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EXPANSIONS OF THE CHARACTERISTIC EXPONENTS AND THE FLOQUET SOLUTIONS FOR THE LINEAR HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS.

118 48 Expansions of the Characteristic Exponents and the Floquet Solutions for the Linear Homogeneous Second Order Differential Equation with Periodic Coefficients.

Nils Vogt-Nilsen MURA/NVN/3 5/1956

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Introduction.

From time to time during the study of free betatron oscillations the need for relatively simple formulas expressing the characteristic exponents and Floquet solutions of the extended Hill equation has been felt. In the following an attempt is made to meet this need. It is believed that in most cases the convergence of the derived expansions will be rapid enough to satisfy the demands of practical calculations.

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*I.C.A. Fellow from the Norwegian Institute of Technology, Trondheim, Norway.

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The general linear homogeneous second order differential equation with periodic coefficients of period T

$$(1)\qquad \begin{aligned} &x'' + P(t)x' + Q(t)x = 0 \\ &P(t+T) = P(t)\ , \quad Q(t+T) = Q(t) \end{aligned}$$

may always, by the substitution

$$(2)\qquad x = y\, e^{-\frac{1}{2}\int_0^t P(\alpha)\,d\alpha}\ ,$$

be transformed to the following standard form, which will serve as a base for our expansions:

$$(3)\qquad y'' + \left[\omega^2 + m(t)\right] y = 0.$$

Here

$$(4)\qquad \begin{aligned} &\omega^2 + m(t) = Q(t) - \frac{1}{2}P'(t) - \frac{1}{4}P(t)^2, \\ &m(t+T) = m(t)\ , \quad \overline{m(t)} = 0, \end{aligned}$$

where the bar denotes the mean value over one period T.

The Floquet solutions to this equation will be written of the form:

$$(5)\qquad \begin{aligned} &y_1(t) = e^{i\frac{\sigma}{T}t}\,\phi_1(t)\ , \quad y_2(t) = e^{-i\frac{\sigma}{T}t}\,\phi_2(t) \\ &\phi_1(t+T) = \phi_1(t)\ , \qquad \phi_2(t+T) = \phi_2(t) \end{aligned}$$

Here $\pm i\frac{\sigma}{T}$ are the characteristic exponents, i.e. σ as usual denotes the phase advance of the oscillation per period T, and $\phi_1(t)$, $\phi_2(t)$ are the two periodic Floquet functions. The two solutions (5) will be linearly independent if and only if $\cos\sigma \neq \pm 1$. In the case of $\cos\sigma = \pm 1$ there exists a solution

$$
(6) \qquad \begin{aligned} y_3(t) &= e^{i\frac{\sigma}{T}t}\left[\frac{t}{T}\phi_1(t) + \phi_3(t)\right], \\ \phi_3(t+T) &= \phi_3(t), \end{aligned}
$$

which is linearly independent with the solutions (5).

The corresponding Floquet solutions to the general eq. (1) are easily written down by using eq. (2):

$$
(7) \qquad \begin{aligned} x_1(t) &= e^{\left(-\frac{1}{2}\overline{P(t)} + i\frac{\sigma}{T}\right)t}\,\psi_1(t) \\ x_2(t) &= e^{\left(-\frac{1}{2}\overline{P(t)} - i\frac{\sigma}{T}\right)t}\,\psi_2(t) \\ x_3(t) &= e^{\left(-\frac{1}{2}\overline{P(t)} + i\frac{\sigma}{T}\right)t}\left[\frac{t}{T}\psi_1(t) + \psi_3(t)\right] \\ \psi_k(t) &= \psi_k(t+T) = \phi_k(t)\, e^{\frac{1}{2}\int_0^t \left[\overline{P(t)} - P(\alpha)\right]d\alpha}, \quad (k=1,2,3), \end{aligned}
$$

where again $x_1(t)$, $x_2(t)$ are linearly independent if and only if $\cos\sigma \neq \pm 1$, while the solution $x_3(t)$ exists only for $\cos\sigma = \pm 1$ and is then linearly independent with $x_1(t)$ and $x_2(t)$. Note that the first derivative term in eq. (1) will not produce any damping or anti-damping if its coefficient P (t) has zero mean value over the period T.

Outline of General Method.

