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ADIABATIC DAMPING OF LARGE-AMPLITUDE PHASE OSCILLATIONS IN A LINEAR ACCELERATOR

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Abstract

Formulas and tables are given for calculating the adiabatic damping of large-amplitude phase oscillations, due to changing the accelerating field, phase angle, and proton velocity. These are applied to determining the detailed energy distribution at the output of a conventional linear accelerator, in terms of input parameters.

The possibility is investigated for accomplishing the normal function of a simple buncher by modifications of the initial parameters in the linear accelerator itself, with the hope of capturing 100% of the injected beam; this is shown to work in principle, but to require an excessively long accelerator in practice.

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The energy distribution in the output of a typical 50 Mev proton Linac using a constant accelerating field and a single-cavity buncher is calculated; 69% of all injected protons are stably accelerated, and of these 89% are ejected with energy spread less than ± 1/2%. Multiplying these factors, 61% of all injected protons will be accelerated to 50 Mev ± 1/2%; 47% will be within ± 0.3% in energy.

This calculation takes account only of losses due to phase oscillations, in a perfectly aligned accelerator.

Contents

Part I - Adiabatic Phase Damping

Part II - 100% Phase Acceptance by Varying Linac Parameters

Part III - Detailed Energy Distribution of Protons from a Conventional Linac with Simple Buncher

Part I - Adiabatic damping of large-amplitude phase oscillations.

The phase equation obeyed by protons in a linear accelerator is, as derived in appendix A:

$$\frac{d}{dn}\left(\beta^2 \frac{d\phi}{dn}\right) = \frac{-2\pi W_o \beta \lambda}{MC^2}(\sin\phi - \sin\phi_o) \qquad (1)$$

where n = ordinal no. of drift tube

β = v/c for the protons, varies with n.

ϕ = phase of RF field, measured from zero field time, when proton passes the center of a gap.

ϕ_o = synchronous phase angle.

W_o = max energy gain in Mev per cm of accelerator, averaged over a complete cell of the machine. W_o is closely related to the peak accelerating electric field strength, which is assumed to change slowly with n.

MC^2 = total energy of proton in Mev.

λ = RF wavelength.

This has not been solved generally, but asymptotic solutions are known for small oscillations and adiabatic variation of parameters.* Since we wish to deal with oscillations up to $\pm 180^o$, we must use a large-amplitude treatment.**

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* See Alvarez et al, UCRL 236, p. 23 (1948) and Blewett, Phys. Rev. 88, 1197 (1952).

** A similar treatment was made of synchrotron oscillations by S. C. Wright in Mac 4, SCW - 1.

Much clarity of concept is gained by thinking of phase oscillations in terms of a mechanical analog first suggested by McMillan,* namely the "biased pendulum" shown:

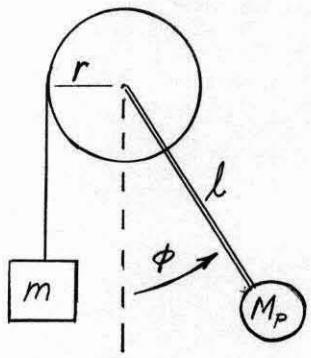

Fig. 1

The differential equation of motion for this system is directly obtained by setting the torque equal to the rate of change of angular momentum:

$$\frac{d}{dt}\left(I\,\frac{d\phi}{dt}\right) = -Mp\,g\ell\,(\sin\phi - \sin\phi_o) \tag{2}$$

where

$$I = Mp\ell^2 + mr^2$$

and

$$\sin\phi_o = mr/Mp\ell$$

so the analogy between (1) and (2) is clear, if we consider β^2 in the linac to be analogous to I of the pendulum system

$$\beta^2 = I \tag{3}$$

and the further analogy

$$2\pi\, W_o\beta\lambda/MC^2 = Mpg\ell \tag{4}$$

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* Edwin M. McMillan, Physical Review 68, 143 (L) (1945).

