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THE MAGNETIC FIELD AND THE EQUATIONS OF MOTION IN SPHERICAL COORDINATES FOR THE MARK V, FFAG ACCELERATOR

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ABSTRACT

Spherical coordinates are especially suited for the expression of the magnetic field of the Mark V. FFAG accelerator. The magnetic induction is assumed given in the median plane in the form of a series of circular harmonics. Expansions are then given for the magnetic induction, and its vector and scaler potentials on both sides of this plane. An exact solution is given by making use of associated Legendre functions. Certain approximations and series expansions are introduced. A particularly simple approximation is worked out in detail. If used for the equations of motion it is always required that any approximation satisfy the Liouville theorem exactly. The Lagrangian and Hamiltonian equations are derived.

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II. The introduction of certain expansions and approximations. Explicit expression of a very simple approximation, which satisfies the theorem of Liouville.

III. Lagrangian and Hamiltonian equations of motion.

I. The use of spherical coordinates for the exact expression of the magnetic field.

Let $\vec{B}$ represent the magnetic induction vector, V its scalar potential and $\vec{A}$ its vector potential. The equations:

$$\vec{B} = \nabla \times \vec{A} = -\nabla V$$

expressed in spherical coordinates are:

$$\begin{aligned} B &= (\nabla \times \vec{A})_r = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right] = -\frac{\partial V}{\partial r}, \\ B_\theta &= (\nabla \times \vec{A})_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial(rA_\phi)}{\partial r}\right] = -\frac{1}{r}\frac{\partial r}{\partial\theta}, \quad (1.1) \\ B_\phi &= (\nabla \times \vec{A})_\phi = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}. \end{aligned}$$

Now replace θ by $\gamma = \frac{\pi}{2} - \theta$, so that $\frac{\partial}{\partial\theta} = -\frac{\partial}{\partial\gamma}$

Also let $A_\gamma = -A_\theta$, $B_\gamma = -B_\theta$. Then:

$$B_r = (\nabla \times \vec{A})_r = \frac{1}{r\cos\gamma}\left[\frac{\partial A_\gamma}{\partial\phi} - \frac{\partial(\cos\gamma A_\phi)}{\partial\gamma}\right] = -\frac{\partial V}{\partial r}$$

$$B_\gamma = (\nabla \times \vec{A})_\gamma = \frac{1}{r\cos\gamma}\left[\cos\gamma\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right] = -\frac{1}{r}\frac{\partial V}{\partial\gamma} \qquad (1.2)$$

$$B_\phi = (\nabla \times \vec{A})_\phi = \frac{1}{r}\left[\frac{\partial A_r}{\partial\gamma} - \frac{\partial(rA_\gamma)}{\partial r}\right] = -\frac{1}{r\cos\gamma}\frac{\partial V}{\partial\phi}$$

We shall choose the gauge so that $A_r = 0$. In this case:

$$\frac{\partial(rA_\phi)}{\partial r} = -\frac{\partial V}{\partial\gamma}; \qquad \cos\gamma\frac{\partial(rA_\phi)}{\partial r} = \frac{\partial V}{\partial\phi} \qquad (1.3)$$

Define $W = \frac{1}{r}\int dr\, V$. Then one finds:

$$A_\gamma = \frac{1}{\cos\gamma}\frac{\partial W}{\partial\phi}, \quad A_\phi = -\frac{\partial W}{\partial\gamma}; \quad V = \frac{\partial(Wr)}{\partial r} \qquad (1.4)$$

If we assume that V is expressible in the form:

$$V = \sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty} V_{m,n}^{(+)} r^n P_n^m(\zeta) e^{im\phi} + \sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty} V_{m,n}^{(-)} r^{-(n+1)} P_n^m(\zeta) e^{im\phi} \qquad (1.5)$$

and if furthermore $V_{m,0}^{(-)} \doteq 0 \quad (m)$, then $\nabla^2 W = 0$.
This condition is valid for the Mark V. unless k = - 2.