Let

$$
(8) \qquad \begin{aligned} y &= \eta_1(t), \quad \eta_1(0) = 1, \quad \eta_1'(0) = 0 \\ y &= \eta_2(t), \quad \eta_2(0) = 0, \quad \eta_2'(0) = 1 \end{aligned}
$$

be two linearly independent solutions to the standard differential equation (3). The general solution and its first derivative are then given by

$$Y(t) = M(t)\, Y(0)$$

(9) $$Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} \eta_1(t) \,, & \eta_2(t) \\ \eta_1'(t) \,, & \eta_2'(t) \end{pmatrix}.$$

By well known results of linear theory one obtains $\cos\sigma$ as half the trace of the matrix $M(T)$:

(10) $$\cos\sigma = \frac{1}{2}\left[\eta_1(T) + \eta_2'(T)\right].$$

Due to the periodicity of the function $M(t)$ the following relation is easily established:

(11) $$M(T) = M(-T)^{-1}$$

or written out

(12) $$\begin{pmatrix} \eta_1(T) \,, & \eta_2(T) \\ \eta_1'(T) \,, & \eta_2'(T) \end{pmatrix} = \begin{pmatrix} \eta_2'(-T) \,, & -\eta_2(-T) \\ -\eta_1'(-T) \,, & \eta_1(-T) \end{pmatrix}.$$

Hence we also have

(13) $$\cos\sigma = \frac{1}{2}\left[\eta_1(T) + \eta_1(-T)\right].$$

Knowing the value of σ, we may proceed to determine the periodic Floquet functions. By eqs. (5), (8) there exists a pair of constants A_1, B_1, not both zero, such that

(14) $$A_1\,\eta_1(t) + B_1\,\eta_2(t) = e^{i\frac{\sigma}{T}t}\,\phi_1(t).$$

Since $\phi_1(t)$ is periodic these constants may be determined

by inserting $t = 0, T$:

$$A_1 = \phi_1(0)$$

$$A_1 \eta_1(T) + B_1 \eta_2(T) = e^{i\sigma} \phi_1(0)$$

Choosing

(15) $$\phi_1(0) = \eta_2(T)$$

one obtains

$$A_1 = \eta_2(T)$$
$$B_1 = e^{i\sigma} - \eta_1(T)$$

and, by eq. (14):

(16) $$\phi_1(t) = e^{-i\frac{\sigma}{T}t} \left\{ \eta_2(T)\eta_1(t) + \left[e^{i\sigma} - \eta_1(T)\right]\eta_2(t) \right\}.$$

In exactly the same manner the second Floquet function is obtained:

(17) $$\phi_2(t) = e^{i\frac{\sigma}{T}t} \left\{ \eta_2(T)\eta_1(t) + \left[e^{-i\sigma} - \eta_1(T)\right]\eta_2(t) \right\}.$$

In the cases where $\cos\sigma = \pm 1$ one must use the third solution (6). Then there existis a pair of constants A_2, B_2, not both zero, such that

(18) $$A_2 \eta_1(t) + B_2 \eta_2(t) = e^{i\frac{\sigma}{T}t} \left[\frac{t}{T}\phi_1(t) + \phi_3(t) \right],$$

where $\phi_1(t)$ is already determined above. Again A_2, B_2 are determined by inserting $t = 0, T$:

$$A_2 = \phi_3(0)$$
$$A_2 \eta_1(T) + B_2 \eta_2(T) = e^{i\sigma} \left[\phi_1(0) + \phi_3(0) \right].$$

Choosing

$$\phi_3(0) = 0,$$

and using eq. (15):

$$A_2 = 0$$
$$B_2 = e^{i\sigma}$$

Hence, by eq. (18):

$$\phi_3(t) = e^{-i\frac{\sigma}{T}t} e^{i\sigma} \eta_2(t) - \frac{t}{T}\phi_1(t). \tag{19}$$

The three Floquet functions determined above are somewhat special in that their initial values at t = 0 are chosen. One realizes that if C_1, C_2, C_3, are constants, C_1 and $C_2 \neq 0$, the following set of Floquet Functions will serve equally well:

$$\begin{aligned} \Phi_1(t) &= C_1 \phi_1(t) \\ \Phi_2(t) &= C_2 \phi_2(t) \\ \Phi_3(t) &= C_1 \phi_3(t) + C_3 \phi_1(t). \end{aligned} \tag{20}$$

<u>A Lemma On Convergence:</u>

In what follows we shall repeatedly make use of the following lemma:

The series of functions of the real variable t

$$y(t) = \sum_{k=0}^{\infty} P_k(t), \quad \text{where}$$

$$P_{k+1}(t) = \int_{\alpha=0}^{t} P_k(\alpha) F(\alpha, t)\, d\alpha, \quad (k = 0, 1, 2, \ldots),$$

will converge absolutely and uniformly in any finite internal $|t| < \tau$, $\tau > 0$ if $|F(\alpha, t)|$ and $|P_0(t)|$ are bounded for $|\alpha|$ and $|t| < \tau$. (21)

Likewise the series $y'(t) = \sum_{k=0}^{\infty} P_k'(t)$ will converge absolutely and uniformly in the same interval if $\left| \frac{\partial F(\alpha, t)}{\partial t} \right|$ is bounded for $|\alpha|$ and $|t| < \tau$.