The phase angle ϕ in the Linac appears identically as the angle of displacement ϕ of the pendulum.

The Adiabatic Theorem says that for adiabatic variation of parameters,

$J = \oint P dq$ is constant during the entire trajectory of a proton, where $\oint$ indicates integration over a single cycle of the motion. In the analog, $P = I\dot{\phi}$, and $q = \phi$, so

$$J = \oint I\dot{\phi} d\phi. \tag{5}$$

We need an expression relating ϕ and $\dot{\phi}$, which is easily obtained by expressing conservation of energy for the pendulum: the potential energy will be formulated to be zero when $\phi = \phi_o$.

Kinetic energy + potential energy = max potential energy.

$$1/2\ I\ \dot{\phi}^2 + Mpg\ell\ (\cos\phi_o - \cos\phi + (\phi_o - \phi)\sin\phi_o) =$$

$$Mpg\ell\ (\cos\phi_o - \cos\phi_m + (\phi_o - \phi_m)\sin\phi_o)$$

where ϕ_m is the extreme swing of the pendulum in the positive direction. This gives

$$\dot{\phi} = \left[\frac{2Mpg\ell}{I}(\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o)\right]^{1/2} \tag{6}$$

Then, substituting $\dot{\phi}$ from (6) into (5),

$$J = [2IMpg\ell]^{1/2} \oint [\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o]^{1/2} d\phi \tag{7}$$

Translating (7) to the Linac case by the use of (3) and (4),

$$J = \left[\frac{4\pi W_o \beta^3 \lambda}{Mc^2}\right]^{1/2} \oint \left[\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o\right]^{1/2} d\phi \quad (8)$$

Calling the integrand Φ, and the integral F, we have

$$\Phi(\phi, \phi_o, \phi_m) \equiv \left[\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o\right]^{1/2}$$

and

$$F(\phi_o, \phi_m) = \oint \Phi(\phi, \phi_o, \phi_m)\, d\phi$$

giving the resultant expression for J

$$J = \left[\frac{4\pi W_o \beta^3 \lambda}{MC^2}\right]^{1/2} F(\phi_o, \phi_m) \quad (9)$$

This is the final form of the equation we will use.

Values of the integral F are calculated numerically and given in table I as a function of ϕ_o and ϕ_m. The results are also plotted in Figs. 2, 3 and 4.

Use of the results then proceeds as follows, to calculate large-angle damping of ϕ_m:

It is assumed that ϕ_o, ϕ_m, W_o, β, and M vary for a particular proton as it goes down the accelerator. If the subscripts 1 and 2 apply to any two stations along the accelerator, then the value of J will be the same at the two points, or

$$J = \left(\frac{4\pi\, W_{o1}\beta_1^3\, \lambda_1}{M_1 C^2}\right)^{1/2} F(\phi_{o1}\phi_{m1}) = \left(\frac{4\pi W_{o2}\beta_2^3\lambda_2}{M_2 C^2}\right)^{1/2} F(\phi_{o2}\phi_{m2})$$

The linacs under consideration are not very relativistic, and λ is constant, so we may simplify the above by saying $\lambda = \lambda_1 = \lambda_2$ and $M_1 = M_2$:

$$W_{o1}^{1/2} \beta_1^{3/2} F(\phi_{o1}\phi_{m1}) = W_{o2}^{1/2}\beta_2^{3/2} F(\phi_{o2}\phi_{m2}) \qquad (10)$$

for small-amplitude oscillations, say $\phi_m - \phi_o < 10^o$, F can be well approximated as:

$$F(\phi_o,\phi_m) \simeq (\phi_m - \phi_o)^2 (1 - 1/3 \tan \phi_o(\phi_m - \phi_o))(\frac{\pi^2}{2} \cos \phi_o)^{1/2}$$

approx. (11)

This approximation was used for the small-amplitude parts of the curves in Figs. 3 and 4. (11) can be further approximated for very small amplitudes to

$$F (\phi_o, \phi_m) \doteq (\phi_m - \phi_o)^2 (\frac{\pi^2}{2} \cos \phi_o)^{1/2} \text{ approx.} \qquad (11.5)$$