One has assumed for the magnetic field in the median plane:

$$B_r = B_0 \left(\frac{r}{r_0}\right)^k \left\{1 + f \sin\left[k \ln \frac{r}{r_0} - N\phi\right]\right.$$

$$B_r = B_\phi = 0 \tag{1.7}$$

We shall assume the following more general form for this field:

$$B_r = B_0 (1+s)^k \left\{1 + \sum_m^{1-\infty} f_m \sin m\left[k \ln(1+s) - N(\phi - \phi_m)\right]\right.$$

$$B_r = B_\phi = 0 \tag{1.8}$$

where $r = r_0 (1 + s)$, and $\phi_m \ (m)$ are constant phase angles. This field is the pure imaginary part of the following complex field:

$$B_r = B_0 (1+s)^k \left\{i + \sum_m f_m e^{im\left[k \ln(1+s) - N(\phi - \phi_m)\right]}\right.$$

$$B_r = B_\phi = 0 \tag{1.9}$$

The complex field, whose B_r in the median plane given by (1.9), will be represented by

$$W = iW_0(1+s)^{k+1}L_{k+1} + (1+s)^{k+1}\sum_m W_m e^{im[k\ln(1+s) - N(\phi-\phi_m')]}L^{(mN)}_{k+1+imk}$$

$$= iW_0(1+s)^{k+1}L_{k+1} + \sum_m W_m (1+s)^{k+1+imk} e^{-imN(\phi-\phi_m')}L^{(mN)}_{k+1+imk} \qquad (1.10)$$

$$= iW_0(1+s)^{k+1}L_{k+1} + (1+s)^{k+1}\sum_m W_m e^{im[k\ln(1+s) - N(\phi-\phi_m') + \chi_m]}L^{-mN}_{k+1+imk}$$

where $L^{mN}_{k+1+imk}(\sin\gamma)$ satisfies the associated Legendre equation

$$\frac{d^2L^{mN}_{k+1+imk}}{d\gamma^2} - \tan\gamma\frac{dL^{mN}_{k+1+imk}}{d\gamma} + \left[(k+1+imk)(k+2+imk) - \frac{m^2N^2}{\cos^2\gamma}\right]L^{(mN)}_{k+1+imk} = 0 \qquad (1.11)$$

and where $L_{k+1} = L^{(0)}_{k+1}$. In order that

$$B_r = B_\phi = 0 \quad \text{for } \gamma = 0,$$

it is required that

$$L_{k+1}(-\sin\gamma) = -L_{k+1}(\sin\gamma)$$
$$L^{mN}_{k+1+imk}(-\sin\gamma) = -L^{mN}_{k+1+imk}(\sin\gamma) \quad (m) \qquad (1.12)$$

Furthermore let

$$L^{(mN)}_{k+1+imK} = \bar{L}^{(mN)}_{k+1+imK}\, e^{i m \chi_m}$$

$$\frac{d}{d\gamma} L^{(mN)}_{k+1+imK} = \bar{L}^{\prime(mN)}_{k+1+imK}\, e^{im\left[\frac{\pi}{2m} - \chi_m'\right]} \qquad (1.13)$$

where $\bar{L}^{mN}_{k+1+imK}$, $\bar{L}^{\prime mN}_{k+1+imK}$, χ_m, χ_m' are real functions of γ. We may sometimes require that $\chi_m(0) = 0\ (m)$. This can always be achieved by multiplying any solution of (1.11) by a suitable complex constant. The $\frac{\pi}{2m}$ was introduced in (1.13) so that χ_m' would be a small quantity. From (1.4) and (1.10), one finds:

$$V = \frac{\partial(W\psi)}{\partial r} = iW_0(k+2)(1+s)^{k+1} L_{k+1} + \sum_m W_m (k+2+imK)(1+s)^{k+1+imK} e^{-imN(\phi-\phi_m')} L^{(mN)}_{k+1+imK} \qquad (1.14)$$

$$= iW_0(k+2)(1+s)^{k+1} L_{k+1} + (1+s)^{k+1} \sum_m W_m g_m^{(k+2)} e^{im\left[K\ln(1+s) - N(\phi-\phi_m') + \frac{\pi}{2m} - \zeta_m^{(k+2)} + \chi_m\right]} \bar{L}^{(mN)}_{k+1+imK}$$

Where $g_m^{(k+2)} = \sqrt{(k+2)^2 + m^2K^2}$ and $\tan m\left(\frac{\pi}{2m} - \zeta_m^{(k+2)}\right) = \frac{mK}{k+2}$ (1.15)

From (1.2) and (1.25), one finds:

$$B_\gamma = -\frac{1}{r}\frac{\partial V}{\partial \gamma} = \frac{-iW_0}{r_0}(k+2)(1+s)^k \frac{dL_{k+1}}{d\gamma} - \sum_m \frac{W_m}{r_0}[k+2+imK](1+s)^{k+imK} e^{-imN(\phi-\phi_m')} \frac{d}{d\gamma} L^{mN}_{k+1+imK}$$

$$= \frac{-iW_0}{r_0}(k+2)(1+s)^k \frac{dL_{k+1}}{d\gamma} + \sum_m \frac{W_m g_m^{(k+2)}}{r_0}(1+s)^k e^{im\left[K\ln(1+s) - N(\phi-\phi_m') - \zeta_m^{(k+2)} - \chi_m'\right]} \bar{L}^{(mN)}_{k+1+imK} \qquad (1.16)$$