Proof: Consider the complex function of the real

variable α

$$f(\alpha) = |f(\alpha)| e^{i\mu(\alpha)}, \quad \mu(\alpha) = \text{real}.$$

Then:

$$\left|\int_0^t f(\alpha)\,d\alpha\right| = \left|\int_0^t |f(\alpha)| \cos\mu\alpha\, d\alpha + i\int_0^t |f(\alpha)| \sin\mu(\alpha)\, d\alpha\right|$$

$$\leq \left|\int_0^t |f(\alpha)| \cos\mu(\alpha)\, d\alpha\right| + \left|\int_0^t |f(\alpha)| \sin\mu(\alpha)\, d\alpha\right| \leq 2\left|\int_0^t |f(\alpha)|\, d\alpha\right|.$$

Hence,

$$\left|\int_0^t f(\alpha)\,d\alpha\right| \leq 2\int_0^t |f(\alpha)|\,|d\alpha|. \tag{22}$$

By the conditions called for in the lemma (21) one may set

$$\begin{aligned} &|F(\alpha,t)| < \tfrac{1}{2} N(\tau) \quad \text{for } |\alpha| \text{ and } |t| < \tau, \\ &|P_0(t)| < Q(\tau) \quad \text{for } |t| < \tau, \end{aligned} \qquad \tau > 0 \tag{23}$$

where $N(\tau)$ and $Q(\tau)$ are positive numbers. Then, by (22):

$$|P_{k+1}(t)| = \left|\int_{\alpha=0}^t P_k(\alpha) F(\alpha,t)\, d\alpha\right| \leq 2\int_0^t |P_k(\alpha)|\,|F(\alpha,t)|\,|d\alpha|$$
$$< N(\tau)\int_0^t |P_k(\alpha)|\,|d\alpha|,$$

and one obtains successively:

$$|P_1(t)| < N(\tau)\int_0^t |P_0(\alpha)|\,|d\alpha| < N(\tau)\, Q(\tau)\, |t|$$

$$|P_2(t)| < N(\tau)\int_0^t |P_1(\alpha)|\,|d\alpha|$$
$$< N(\tau)^2 Q(\tau)\int_0^t |\alpha|\,|d\alpha| = N(\tau)^2 Q(\tau)\, \frac{|t|^2}{2}$$

$$|P_3(t)| < N(\tau)\int_0^t |P_2(\alpha)|\,|d\alpha|$$
$$< N(\tau)^3 Q(\tau)\int_0^t \frac{|\alpha|^2}{2}\,|d\alpha| = N(\tau)^3 Q(\tau)\, \frac{|t|^3}{3!}$$

.....

and in general

$$|P_k(t)| < N(\tau)^k Q(\tau)\, \frac{|t|^k}{k!} < N(\tau)^k Q(\tau)\, \frac{\tau^k}{k!} = M_k(\tau), \quad (k = 1, 2, 3, \cdots). \tag{24}$$

But since

$$\lim_{k\to\infty} \frac{M_{k+1}(\tau)}{M_k(\tau)} = \lim_{k\to\infty} \frac{N(\tau)\,\tau}{k+1} = 0$$

for finite τ, the series $\sum_{k=1}^{\infty} M_k(\tau)$ converges, and since each term $M_k(\tau)$ is positive, independent of t and larger than the absolute value of the corresponding term in the series $y(t) = \sum_{k=1}^{\infty} P_k(t)$, the first statement in the lemma is true.

Now, if $\left|\frac{\partial F(\alpha,t)}{\partial t}\right| < \frac{1}{2} S(\tau)$, $S(\tau) > 0$, for $|\alpha|$ and $|t| < \tau$, one obtains from eqs. (22) to (24):

$$|P'_{k+1}(t)| = \left|\frac{d}{dt}\int_{\alpha=0}^{t} P_k(\alpha) F(\alpha,t)\,d\alpha\right| \leq |P_k(t)|\,|F(t,t)|$$

$$+ \left|\int_{\alpha=0}^{t} P_k(\alpha)\frac{\partial F(\alpha,t)}{\partial t}\,d\alpha\right| < \left[\frac{1}{2}N(\tau) + \tau S(\tau)\right] M_k(\tau).$$

Hence, for the same reasons as above, the series $y'(t) = \sum_{k=0}^{\infty} P'_k(t)$ will also converge abosclutely and uniformly in $|t| < \tau$.