Since one is usually more interested in the relative maximum energy deviations $\frac{\Delta Em}{E}$ of the machine than in the phase errors, one needs the relation between energy deviation and phase amplitude: (derived in appendix B)

$$\left(\frac{\Delta Em}{E}\right)^2 = \frac{4\lambda W_o}{\pi\beta MC^2} (\cos \phi_o - \cos \phi_m + (\phi_o - \phi_m) \sin \phi_o) \qquad (12)$$

where ΔE_m is the maximum energy deviation corresponding to ϕ_m. (12) can be approximated for small phase amplitudes to:

$$\left(\frac{\Delta E_m}{E}\right)^2 \simeq \frac{2\lambda W_o}{\pi\beta MC^2} \cos \phi_o (\phi_m - \phi_o)^2 \text{ approx.} \qquad (13)$$

We will now apply this to determining the maximum energy deviation of a 50 Mev linear accelerator of conventional design. Parameters are taken as follows:

E output = 50 Mev

E input = 0.5 Mev

W_o = .018 Mev/cm (constant)

ϕ_o = 60° (constant)

β input = .032

β output = .35

λ = 148 cm

The extreme phase oscillations that can be accepted from the injector are given by $\phi_m = \pi - \phi_o = 120^o$. We will set the above into (10) to find the output value of the phase amplitude:

$$(.018)^{1/2}(.032)^{3/2}F(60^o, 120^o) = (.018)^{1/2}(.35)^{3/2}F(60^o, \phi_{m2})$$

Using the curves in Fig. 3 for values of F, one finds $(\phi_{m2} - \phi_{o2}) = 6.08^o$, or $\phi_{m2} = 66.08^o$. This is a small enough amplitude that one may then use (13) without appreciable error to get the maximum energy deviation:

$$\frac{\Delta E_m}{E} = \left(\frac{2 \times 148 \times .018}{\pi \times .35 \times 938} \cos 60^o\right)^{1/2} \left(\frac{66.08^o - 60^o}{57.3}\right) = .0054 \text{ or } .54\%$$

This is the maximum energy deviation at the 50 Mev end of the accelerator, the majority of the protons having considerably smaller ones. A detailed calculation of the output energy distribution will be made in Part III.

Part II - Phase bunching and 100% phase acceptance by varying Linac parameters.

It is recognized from (10) that if one starts with a low value of W_o one can realize some extra phase damping; also, if one changes ϕ_o, the equilibrium phase angle, so that the accelerator starts with $\phi_o = 0$, no injected protons should be rejected because of phase of arrival; all initial phases should be stable, the extreme ones oscillating with $\pm 180^o$ amplitude. Slowly increasing W_o should cause phase damping, and then ϕ_o can be slowly increased above zero, so some acceleration can take place, until full constant values of W_o and ϕ_o are reached; thus one should have a 100% efficient buncher built into the accelerator itself.

To see how far we can take this, and at what price, let us choose the initial value of W_o such that a 2% maximum energy oscillation* can be contained at injection; putting this into (12) with $\phi_o = 0$, and using other initial parameters as listed, $(.02)^2 = \frac{4\ (148)\ W_o}{\pi(.032)938}\ (1 - 0 + 0)$, calling for an initial field of $W_o = 6.4 \times 10^{-5}$ Mev/cm. Application of (10) then tells what will be the corresponding ϕ_m when 50 Mev is reached, using the parameters:

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* The energy tolerance on the injector must then be considerably smaller than this value or perhaps 0.2%.

$W_{o1} = 6.4 \times 10^{-5}$ $\qquad$ $W_{o2} = .018$ Mev/cm

$\phi_{o1} = 0$ $\qquad$ $\phi_{o2} = 60^{o}$

$\beta_1 = .032$ $\qquad$ $\beta_2 = 0.35$

$\lambda = 148$ cm

This gives $\phi_m = 66.25^{o}$, corresponding to $\frac{\Delta E_m}{E} = \pm .554\%$ (about the same as with a buncher).