Comparing (1.9) and (1.27), one finds:

$$B_0 = -\frac{W_0}{r_0}(k+2)\,\zeta'_{k+1}$$

$$\text{and } \phi'_m = \phi_m + \frac{\xi_m^{(k+2)} + \chi'_{m,0}}{N} \quad \text{where } \chi'_{m,0} = \chi'_m(0)$$

$$B_0 f_m = \frac{W_m g_m}{r_0}\,\zeta'_{k+1+imK}$$

where $\zeta'_{k+1} = \left[\frac{dL_{k+1}}{d\gamma}\right]_{\gamma=0}$; $\zeta'_{k+1+imK} = \left[\bar{L}'^{mN}_{k+1+imK}\right]_{\gamma=0}$ (1.17)

Solving, we find

$$W_0 = \frac{-B_0 r_0}{\zeta'_{k+1}(k+2)}, \quad W_m = \frac{B_0 r_0}{g_m^{(k+2)}\zeta'_{k+1+imK}} f_m \quad (m)$$

Substituting the above values of W_m and ϕ'_m into (1.10), we find:

$$\frac{W}{B_0 r_0} = -\frac{i(1+s)^{k+1}}{k+2}\frac{L_{k+1}}{\zeta'_{k+1}} + \sum_m \frac{f_m}{g_m^{k+2}}(1+s)^{k+1+imK} e^{im[-N(\phi-\phi'_m)+\xi_m^{(k+2)}+\chi'_{m,0}]}\frac{L^{(mN)}_{k+1+imK}}{\zeta'_{k+1+imK}} \quad (1.18)$$

$$= -\frac{i(1+s)^{k+1}}{k+2}\frac{L_{k+1}}{\zeta'_{k+1}} + (1+s)^{k+1}\sum_m \frac{f_m}{g_m^{(k+2)}} e^{im[K\ln(1+s)-N(\phi-\phi_m)+\xi_m^{(k+2)}+\chi'_{m,0}+\chi_m]}\frac{\bar{L}^{mN}_{k+1+imK}}{\zeta'_{k+1+imK}}$$

We now compute the V and the components of $\vec{B}$ and $\vec{A}$ in terms of the B_0, f_m and $\phi_m(m)$:

From (1.14), we find

$$\frac{V}{B_0 r_0} = -i(1+s)^{k+1}\frac{L_{k+1}}{l'_{k+1}} + \sum_m \frac{f_m}{g_m^{(k+2)}}[k+2+imK](1+s)^{k+1+imK} e^{-im[N(\phi-\phi_m)-\zeta_m^{k+2}-\chi'_{m,0}]}\frac{L^{(mN)}_{k+1+imK}}{l'_{k+1+imK}}$$

$$= -i(1+s)^{k+1}\frac{L_{k+1}}{l'_{k+1}} + i(1+s)^{k+1}\sum_m f_m e^{im[K\ln(1+s)-N(\phi-\phi_m)+\chi_m+\chi'_{m,0}]}\frac{L^{mN}_{k+1+imK}}{l'_{k+1+imK}} \qquad (1.19)$$

From (1.16), we find:

$$\frac{B_r}{B_0} = \frac{i(1+s)^k}{l'_{k+1}}\frac{dL_{k+1}}{d\gamma} + (1+s)^k\sum_m f_m e^{im[K\ln(1+s)-N(\phi-\phi_m)+\chi'_{m,0}-\chi'_m]}\frac{L^{(mN)}_{k+1+imK}}{l'_{k+1+imK}} \qquad (1.20)$$

From (1.2) and (1.19), one finds

$$\frac{B_r}{B_0} = i(k+1)(1+s)^k\frac{L_{k+1}}{l'_{k+1}} + (1+s)^k\sum_m f_m g_m^{(k+1)} e^{im[K\ln(1+s)-N(\phi-\phi_m)+\chi_m+\chi'_{m,0}-\zeta^{k+1}]}\frac{L^{(mN)}_{k+1+imK}}{l'_{k+1+imK}}$$

$$(1.21)$$

From (1.2) and (1.19) we find

$$\frac{B_\phi}{B_0} = -\frac{1}{(1+s)\cos\gamma}\frac{\partial}{\partial\phi}\left(\frac{V}{B_0 r_0}\right) = -\frac{N(1+s)^k}{\cos\gamma}\sum_m m f_m e^{im[k\ln(1+s)-N(\phi-\phi_m)+\lambda_m+\lambda'_{m0}]}\frac{L^{mN}_{k+1+imk}}{l'_{k+1+imk}} \tag{1.22}$$