Hereby the lemma is proved.

Expansions for $\eta_1(t)$ and $\eta_2(t)$.

If one confines oneself to the case $\omega \neq 0$ and treats the term $m(t)y$ in eq. (3) as a regular forcing term, the method of variation of constants gives

$$y = A\cos\omega t + B\sin\omega t + \frac{1}{\omega}\int_{\alpha=0}^{t} m(\alpha)y(\alpha)\sin\omega(\alpha - t)\,d\alpha, \tag{25}$$

A, B being the arbitrary integretion constants. By an infinite succession of iterations of this expressio one finally ends up with the following series for the general solution to eq. (3):

$$y(t) = \sum_{k=0}^{\infty} P_k(t)$$

(26)
$$P_0(t) = A\cos\omega t + B\sin\omega t$$

$$P_{k+1}(t) = \frac{1}{\omega}\int_{\alpha=0}^{t} m(\alpha)\sin\omega(\alpha-t)\,P_k(\alpha)\,d\alpha\,,\quad (k=0,1,2,\cdots).$$

By the lemma (21) this series will converge absolutely and uniformly in any finite interval $|t|<\tau$ where $|m(t)|$ is bounded. Term by term differentiation gives

(27)
$$y'(t) = \sum_{k=0}^{\infty} P_k'(t)$$

$$P_0'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$P_{k+1}'(t) = -\int_{\alpha=0}^{t} m(\alpha)\cos\omega(\alpha-t)\,P_k(\alpha)\,d\alpha,\quad (k=0,1,2,\cdots),$$

provided that this series converges uniformly in the interval considered. Again this is true by the lemma (21).

Comparing these results with the eqs. (8), it is realized that the basic solutions $\eta_1(t)$, $\eta_2(t)$ are given for bounded $|m(t)|$ by the absolutely and uniformly convergent series:

(28)
$$\eta_1(t) = \sum_{k=0}^{\infty} U_k(t)\,,\quad U_0(t) = \cos\omega t$$

$$U_{k+1}(t) = \frac{1}{\omega}\int_{\alpha=0}^{t} m(\alpha)\sin\omega(\alpha-t)\,U_k(\alpha)\,d\alpha,\quad (k=0,1,2,\cdots).$$

(29)
$$\eta_2(t) = \sum_{k=0}^{\infty} V_k(t)\,,\quad V_0(t) = \frac{1}{\omega}\sin\omega t$$

$$V_{k+1}(t) = \frac{1}{\omega}\int_{\alpha=0}^{t} m(\alpha)\sin\omega(\alpha-t)\,V_k(\alpha)\,d\alpha\,,\quad (k=0,1,2,\cdots).$$

The above expansions for the solutions to eq. (3) will hold for quite arbitrary real or complex $\omega \neq 0$ and $m(t)$. Thus $m(t)$ need not be periodic nor have zero mean value in the above equations.

Similar formulas may be derived for the case $\omega = 0$. However, since the solutions of eq. (3) will be continuous functions of the parameter ω, it will be far simpler to derive the formulas for $\cos \sigma$ and the Floquet functions in this case by performing the limit operation $\omega \to 0$ in the corresponding formulas for the case $\omega \neq 0$.

The periodic function $m(t)$ will usually be given as a Fourier series

$$m(t) = \sum_{k=1}^{\infty} \left(A_k \cos k\Omega t + B_k \sin k\Omega t \right), \quad \Omega = \frac{2\pi}{T}. \tag{30}$$

However, most of the formulas to be derived will gain considerably in simplicity by using the equivalent complex form:

$$m(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} a_k e^{ik\Omega t}, \quad \Omega = \frac{2\pi}{T}. \tag{31}$$

Here:

$$\begin{aligned} a_k &= \tfrac{1}{2}(A_k - iB_k) \\ a_{-k} &= \tfrac{1}{2}(A_k + iB_k) \end{aligned} \qquad (k = 1, 2, 3, \ldots). \tag{32}$$