To estimate the length of an accelerator as above in which W_o increases a factor of 300 <u>adiabatically</u>, we note that the wave length of phase oscillations is given approximately for small oscillations as

$$\lambda_n = \left(\frac{2\pi\beta MC^2}{W_o \lambda \cos\phi_{o_a}} \right)^{1/2} \tag{14}$$

where λ is measured in numbers of cells of the machine. This makes $\lambda = 300$ cells (= 47 feet) at the beginning and = 40 cells (68 feet) at the end of the machine so that the machine would be many hundreds of feet long to even approach the adiabatic condition by increasing W_o a factor of two in each wave length.

It seems very likely that even if one were willing to go to the expense of building such a long adiabatic machine, the alignment errors in its many drift tubes would excite enough radial oscillations to lose as many protons as would a conventional linac with a simple buncher.

It was thought possible that although the adiabatic case requires an impractically long accelerator, comparable

results might be achieved in an accelerator employing a similar range of variation of W_o, (accelerating field) but allowing the field to increase to full value in only a hundred feet, so that the overall length of the accelerator might be only 150 feet. Numerical trajectories were calculated for several such cases, but in general the amplitudes of oscillation increased instead of damping, and no cases were found which preserved as many protons as a conventional linac with a buncher.

Part III - Detailed energy distribution of protons from a conventional linac with simple buncher.

The maximum energy deviation in the output energy was calculated in Part I and this is independent of whether or not a buncher is used; but it is desirable to know the entire energy distribution, since most protons will deviate much less than the maximum. Using the same parameters listed for the above-mentioned calculation, individual proton histories are traced as follows:

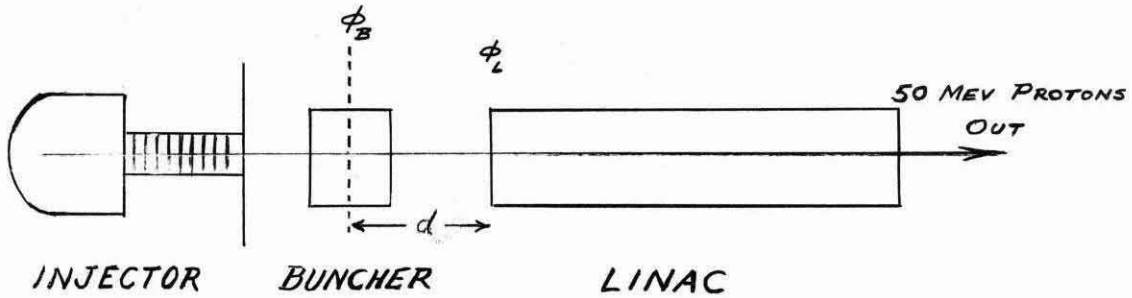

Fig. 5

The master reference for phase is that of the RF $\mathcal{E}$ vector in the Linac tank, i.e., $\phi = 0$ when $\mathcal{E} = 0$, increasing in the tank. Thirty-six protons are traced, one starting out from the injector every ten degrees of phase during one RF cycle. The RF phase when each of these passes the center of the buncher gap will be called ϕ_{Bi} for that proton and ϕ_{Li} will be the phase when each reaches the first gap in the linac. Calling V_o the injector energy and V_1 the maximum energy gain the buncher can give a proton and assuming the RF in the buncher is in phase with the tank, the phase change for a given proton at the Linac input occasioned by the buncher is

$$\Delta\phi_i = -\pi \frac{V_1 d}{V_o \beta_o \lambda} \sin \phi_{Bi} \tag{15}$$

where d is the bunching distance, $C\beta_o$ is the injector velocity, and λ is the RF wave length in vacuo. At the same time, the proton velocity is varied from the injector velocity by an amount

$$\Delta v_i = \frac{C\beta_o}{2} \frac{V_1}{V_o} \sin \phi_{Bi} \tag{16}$$

which causes a rate of phase error in the beginning of the tank

$$\dot{\phi}_{Li} = 2\pi \frac{\Delta v_i}{C\beta_o} = \pi \frac{V_1}{V_o} \sin \phi_{Bi} \tag{17}$$