From (1.4) and (1.18) we find

$$\frac{A_\gamma}{B_0 r_0} = \frac{1}{\cos\gamma}\frac{\partial}{\partial\phi}\left(\frac{W}{B_0 r_0}\right) = \frac{-iN(1+s)^{k+1}}{\cos\gamma}\sum_m \frac{m f_m}{g_m^{(k+2)}} e^{im[k\ln(1+s)-N(\phi-\phi_m)+\lambda_m+\lambda'_{m0}+\zeta_m^{k+2}]}\frac{L^{(mN)}_{k+1+imk}}{l'_{k+1+imk}} \tag{1.23}$$

In like manner, we find

$$\frac{A_\phi}{B_0 r_0} = -\frac{\partial}{\partial\gamma}\left(\frac{W}{B_0 r_0}\right)$$

$$= \frac{i(1+s)^{k+1}}{k+2}\frac{1}{l'_{k+1}}\frac{d}{d\gamma}L_{k+1} - i(1+s)^{k+1}\sum_m \frac{f_m}{g_m^{(k+2)}} e^{im[k\ln(1+s)-N(\phi-\phi_m)-\lambda'_m+\lambda'_{m0}+\zeta_m^{k+2}]}\frac{L'^{(mN)}_{k+1+imk}}{l'_{k+1+imk}} \tag{1.24}$$

In order to find the field corresponding to the real field (1.8) in the median plane, we take the pure imaginary,

parts of the above expression, and find:

$$\frac{W}{B_0 r_0} = -\frac{(1+s)^{k+1}}{k+2}\frac{L_{k+1}}{l'_{k+1}} + (1+s)^{k+1}\sum_m \frac{f_m}{g_m^{k+2}} \sin m\left\{K\ln(1+s) - N(\phi-\phi_m) + \xi_m^{(k+2)} + \lambda'_{m,0} + \lambda_m\right\} \frac{\bar{L}^{mN}_{k+1+imK}}{l'_{k+1+imK}}$$

$$\frac{V}{B_0 r_0} = -(1+s)^{k+1}\frac{L_{k+1}}{l'_{k+1}} + (1+s)^{k+1}\sum_m f_m \cos m\left\{K\ln(1+s) - N(\phi-\phi_m) + \lambda'_{m,0} + \lambda_m\right\} \frac{\bar{L}^{mN}_{k+1+imK}}{l'_{k+1+imK}}$$

$$\frac{B_\gamma}{B_0} = \frac{(1+s)^k}{l'_{k+1}}\frac{dL_{k+1}}{d\gamma} + (1+s)^k\sum_m f_m \sin m\left\{K\ln(1+s) - N(\phi-\phi_m) + \lambda'_{m,0} - \lambda'_m\right\} \frac{\bar{L}'^{(mN)}_{k+1+imK}}{l'_{k+1+imK}}$$

$$\frac{B_r}{B_0} = (k+1)(1+s)^k\frac{L_{k+1}}{l'_{k+1}} + (1+s)^k\sum_m f_m g_m^{(k+1)} \sin m\left\{K\ln(1+s) - N(\phi-\phi_m) + \lambda_m + \lambda'_{m,0} - \xi_m^{k+1}\right\} \frac{\bar{L}^{(mN)}_{k+1+imK}}{l'_{k+1+imK}}$$

$$\frac{B_\phi}{B_0} = -\frac{N(1+s)^k}{\cos\gamma}\sum_m m f_m \sin m\left\{K\ln(1+s) - N(\phi-\phi_m) + \lambda_m + \lambda'_{m,0}\right\} \frac{\bar{L}^{(mN)}_{k+1+imK}}{l'_{k+1+imK}}$$

$$\frac{A_\phi}{B_0 r_0} = \frac{(1+s)^{k+1}}{(k+2)\,l'_{k+1}}\frac{d}{d\gamma}L_{k+1} - (1+s)^{k+1}\sum_m \frac{f_m}{g_m^{(k+2)}} \cos m\left\{K\ln(1+s) - N\phi - \phi_m + \lambda'_{m,0} - \lambda'_m + \xi_m^{k+2}\right\} \frac{\bar{L}'^{(mN)}_{k+1+imK}}{l'_{k+1+imK}}$$

$$\frac{A_\gamma}{B_0 r_0} = -\frac{N(1+s)^{k+1}}{\cos\gamma}\sum_m \frac{m f_m}{g_m^{(k+2)}} \cos m\left\{K\ln(1+s) - N(\phi-\phi_m) + \lambda_m + \lambda'_{m,0} + \xi_m^{k+2}\right\} \frac{\bar{L}^{mN}_{k+1+imK}}{l'_{k+1+imK}}$$