Including up to second order terms in the coefficients a_k we obtain by simple integration the following approximate formulas for $\eta_1(\pm T), \eta_2(\pm T)$ which occur in the general

formulas for cos σ and the Floquet functions:

$$\eta_1(\pm T) = \cos\omega T \mp \frac{iT\sin\omega T}{2\pi\omega}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{k a_k}{k^2-p^2}$$

$$(33)\qquad - \frac{T^3\sin\omega T}{8\pi^2\omega}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{a_k a_{-k}}{k^2-p^2}$$

$$\mp \frac{iT^3\sin\omega T}{8\pi^3\omega}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\sum_{\substack{l=-\infty\\l\neq 0}}^{+\infty} a_k a_l \frac{(k+l)(k^2+kl+l^2-3p^2)}{(k^2-p^2)(l^2-p^2)((k+l)^2-p^2)}$$

$$+\cdots\cdots,\ \left(\omega\neq 0,\ p=\frac{\omega T}{\pi}\neq \text{integer}\right)$$

$$\eta_2(\pm T) = \frac{1}{\omega}\sin\omega T + \frac{T^2\sin\omega T}{2\pi^2\omega}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{a_k}{k^2-p^2}$$

$$(34)\qquad + \frac{T^3\cos\omega T}{8\pi^2\omega^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{a_k a_{-k}}{k^2-p^2}$$

$$+ \frac{T^4\sin\omega T}{8\pi^4\omega}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\sum_{\substack{l=-\infty\\l\neq 0}}^{+\infty} a_k a_l \frac{k^2+kl+l^2-3p^2}{(k^2-p^2)(l^2-p^2)((k+l)^2-p^2)}$$

$$+\cdots\cdots,\ \left(\omega\neq 0,\ p=\frac{\omega T}{\pi}\neq \text{integer}\right)$$

<u>Expansions for cos σ .</u>

By the eqs. (13), (28) one may now write down the following absolutely convergent series for cos σ when $\omega\neq 0$:

$$\cos\sigma = \frac{1}{2}\left[\eta_1(T)+\eta_1(-T)\right]$$

$$= \frac{1}{2}\cos\omega T + \frac{1}{2\omega}\int_{\alpha=0}^{T} m(\alpha)\sin\omega(\alpha-T)\cos\omega\alpha\, d\alpha$$

$$+ \frac{1}{2\omega^2}\int_{\alpha_1=0}^{T}\int_{\alpha_2=0}^{\alpha_1} m(\alpha_1)m(\alpha_2)\sin\omega(\alpha_1-T)\sin\omega(\alpha_2-\alpha_1)\cos\omega\alpha_2\, d\alpha_1 d\alpha_2$$

$$+\cdots\cdots$$

$$+ \frac{1}{2}\cos\omega T + \frac{1}{2\omega}\int_{\alpha=0}^{-T} m(\alpha)\sin\omega(\alpha+T)\cos\omega\alpha\, d\alpha$$

$$+ \frac{1}{2\omega^2}\int_{\alpha_1=0}^{-T}\int_{\alpha_2=0}^{\alpha_1} m(\alpha_1)m(\alpha_2)\sin\omega(\alpha_1+T)\sin\omega(\alpha_2-\alpha_1)\cos\omega\alpha_2\, d\alpha_1 d\alpha_2+\cdots$$

where the single integrals cancel since $\overline{m(t)} = 0$.

Then, by reversing the signs of the integration variables involved in the last series:

$$
\begin{aligned}
\cos\sigma &= \cos\omega T \\
&+ \frac{1}{2\omega^2}\int_{\alpha_1=0}^{T}\int_{\alpha_2=0}^{\alpha_1}\left[m(\alpha_1)m(\alpha_2)+m(-\alpha_1)m(-\alpha_2)\right]\sin\omega(\alpha_1-T)\sin\omega(\alpha_2-\alpha_1)\cdot \\
&\qquad\qquad \cdot\cos\omega\alpha_2\, d\alpha_1 d\alpha_2 \\
&+ \cdots\cdots \\
&+ \frac{1}{2\omega^k}\int_{\alpha_1=0}^{T}\int_{\alpha_2=0}^{\alpha_1}\cdots\int_{\alpha_k=0}^{\alpha_{k-1}}\left[m(\alpha_1)m(\alpha_2)\cdots m(\alpha_k)+m(-\alpha_1)m(-\alpha_2)\cdots m(-\alpha_k)\right]\cdot \\
&\quad \cdot\sin\omega(\alpha_1-T)\sin\omega(\alpha_2-\alpha_1)\cdots\sin\omega(\alpha_k-\alpha_{k-1})\cos\omega\alpha_k\, d\alpha_1 d\alpha_2\cdots d\alpha_k \\
&+ \cdots\cdots \qquad\qquad (k=2,3,4,\cdots).
\end{aligned}
\tag{35}
$$