For each of the 36 protons, then, one calculates the phase of arrival at the linac tank, ϕ_{Li}, and $\dot{\phi}_{Li}$, and these are

plotted in Fig. 6. Also plotted on the same axis are the ϕ vs. $\dot{\phi}$ trajectories typical of the accelerator, so one can determine for each proton the value of ϕ_m for the curve it lands on. Table II then tabulates for each proton its ϕ_m value in the initial Linac; the phase damping is then calculated using the method outlined in Part I, giving the corresponding value of ϕ_m for the proton at the Linac output. From the latter the maximum energy deviation for the proton, $(\Delta E/E)_m$ is calculated by the approximate small-amplitude relation (13) and plotted in Table II.

$$(\Delta E/E)_m \simeq \sqrt{\frac{2W_o \lambda \cos\phi_o}{\pi\beta MC^2}} \quad (\phi_m - \phi_o) \tag{13}$$

The above calculation would give the energy distribution in the Linac output if every proton came out at the extreme phase in its phase oscillation cycle; it must be weighted then to allow for a sinusoidal distribution of energy probability for each proton, with the above values as the amplitude of the sine wave. (Recall that at the output the phase oscillations are damped to an amplitude of $\sim 6^o$, hence they can be regarded as sinusoidal oscillations to good approximation.

Applying such a weighting to the energy distribution results in the final energy distribution for the accelerator, as shown in Fig. 7. The same data are tabulated in Table III. The results show that while the maximum energy deviation is $\pm$ .63%, 68% of the output protons will have energy deviation less than $\pm$ 0.3%.

TABLE II

Proton #	ϕB	ϕ_m ½ Mev	F .5 Mev	F 50 Mev	$(\phi_m - \phi_o)$ Mev	$\Delta E/E$ Max
- 18	- 180°	>120°				
- 17	- 170°	"				
- 16	- 160	"	PROTONS LOST			
- 15	- 150	"				
- 14	- 140	"				
- 13	- 130	101	.470	.0149	5.7°	.54 %
- 12	- 120	67	.020	.00063	1.1	.10
- 11	- 110	71	.050	.0016	1.9	.18
- 10	- 100	86	.152	.0048	3.2	.30
- 9	- 90	98	.426	.0135	5.4	.51
- 8	- 80	106	.540	.0171	6.1	.56
- 7	- 70	112	.616	.0195	6.5	.61
- 6	- 60	114	.640	.0202	6.6	.62
- 5	- 50	113	.628	.0199	6.5	.61
- 4	- 40	110	.572	.0181	6.3	.59
- 3	- 30	105	.528	.0167	6.0	.57
- 2	- 20	98	.426	.0135	5.4	.51
- 1	- 10	89	.285	.0090	4.4	.41
0	0	82	.177	.0056	3.5	.33

TABLE II (continued)

Proton #	ϕB	ϕ_m $\frac{1}{2}$ Mev	F .5 Mev	F 50 Mev	$(\phi_m - \phi_o)$ Mev	$\Delta E/E$ Max
+ 1	10	74	.082	.0026	2.3	.22 %
+ 2	20°	70	.043	.0014	1.7	.16
+ 3	30°	72	.060	.0019	2.0	.19
4	40	77				
5	50	80				
6	60	82				
7	70	80				
8	80	77				
9	90	75				
10	100	83				
11	110	93				
12	120	120				
13	130	>120°	PROTONS LOST			
14	140	>120°				
15	150	>120°				
16	160	>120°				
17	170	>120°				
18	+ 180°	>120°				

TABLE III.