(1.23)

The above expressions are exact. However, approximations may afterwards be introduced through $L^{mN}_{k+1+imK}$. It may be desired to use these approximations in the equations of motion, in such a way that the theorem of Liouville is satisfied. In order to achieve this, the same or different approximations may be made in each of the A_r, A_γ and A_ϕ, but after that the B_r, B_θ and B_ϕ must be recomputed by substituting these approximate values of A_r, A_γ and A_ϕ in the exact relations (1.2).

II. The introduction of certain expansions and approximations.

Explicit expression of a very simple approximation, which satisfies the theorem of Liuoville

In the previous section, exact expressions have been found for the magnetic induction, and its scalar and vector potentials for the Mark V. accelerator. However, these expressions require a knowledge of the function $L^{mN}_{k+1+imK}(\sin\gamma)$ which satisfies the associated Legendre equation:

$$\frac{d^2}{d\gamma^2}L^{mN}_{k+1+imK} - \tan\gamma\frac{d}{d\gamma}L^{mN}_{k+1+imK} + \left[(k+1+imK)(k+2+imK) - \frac{m^2N^2}{\cos^2\gamma}\right]L^{mN}_{k+1+imK} = 0$$

$$(m = 0, 1, 2 \cdots\cdots) \qquad (2.1)$$

with the additional condition that

$$L^{mN}_{k+1+imK}(-\sin\gamma) = -L^{mN}_{k+1+imK}(\sin\gamma) \qquad (2.2)$$

This function can be expressed in terms of the associated Legendre functions as follows:

For the case m = 0

$$\overset{0}{L}_{k+1} = P_{k+1}(\sin\gamma) \quad \text{for } k \text{ even}$$
$$\overset{0}{L}_{k+1} = Q_{k+1}(\sin\gamma) \quad \text{for } k \text{ odd} \qquad (2.3)$$

For the case m = 1, 2, 3 - -

$$L^{mN}_{k+1+imK}(\sin\gamma) = P^{mN}_{k+1+imK}(\sin\gamma) - P^{mN}_{k+1+imK}(-\sin\gamma) \qquad (2.4)$$

In spite of the great amount of material in the literature, relating to associated Legendre functions, this material is not directly concerned with the present situation . It has been found best to deal directly with the equation (2.1), for small γ , but not necessarily ($\omega\gamma$), this equation may be solved by successive approximations. Replace γ by $\lambda\gamma$ in (2.1) and let

$$L^{mN}_{k+1+imK} = \overset{(0)}{L}{}^{mN}_{k+1+imK} + \lambda \overset{(1)}{L}{}^{mN}_{k+1+imK} + \lambda^2 \overset{(2)}{L}{}^{mN}_{k+1+imK} + \cdots\cdots \qquad (2.5)$$

Substituting (2.5) in (2.1) and equating, like powers of λ , we find

$$\frac{d^2}{d\gamma^2} L^{(0)mN}_{k+1+imK} + \omega^2 L^{(0)mN}_{k+1+imK} = 0$$

$$\frac{d^2}{d\gamma^2} L^{(1)mN}_{k+1+imK} + \omega^2 L^{(1)mN}_{k+1+imK} = 0 \qquad (2.6)$$

where

$$\omega^2 = (k+1+imK)(k+2+imK) - m^2N^2 = a_1 + i a_2$$

$$a_1 = k_x^2 - \alpha^2 m^2 K^2 \;, \quad a_2 = mKk_+ \qquad (2.7)$$

with

$$k_x^2 = (k+1)(k+2)$$

$$k_+^2 = 2k+3$$

$$\alpha^2 = 1 + \frac{N^2}{K^2}$$

One finds:

$$L^{(0)mN}_{k+1+imK} = \sin \omega\gamma$$

$$L^{(1)mN}_{k+1+imK} = \frac{1}{4}\left[\frac{\gamma}{\omega}\cos\omega\gamma + \gamma^2 \sin\omega\gamma\right]$$

$$L^{(2)mN}_{k+1+imK} = \left[\frac{1}{4\omega}\left(\frac{1}{8} - m^2\right)\gamma^2 + \frac{\gamma^4}{32}\right]\sin\omega\gamma + \qquad (2.8)$$

$$\left[\frac{1}{4\omega^2}\left(\frac{1}{8} - m^2\right)\gamma + \frac{1}{6}\left(\frac{1}{8} + m^2\right)\gamma^3\right]\cos\omega\gamma$$

This expansion appears to be very rapidly convergent when γ is small and $|\omega|$ is large. This is true for the Mark V accelerater.