Adding the series for $\eta_1(T)$ and $\eta_2'(T)$ according to eq. (10), one obtains the same series in a slightly more compact form:

$$
\begin{aligned}
\cos\sigma &= \cos\omega T \\
&+ \frac{1}{2\omega^2}\int_{\alpha_1=0}^{T}\int_{\alpha_2=0}^{\alpha_1} m(\alpha_1)m(\alpha_2)\sin\omega(\alpha_1-T-\alpha_2)\sin\omega(\alpha_2-\alpha_1)\, d\alpha_1 d\alpha_2 \\
&+ \cdots\cdots \\
&+ \frac{1}{2\omega^k}\int_{\alpha_1=0}^{T}\int_{\alpha_2=0}^{\alpha_1}\cdots\int_{\alpha_k=0}^{\alpha_{k-1}} m(\alpha_1)m(\alpha_2)\cdots m(\alpha_k)\sin\omega(\alpha_1-T-\alpha_k)\cdot \\
&\qquad \cdot\sin\omega(\alpha_2-\alpha_1)\cdots\sin\omega(\alpha_k-\alpha_{k-1})\, d\alpha_1 d\alpha_2\cdots d\alpha_k \\
&+ \cdots\cdots \qquad\qquad (k=2,3,4,\cdots).
\end{aligned}
\tag{36}
$$

Introducing the function $m(t)$ by its Fourier expansion, both series above for $\cos\sigma$ will come out as power series in the Fourier coefficients of $m(t)$. Since this power series is uniquely defined, the two series (35), (36) are identical.

By the expansion (35) it is immediately seen that if $m(t)$ is an odd function about $t = 0$ all multiple integrals

of odd order $k = 3, 5, 7, \ldots\ldots$ will vanish. By shifting the origin of the time scale it is further realized, that if there exists a time t_o such that $n(t)$ is odd about $t = t_o$, then also all odd order multiple integrals will vanish. In such cases (e.g. the Mathieu equation) $\cos\sigma$ is given as a power series containing only even order terms in the Fourier coefficients of $n(t)$.

In the general case where there does not exist a time $t = t_o$ as described above, one will have contributions from odd order terms in the expansion for $\cos\sigma$. Hill's equation and the case $n(t) = B\cos 2t + C\cos 4t$ treated in the Laslett, Snyder, Hutchinson tables are of this category.

However, even in the general case it is easily understood from eq. (35), that only terms which are even in the coefficients B_k, $(k = 1, 2, 3, \ldots)$, of the odd part of $n(t)$ in eq. (30) can contribute to $\cos\sigma$. From this fact it is realized that the even and odd order multiple integrals in eqs. (35), (36) will be respectively even and odd in the coefficients A_k, $(k = 1, 2, 3, \ldots)$ of the even part of $n(t)$. One must bear in mind here that the even and odd parts of $n(t)$ will depend on the choice of origin for the time scale.

Of course the expressions for $\cos\sigma$ must be even in T and ω.

By the eqs. (13), (33) one may write down the following approximate formula for $\cos\sigma$ which includes second order terms in the Fourier coefficients for $n(t)$:

$$\cos\sigma = \frac{1}{2}\left[\eta_1(T) + \eta_1(-T)\right] = \cos\omega T$$

$$(37) \qquad -\frac{T^3 \sin\omega T}{8\pi^2\omega} \sum_{\substack{k=-\infty \\ k\neq 0}}^{k=+\infty} \frac{a_k a_{-k}}{k^2 - p^2} + \ldots\ldots$$

$$\left(\omega \neq 0,\ p = \frac{\omega T}{\pi} \neq \text{any integer } k \text{ for which } a_k a_{-k} \neq 0\right)$$

Introducing the real Fourier coefficients A_k, B_k, ($k = 1, 2, 3, \ldots.$) from eqs. (32) one obtains:

(38) For $\omega \neq 0$, $p = \frac{\omega T}{\pi} \neq$ any integer k for which $A_k^2 + B_k^2 \neq 0$:

$$\cos\sigma = \cos\omega T - \frac{T^3 \sin\omega T}{16\pi^2\omega} \sum_{k=1}^{\infty} \frac{A_k^2 + B_k^2}{k^2 - p^2} + \ldots.$$

(39) For $\omega \neq 0$, $p = \frac{\omega T}{\pi} =$ integer k for which $A_k^2 + B_k^2 \neq 0$:

$$\cos\sigma = (-1)^p \left[1 + \frac{T^2}{32\omega^2}\left(A_p^2 + B_p^2\right)\right] + \ldots.$$

(40) For $\omega = 0$:

$$\cos\sigma = 1 - \frac{T^4}{16\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}\left(A_k^2 + B_k^2\right) + \ldots.$$

The two last formulas are obtained from the first by applying the limits $p \rightarrow$ integer and $\omega \rightarrow 0$ respectively.

The above approximate formulas for $\cos\sigma$ have been derived also by W. Walkinshaw [1] at the Harwell Laboratory and by F. T. Adler and D. Baroncini [2] at the Carnegie Institute of Technology.

