right\}$$

$$B_n = \frac{-B_1^2}{8w^2\left[\nu_x^2 - \frac{n^2 N^2}{9}\right]} \left\{ f_{\frac{n-2}{3}} \left(\frac{n-2}{3}\right)^2 + f_{\frac{n+2}{3}} \left(\frac{n+2}{3}\right)^2 \right\}$$

since $B2\text{-}3_n$ and $B\text{-}3_n$ never contribute. We now obtain a single equation for B_1:

$$\left(\nu_x^2 - \frac{N^2}{9}\right) = -\frac{f_1}{3w^2} B_1 - \frac{B_1^2}{32 w^4} \sum_{n=1}^{\infty} f_n \, n^2 \left\{ \cdots \right\}$$

$$\left\{ \cdots \right\} = \frac{f_{n-2/3}\left(n-\frac{2}{3}\right)^2}{\left[\nu_x^2 - n^2 N^2\right]} + \frac{f_{n+2/3}\left(n+\frac{2}{3}\right)^2}{\left[\nu_x^2 - n^2 N^2\right]} - \frac{f_{n-4/3}\left(n-\frac{4}{3}\right)^2}{\left[\nu_x^2 - \frac{(3n-2)^2 N^2}{9}\right]} +$$

$$- \frac{f_n \, n^2}{\left[\nu_x^2 - \frac{(3n-2)^2 N^2}{9}\right]} \qquad (21)$$

However, since the subscripts of f must be integers, only the last term contributes, and one obtains:

$$\left(\nu_x^2 - \frac{N^2}{9}\right) = -\frac{f_1}{8w^2} B_1 + \frac{B_1^2}{32 w^4} \sum_{n=2}^{\infty} \frac{f_n^2 \, n^4}{\left[\nu_x^2 - \frac{(3n-2)^2 N^2}{9}\right]} \qquad (22)$$

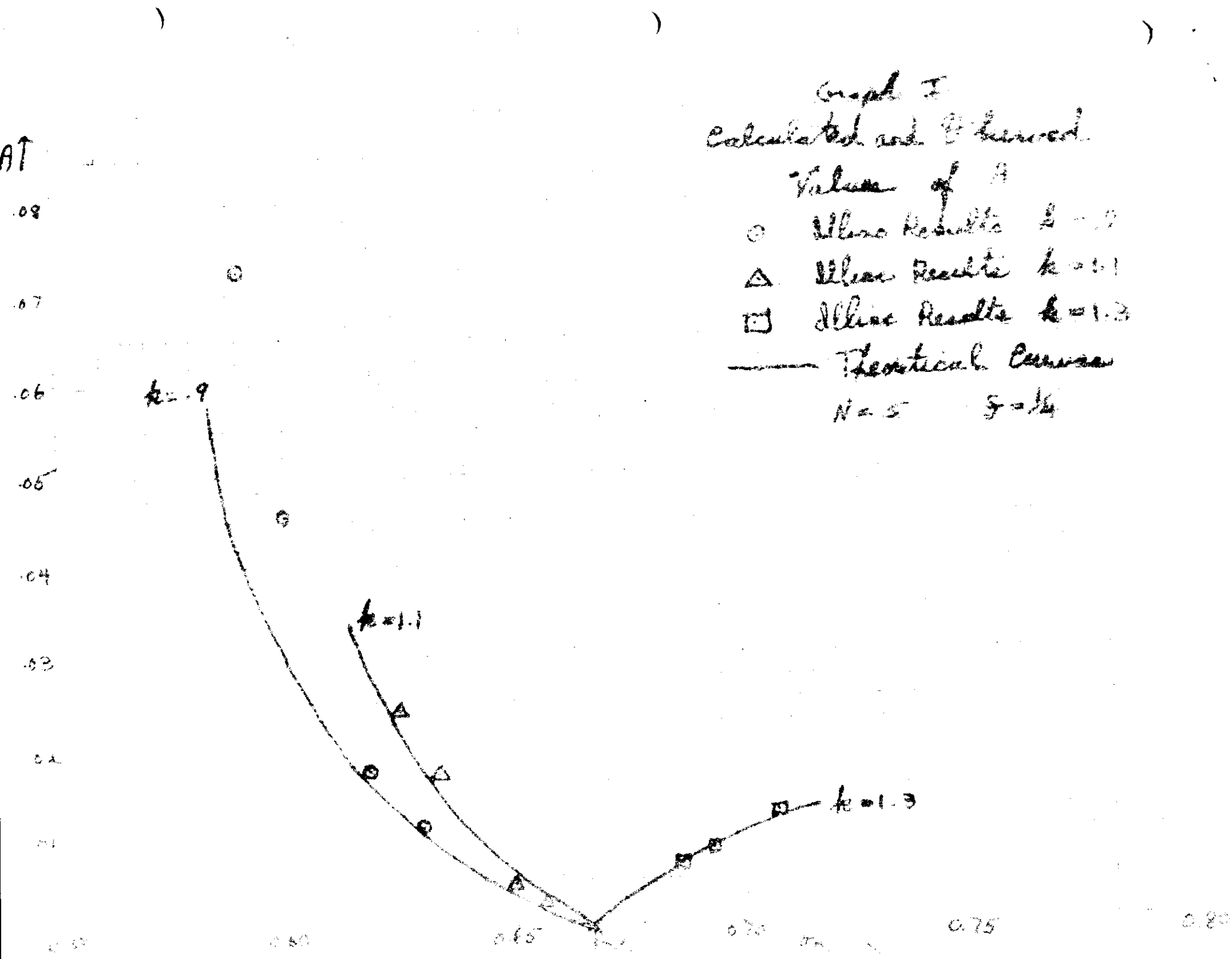
We note that the last term is small, and hence we obtain the approximate formula for B_1:

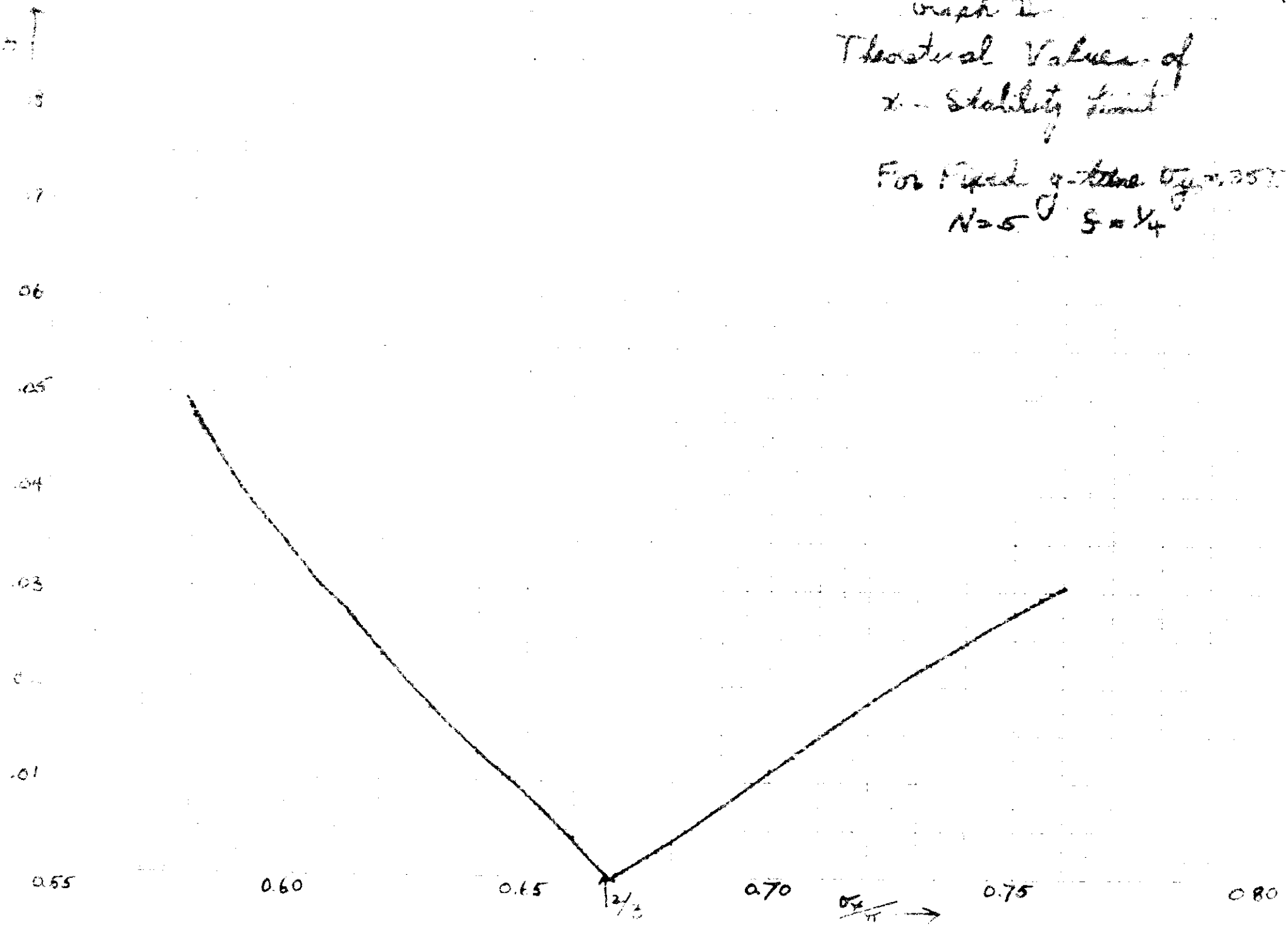
$$B_1 = \frac{8w^2}{f_1}\left(\frac{N^2}{9} - \nu_x^2\right) + \frac{16 w^2}{f_1^3}\left[\frac{N^2}{9} - \nu_x^2\right]^2 \sum_{n=2}^{\infty} \frac{f_n^2 \, n^4}{\left[\nu_x^2 - \frac{(3n-2)^2 N^2}{9}\right]}$$

This is approximately:

$$B_1 \simeq \frac{8w^2}{f_1}\left(\frac{N^2}{9} - \nu_x^2\right)\left\{1 - \frac{2}{f_1^2 N^2}\left(\frac{N^2}{9} - \nu_x^2\right)\sum_{n=2}^{\infty} f_n^2 \, n^2\right\} \qquad (23)$$

For contemplated structures the second term makes very little change in B_1, due to the weighting of higher f_n terms.


Graph I
Calculated and Observed
Values of A
Illinois Results k = .9
Illinois Results k = 1.1
Illinois Results k = 1.3
Theoretical Curves
N = 5 f = 1/4
A
.08
.07
.06
.05
.04
.03
.02
.01
k = .9
k = 1.1
k = 1.3
0.60
0.65
0.70
0.75
0.80


Graph II
Theoretical Values of
x - Stability Limit
For Fixed y-time σ_y = .357
N = 5 S = 1/4
.06
.05
.04
.03
.01
0.55
0.60
0.65
2/3
0.70
0.75
0.80
σ_x/π