In the following table are given values of N, k, κ and f that have been suggested in proposals for two large accelerators and one model:

	N	k	κ	f
Proposal No. 1	33	82.5	1320	1/4
Proposal No. 2	38	82.5	273	1.28
Model	5	.8	23	1/4

These values together with equations (2.7) and (2.8) suggest series expansions in inverse powers of mK. Let $\omega = c + ie$, where c and e are real. Then

$$c = \sqrt{\frac{\sqrt{a_1^2 + a_2^2} + a_1}{2}} \; ; \qquad e = \sqrt{\frac{\sqrt{a_1^2 + a_2^2} - a_1}{2}} \qquad (2.9)$$

Expanding in inverse powers of $m\kappa$, we find

$$c = \frac{k_+}{2\alpha}\left[1 + \frac{4k_x^2\alpha^2 - k_+^2}{8\alpha^4 m^2 K^2} + \cdots\cdots\right]$$

$$e = \alpha m K\left[1 - \frac{4k_x^2\alpha^2 - k_+^2}{8\alpha^4 m^2 K^2} + \cdots\cdots\right] \qquad (2.10)$$

In this report, we shall only be concerned with the zeroth order approximation.

$$L^{mN}_{k+1+imK}(\sin\gamma) = \sin\omega\gamma \qquad (2.11)$$

which actually represents a relatively good approximation for the Mark V. We have two cases to consider:

$$(1)\ m=0,\ \omega=k_x,\ L_{k+1}\sin k_x\gamma,\ \frac{dL_{k+1}}{d\gamma}=k_x\cos k_x\gamma \qquad (2.12)$$

$$(2)\ L^{mN}_{k+1+imK} = -i\sin\omega\gamma = \cos c\gamma\sinh e\gamma - i\sin c\gamma\cosh e\gamma$$

$$= \bar{L}^{mN}_{k+1+imK}\, e^{i\chi_m} \qquad (2.13)$$

where

$$\bar{L}^{(mN)}_{k+1+imK} = \sinh e\gamma\sqrt{1+\frac{\sin^2 c\gamma}{\sinh^2 c\gamma}}$$

$$\tan m\chi_m = -\frac{\tan c\gamma}{\tanh e\gamma} \qquad (2.14)$$

The multiplicative constant has been introduced into the solution (2.13) to make X_m (m) small. Expanding the second

equation of (2.14), we find:

$$m\chi_m = -\arctan\frac{c}{e} - \frac{ce}{3}\gamma^2 + \cdots\cdots \qquad (2.15)$$

which suggests that we choose a different multiplicative constant, so that

$$L^{(mN)}_{k+1+imK} = -ie^{i\arctan\frac{c}{e}}\sin\omega\gamma, \text{ so that} \qquad (2.16)$$

$$L^{mN}_{k+1+imK} = \sin e\gamma\sqrt{1+\frac{\sin^2 c\gamma}{\sinh^2 e\gamma}} \text{ as before} \qquad (2.17)$$

$$\text{and } m\chi_m = -\frac{ce}{3}\gamma^2 + \cdots\cdots \qquad (2.18)$$

This at once suggests the approximate but very simple solution:

$$L^{(mN)}_{k+1+imK} = \sinh\alpha mK\gamma$$

$$L^{1(mN)}_{k+1+imK} = \alpha mK\cosh\alpha mK\gamma \qquad (2.19)$$

$$\zeta^{1(mN)}_{k+1+imK} = \alpha mK$$

This approximation consists in omitting first and higher powers of γ and $C\gamma$ in the expansions, but keeping all powers of $e\gamma$. Substituting (2.19) into equations (1.19-1.23), we find

$$\frac{V}{B_0 r_0} = -(1+s)^{k+1}\frac{\sin k_x \gamma}{k_x} + (1+s)^{k+1}\sum_m \frac{f_m}{\alpha m K}\cos m\{K\ln(1+s) - N(\phi-\phi_m)\}\sinh \alpha m K\gamma \tag{2.20}$$

$$\frac{B_r}{B_0} = (1+s)^k \cos k_x \gamma + (1+s)^k \sum_m f_m \sin m\{K\ln(1+s) - N(\phi-\phi_m)\}\cosh \alpha m K\gamma \tag{2.21}$$