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[1] Private communication.

[2] To be published in Nuovo Cimento?

The third and higher order terms in the general $\cos\sigma$ expansion seem to be very complicated. For this reason the cases where higher order terms are needed are most easily treated individually. Below formulas for $\cos\sigma$ including the third order term are given for the case

$$m(t) = B\cos\Omega t + C\cos 2\Omega t\ , \quad \Omega = \frac{2\pi}{T} \tag{41}$$

which is treated in the Laslett, Snyder, Hutchinson tables of April, 1955:

For $\omega \neq 0$, $p = \frac{\omega T}{\pi} \neq 1, 2$:

$$\cos\sigma = \cos\omega T - \frac{T^3 \sin\omega T}{16\pi^2\omega}\left(\frac{B^2}{1-p^2} + \frac{C^2}{4-p^2} + \frac{3B^2C\,T^2}{4\pi^2(1-p^2)(4-p^2)}\right) + \cdots\cdots \tag{42}$$

For $\omega \neq 0$, $p = \frac{\omega T}{\pi} = 1$:

$$\cos\sigma = -\left[1 + \frac{T^4B^2}{32\pi^2}\left(1 + \frac{C\,T^2}{4\pi^2}\right)\right] + \cdots\cdots \tag{43}$$

For $\omega \neq 0$, $p = \frac{\omega T}{\pi} = 2$:

$$\cos\sigma = 1 + \frac{T^4C}{128\,\pi^2}\left(C - \frac{B^2T}{4\pi^2}\right) + \cdots\cdots \tag{44}$$

For $\omega = 0$:

$$\cos\sigma = 1 - \frac{T^4}{16\pi^2}\left(B^2 + \frac{C^2}{4} + \frac{3B^2C\,T^2}{16\pi^2}\right) + \cdots\cdots \tag{45}$$

These approximate formulas for $\cos\sigma$ have been tested against a new and accurate set of tables[3] prepared by numerical integration on the digital computor at the University of Illinois by G. Belford, L. Jackson Laslett and J. N. Snyder. The range of the parameters in these tables are in the interval $[-.5, +.5]$ for ϕ^2 and C and in $[0, 2]$ for B. The result of this test is tabulated below. One will observe that there is an excellent agreement between the approximate and accurate values of $\cos\sigma$ for small values of the parameter B; while for larger B the accurate values are somewhat larger (more positive) than the approximate ones. The largest error in the range tested occurs in the third significant figure.

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[3] To be published as a MURA report.

$T = \pi$

$A = \omega^2$	B	Cos σ					
		C = -.5		C = 0		C = +.5	
		FORMULA (42) or (45)	ACCURATE VALUE	FORMULA (42) or (45)	ACCURATE VALUE	FORMULA (42) or (45)	ACCURATE VALUE
-.5	0	4.5944	4.5945	4.6647	4.6645	4.5944	4.5945
	.4	4.4707	4.4707	4.5298	4.5297	4.4482	4.4482
	.8	4.0996	4.1006	4.1249	4.1263	4.0096	4.0108
	1.2	3.4810	3.4874	3.4502	3.4578	3.2787	3.2858
	1.6	2.6150	2.6366	2.5056	2.5301	2.2553	2.2794
	2.0	1.5016	1.5558	1.2911	1.3512	0.9396	1.0000
0	0	0.9614	0.9616	1.0000	1.0000	0.9614	0.9616
	.4	0.8720	0.8721	0.9013	0.9014	0.8535	0.8537
	.8	0.6037	0.6046	0.6052	0.6065	0.5297	0.5307
	1.2	0.1565	0.1618	0.1117	0.1183	-0.0101	-0.0041
	1.6	-0.4696	-0.4518	-0.5791	-0.5583	-0.7657	-0.7456
	2.0	-1.2746	-1.2296	-1.4674	-1.4167	-1.7373	-1.6866
.5	0	-0.6215	-0.6213	-0.6057	-0.6057	-0.6215	-0.6213
	.4	-0.6846	-0.6845	-0.6764	-0.6763	-0.6998	-0.6996
	.8	-0.8740	-0.8733	-0.8885	-0.8874	-0.9346	-0.9337
	1.2	-1.1897	-1.1853	-1.2420	-1.2365	-1.3259	-1.3210
	1.6	-1.6317	-1.6168	-1.7370	-1.7196	-1.8739	-1.8572
	2.0	-2.1999	-2.1624	-2.3733	-2.3309	-2.5783	-2.5362

Expansions for the Floquet Solutions.