Energy Distribution of 50 Mev Protons from Linac with Simple Bunches

$\Delta E/E$	% of Protons
0 - .05	10.63 %
.05 - .10	11.77
.10 - .15	11.74
.15 - .20	12.61
.20 - .25	10.40
.25 - .30	11.27
.30 - .35	6.54
.35 - .40	4.77
.40 - .45	4.56
.45 - .50	4.61
.50 - .55	4.52
.55 - .60	4.42
.60 - .65	2.16
.65 - .70	0.
Total	100.00 %

TABLE I

VALUES OF $F(\phi_o, \phi_m)$

$\phi_o \downarrow$ / $\phi_m \rightarrow$	180°	170°	160°	150°	140°	130°	120°	110°	100°	90°	80°	70°	60°	50°	40°	30°	20°	10°
0°	11.24	11.08	10.60	10.00	9.13	8.25	7.52	6.64	5.65	4.68	3.67	2.76	2.03	1.41	.94	.586	.264	.070
10°		7.80	7.67	7.38	6.94	6.31	5.73	5.05	4.25	3.45	2.75	2.080	1.45	.872	.571	.265	.067	
20°			5.51	5.44	5.23	4.77	4.22	3.66	3.124	2.52	1.916	1.36	.894	.55	.259	.066		
30°				3.72	3.60	3.32	2.98	2.64	2.15	1.68	1.23	.85	.504	.230	.060			
40°					2.37	2.26	2.07	1.78	1.44	1.07	.744	.456	.214	.0547				
50°						1.34	1.25	1.09	0.87	.616	.387	.187	.052					
60°							.666	.592	.459	.300	.153	.044						
70°								.248	.196	.111	.033							
80°									.0442	.019								

$$J = \oint P\,dq = \left(\frac{4\pi W_o B^3 \lambda}{MC^2}\right)^{1/2} F(\phi_o, \phi_m)$$

Appendix A: Derivation of the Phase Equation for the Linear Accelerator.

Assume that the electric field in the accelerator can supply a peak energy gain of W_o Mev per centimeter of axial distance, averaged over many drift tubes, and that W_o varies slowly with n, the ordinal number of successive drift tubes. L_n is the drift tube repeat length. A proton passing the n'th gap at phase ϕ_n will gain energy

$$\Delta E_n = W_o \sin \phi_n L_n \tag{1}$$

while a proton passing through the gap at phase ϕ_o (design phase) will pick up the designed amount of energy, or

$$\Delta E_{on} = W_o \sin \phi_{on} L_n \tag{2}$$

The change in the proton's excess energy (excess over the design value) will then be

$$\Delta E'_n = (\Delta E_n - \Delta E_{on}) = W_o L_n (\sin \phi_n - \sin \phi_{on}) \tag{3}$$

We now wish to relate this change of excess energy to a change of excess velocity as follows: Assume we are in the non-relativistic region, so we can say

$$E = 1/2\ MC^2 \beta^2 .$$

Let β_{on} be the design velocity, and β'_n the excess velocity at the nth gap. Then $\beta_n = \beta_{on} + \beta'_n$. Similarly let E_{on} be the design energy, and E'_n be the excess energy. $E_n = E_{on} + E'_n$.

(a) $E_{on} = \frac{1}{2} MC^2 (\beta_{on})^2$

(b) $E_{o,n+1} = \frac{1}{2} MC^2 (\beta_{o,n+1})^2$

(c) $E_n = \frac{1}{2} MC^2 (\beta_n)^2 = \frac{1}{2} MC^2 (\beta_{on} + \beta'_n)^2$

(d) $E_{n+1} = \frac{1}{2} MC^2 (\beta_{n+1})^2 = \frac{1}{2} MC^2 (\beta_{o,n+1} + \beta'_{n+1})^2$