$$\frac{B_r}{B_0} = (k+1)(1+s)^k \frac{\sin k_x \gamma}{k_x} + (1+s)^k \sum_m \frac{g_m f_m}{\alpha m K}\cos m\{K\ln(1+s) - N(\phi-\phi_m)\}\sinh \alpha m K\gamma \tag{2.22}$$

$$\frac{B_\phi}{B_0} = -\frac{N(1+s)^k}{\cos\gamma}\sum_m \frac{f_m}{\alpha K}\sin m\{K\ln(1+s) - N(\phi-\phi_m)\}\sinh \alpha m K\gamma \tag{2.23}$$

$$\frac{A_r}{B_0 r_0} = -\frac{N(1+s)^k}{\cos\gamma}\sum_m \frac{f_m}{\alpha K} g_m^{k+2}\cos m\{K\ln(1+s) - N(\phi-\phi_m)\}\sinh \alpha m K\gamma \tag{2.24}$$

$$\frac{A_\phi}{B_0 r_0} = \frac{(1+s)^{k+1}}{k+2}\cos k_x \gamma - (1+s)^{k+1}\sum_m \frac{f_m}{g_m^{k+2}}\cos m\{K\ln(1+s) - N(\phi-\phi_m)\}\cosh \alpha m K\gamma \tag{2.25}$$

$$A_r = 0.$$

If instead of using the above B_r, one computes this quantity using the above values of A_γ , A_ϕ , and A_r , this B_r together with the above expressions for B_r, B_γ , B_ϕ , A_r, A_γ , A_ϕ represent exactly a Liouvillian system. There is a great deal of material that might well be added to this section. Only expansions suitable for values of the constants given in the above mentioned proposals have been considered. For instance, the cases $c > e$ and $c \simeq e$ would require separate consideration.

III. Lagrangian and Hamiltonian Equations of Motion in Spherical Coordinate

Following the report of Nilsen*, we shall derive the equations of motion from Jocobi's principle

$$\delta \int_{(1)}^{(2)} (p\,ds + q\vec{A}\cdot d\vec{s}) = 0 \qquad (3.1)$$

In spherical coordinates (r, γ, ϕ).

$$d\vec{s} = \vec{l}_r dr + \vec{l}_\gamma r\,d\gamma + \vec{l}_\phi r\cos\gamma\, d\phi \qquad (3.2)$$

$$ds = \sqrt{dr^2 + r^2 d\gamma^2 + r^2\cos^2\gamma\, d\phi^2}$$

$$= d\phi\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2} \qquad (3.3)$$

where $\dot{r} = \frac{dr}{d\phi}$, $\dot{\gamma} = \frac{d\gamma}{d\phi}$

Also let $\vec{a} = \frac{q}{p}\vec{A}$, $\vec{b} = \frac{q}{p}\vec{B}$ (3.4)

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Then (2.1) becomes:

$$\delta \int_{(1)}^{(2)} d\phi \, \mathcal{L} = 0 \qquad (3.5)$$

where

$$\mathcal{L} = \sqrt{r^2 \cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2} + a_r \dot{r} + r a_\gamma \dot{\gamma} + r \cos\gamma \, a_\phi \qquad (3.6)$$

Then

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\dot{r}}{\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2}} + a_r$$

$$p_\gamma = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = \frac{r^2\dot{\gamma}}{\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2}} + r a_\gamma \qquad (3.7)$$

The Lagrangian equations are:

$$\frac{d}{d\phi}\left[\frac{\dot{r}}{\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2}} + a_r\right] - \frac{r[\cos^2\gamma + \dot{\gamma}^2]}{\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2}} - \dot{r}\frac{\partial a_r}{\partial r} - \dot{\gamma}\frac{\partial (r a_\gamma)}{\partial r} - \cos\gamma \frac{\partial (r a_\phi)}{\partial r} = 0$$

$$(3.8)$$

$$\frac{d}{d\phi}\left[\frac{r^2\dot{\gamma}}{\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2}} + r a_\gamma\right] + \frac{r^2 \sin\gamma \cos\gamma}{\sqrt{r^2\cos^2\gamma + \dot{r}^2 + r^2\dot{\gamma}^2}} - \dot{r}\frac{\partial a_r}{\partial \gamma} - r\dot{\gamma}\frac{\partial a_\gamma}{\partial \gamma} - r\frac{\partial}{\partial \gamma}\left|\cos\gamma \, a_\phi\right| = 0$$