It is a simple matter to write down absolutely and uniformly convergent series for the Floquet Solutions $y_j(t)$, $(j = 1, 2, 3,)$ defined by eqs. (5), (6). By the results (14) to (19) and (26) the special set of Floquet solutions with initial values $y_1(0) = y_2(0) = \eta_2(T)$, $y_3(0) = 0$ are given for $\omega \neq 0$ by the recursion formulas:

$$y_j(t) = \sum_{k=0}^{\infty} Y_{j,k}(t) \qquad (j=1,2,3)$$

$$(46) \qquad Y_{j,k+1}(t) = \frac{1}{\omega}\int_{\alpha=0}^{t} n(\alpha)\sin\omega(\alpha-t)\, Y_{j,k}(\alpha)\, d\alpha \qquad (k=0,1,2,\cdots)$$

$$Y_{1,0}(t) = \eta_2(T)\cos\omega t + \frac{1}{\omega}\left[e^{i\sigma} - \eta_1(T)\right]\sin\omega t$$

$$Y_{2,0}(t) = \eta_2(T)\cos\omega t + \frac{1}{\omega}\left[e^{-i\sigma} - \eta_1(T)\right]\sin\omega t$$

$$Y_{3,0}(t) = (-1)^q \frac{1}{\omega}\sin\omega t, \quad \sigma = q\pi.$$

Similarly, series for the Floquet functions $\phi_k(t)$, $(k = 1, 2, 3)$ are given by inserting the series (28), (29) into the equations (16, (17), (19). These functions may subsequently be Fourier-analyzed if desired. However, in the presence of a fairly complicated function $n(t)$, these compulations are apt to be very involved. There is also the possibility that there exist simpler series for the Floquent solutions and functions that do not depend explicity on the constant σ as do the series (46).

Outline of a different method.[4)]

By introducing the new dependent variable z(t) defined by

$$y = e^{-\int_{t_0}^{t} z(t)dt}, \quad t_0 = \text{arbitrary constant}, \tag{47}$$

into the basic equation (3) one obtains Riccati's differential equation

$$z' = z^2 + \omega^2 + m(t). \tag{48}$$

Hence, the general solution of the basic eq. (3) is intimately connected with the general solution of the first order non-linear Riccati's equation (48).

Now, for some particular solution $z = w(t)$ of Riccati's equation (48) the original dependent function y of eq. (3) will become the Floquet solution $y_1(t)$:

$$y_1(t) = e^{i\frac{\sigma}{T}t}\phi_1(t) = e^{-\int_{t_0}^{t} w(t)dt}, \tag{49}$$

whereby

$$\phi_1(t) = e^{-i\frac{\sigma}{T}t - \int_{t_0}^{t} w(t)dt}$$

It is a simple matter to determine the conditions that must be fulfilled by the particular solution $w(t)$ such that $\phi_1(t)$ becomes periodic with period T and

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4) Derived independently also by Dr. F. I. Cole.

and that σ becomes a constant. First $\phi_1(t+T) = \phi_1(t)$ if

(51) $$-i\frac{\sigma}{T}t - \int_{t_0}^{t} w(t)\,dt = -i\frac{\sigma}{T}(t+T) - \int_{t_0}^{t+T} w(t)\,dt + 2\pi i k \quad (k = \text{integer})$$

whereby

(52) $$\sigma = i\int_{t}^{t+T} w(t)\,dt + 2\pi k,$$

and this is a constant if and only if the integrand w(t) is periodic of period T. Hence, the following statement is true:

If z = w(t) is a periodic solution with period T of Riccati's equation (48), then the constant σ is given by

(53) $$\sigma = i\,T\,\overline{w(t)} + 2\pi k, \quad (k = \text{integer})$$

and the condition for stability is

(54) $$\overline{w(t)} = \text{pure imaginary.}$$

Due to the fact that the Riccati equation is of first order while the original eq. (3) is of second order, there will generally exist two periodic solutions $z = w_j(t)$, $(j = 1,2)$ for each initial condition for z. Corresponding to the two ordinary Floquet solutions (5) one must have

(55) $$\overline{w_1(t)} = -\overline{w_2(t)}.$$

Also by eq. (48):

(56) $$\overline{w_j(t)^2} = -\omega^2, \quad (j = 1,2).$$

A method for obtaining successive qpproximations to the periodic solution w(t) has been developed. A fairly rapid

convergence was supposedly obtained by satisfying the condition (56) at each step of the proceedure. However, a strict mathematical proof of the convergence has not been obtained. In any case the approximations converge more slowly than the series for $\cos\mathcal{C}$ derived earlier in this paper. There might, however, exist more ingenious methods based on eq. (53) that lead to valuable results.