Making use of equation (1.5) - (1.7) together with the relation

$$\frac{d}{d\phi}(\ ) = \dot{r}\frac{\partial}{\partial r}(\ ) + \dot{\gamma}\frac{\partial}{\partial \gamma}(\ ) + \frac{\partial}{\partial \phi}(\ ),$$

the Lagrangian equations become:

$$\frac{d}{d\phi}\left[\frac{\dot{r}}{\sqrt{r^2\cos^2\gamma+\dot{r}^2+r^2\dot{\gamma}^2}}\right] - \frac{r[\cos^2\gamma+\dot{\gamma}^2]}{\sqrt{r^2\cos^2\gamma+\dot{r}^2+r^2\dot{\gamma}^2}} + r\left[b_\phi\dot{\gamma} - \cos\gamma\, b_\gamma\right] = 0$$

$$\frac{d}{d\phi}\left[\frac{r^2\dot{\gamma}}{\sqrt{r^2\cos^2\gamma+\dot{r}^2+r^2\dot{\gamma}^2}}\right] + \frac{r^2\sin\gamma\cos\gamma}{\sqrt{r^2\cos^2\gamma+\dot{r}^2+r^2\dot{\gamma}^2}} - r\left[\dot{r}\, b_\phi - r\cos\gamma\, b_r\right] = 0 \qquad (3.9)$$

In order to obtain the Hamiltonian, we first express $\dot{r}$ and $\dot{\gamma}$ as functions of p_r and p_γ by making use of (2.7). We find:

$$\dot{r} = \frac{r^2\cos\gamma(p_r - a_r)}{\sqrt{r^2 - r^2(p_r - a_r)^2 - (p_\gamma - r a_\gamma)^2}}$$

$$\dot{\gamma} = \frac{(p_\gamma - r a_\gamma)\cos\gamma}{\sqrt{r^2 - r^2(p_r - a_r)^2 - (p_\gamma - r a_\gamma)^2}} \qquad (3.10)$$

Then

$$H = p_r \dot{r} + p_\gamma \dot{\gamma} - \mathcal{L}$$

$$= -r\cos\gamma\left\{\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}+a_\phi\right\} \quad (3.11)$$

The Hamiltonian equations are:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{r\cos\gamma(p_r-a_r)}{\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}}$$

$$(3.12)$$

$$\dot{\gamma} = \frac{\partial H}{\partial p_\gamma} = \frac{r\cos\gamma(p_\gamma-ra_\gamma)}{\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \cos\gamma\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}+r\cos\gamma\frac{(p_r-a_r)\frac{\partial a_r}{\partial r}+\left(\frac{p_\gamma}{r}-a_\gamma\right)\left(\frac{p_\gamma}{r^2}+\frac{\partial a_\gamma}{\partial r}\right)}{\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}}+\cos\gamma\frac{\partial(ra_\phi)}{\partial r}$$

$$(3.13)$$

$$\dot{p}_\gamma = -\frac{\partial H}{\partial \gamma} = -r\sin\gamma\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}+r\cos\gamma\frac{(p_r-a_r)\frac{\partial a_r}{\partial \gamma}+\left(\frac{p_\gamma}{r}-a_\gamma\right)\frac{\partial a_\gamma}{\partial \gamma}}{\sqrt{1-(p_r-a_r)^2-\left(\frac{p_\gamma}{r}-a_\gamma\right)^2}}+r\frac{\partial}{\partial \gamma}(\cos\gamma\, a_\phi)$$

Introducing the notation of Nilsen:

$$F = \frac{r}{r_0} \frac{1}{\sqrt{1-(p_r-a_r)^2-(p_\varphi-a_\varphi)^2}} \tag{3.14}$$

the Hamiltonian equations become

$$\frac{\dot{r}}{r_0} = F(p_r - a_r)\cos\gamma$$

$$\dot{\gamma} = r_0 F \cos\gamma (p_\varphi - r a_\varphi)$$

$$\dot{p}_r = \frac{r}{r_0 F}\cos\gamma + F\left\{(p_r - a_r)\frac{\partial(r a_r)}{\partial r} + \left(\frac{p_\varphi}{r} - a_\varphi\right)\left(\frac{p_\varphi}{r} + \frac{\partial(r a_\varphi)}{\partial r}\right)\right\}\cos\gamma \frac{\partial(r a_r)}{\partial r}$$

$$\dot{p}_\varphi = -\frac{r^2 \sin\gamma}{r_0 F} + r_0 F \cos\gamma\left\{(p_r - a_r)\frac{\partial a_r}{\partial \varphi} + \left(\frac{p_\varphi}{r} - a_\varphi\right)\frac{\partial a_\varphi}{\partial \varphi}\right\} + r\frac{\partial}{\partial \varphi}(\cos\gamma\, a_\varphi) \tag{3.15}$$