The change in excess energy in going across the nth gap will be

$$\Delta E'_n = (E_{n+1} - E_{o,n+1}) - (E_n - E_{o,n})$$

or

$$\Delta E'_n = \frac{1}{2} MC^2 \left[2\beta_{o,n+1}\beta'_{n+1} + (\beta'_{n+1})^2 - 2\beta_{on}\beta'_n - (\beta'_n)^2 \right]$$

Neglecting terms of the order of $(\beta'_n)^2$ since β' is always assumed small compared to β_o, we have

$$\Delta E'_n = \frac{1}{2} MC^2 \left[2\beta_{o,n+1}\beta'_{n+1} - 2\beta_{on}\beta'_n \right].$$

The expression in brackets is just the change in the quantity ($\beta_{on} \cdot \beta'_n$) in passing one gap, which we call $\Delta (\beta_{on} \cdot \beta'_n)$, or

$$\Delta E'_n = MC^2 \quad \Delta (\beta_{on} \cdot \beta'_n) \tag{4}$$

equating (3) and (4),

$$MC^2 \quad \Delta (\beta_{on} \cdot \beta'_n) = W_o L_n (\sin \phi_n - \sin \phi_{on}). \tag{5}$$

Changing from differences to differentials, (5) becomes

$$\frac{d}{dn} (\beta_{on} \cdot \beta'_n) = \frac{W_o L_n}{MC^2} (\sin \phi_n - \sin \phi_{on}) . \tag{6}$$

The phase change in passing from one gap to the next is related to the excess velocity by

$$\frac{d\phi_n}{dn} = - 2\pi \frac{\beta'_n}{\beta_{on}} \tag{7}$$

which gives

$$\beta'_n = - \frac{\beta_{on}}{2\pi} \frac{d\phi_n}{dn} .$$

Setting this into the left-hand member of (6) gives:

$$\frac{d}{dn} (\beta_{on}^2 \cdot \frac{d\phi_n}{dn}) = - \frac{2\pi \; W_{on} \; \beta_{on} \; \lambda}{MC^2} (\sin \phi_n - \sin \phi_{on}) \tag{8}$$

where we have used the synchronous condition $L_n = \beta_{on} \lambda$, where λ is the RF wavelength in free space.

Appendix B: Expression of Energy Oscillation Amplitude in Terms of the Phase Oscillation Amplitude.

Eq. (6) in Part I gives, for the biased Pendulum Analog,

$$\text{(a)} \quad \dot{\phi} = \left[\frac{2\,Mpg\ell}{I}\left(\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o\right)\right]^{\frac{1}{2}}$$

which, translated to the Linac by using (3) and (4), reads

$$\text{(b)} \quad \dot{\phi} = \left[\frac{4\pi\, W_o\, \lambda}{\beta\, MC^2}\left(\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o\right)\right]^{\frac{1}{2}}$$

$\dot{\phi}$ is connected with excess velocity and excess energy as follows:

$$\text{(c)} \quad \dot{\phi} \equiv \frac{d\phi}{dn} = -\,2\pi\,\frac{\Delta\beta}{\beta}$$

and since $E \propto \beta^2$,

$$\frac{\Delta E}{E} = 2\,\frac{\Delta\beta}{\beta}, \quad \text{we get}$$

$$\text{(d)} \quad \dot{\phi} = -\,\pi\,\frac{\Delta E}{E}$$

and substituting (d) into (b), we get

$$\text{(e)} \quad \frac{\Delta E}{E} = \left[\frac{4\,\lambda\, W_o}{\pi\,\beta\, MC^2}\left(\cos\phi - \cos\phi_m + (\phi - \phi_m)\sin\phi_o\right)\right]^{\frac{1}{2}}$$

ΔE has its maximum where $\phi = \phi_o$, so

$$\text{(f)} \quad \left(\frac{\Delta E}{E}\right)_{max} = \left[\frac{4\,\lambda\, W_o}{\pi\,\beta\, MC^2}\left(\cos\phi_o - \cos\phi_m + (\phi_o - \phi_m)\sin\phi_o\right)\right]^{\frac{1}{2}}.$$

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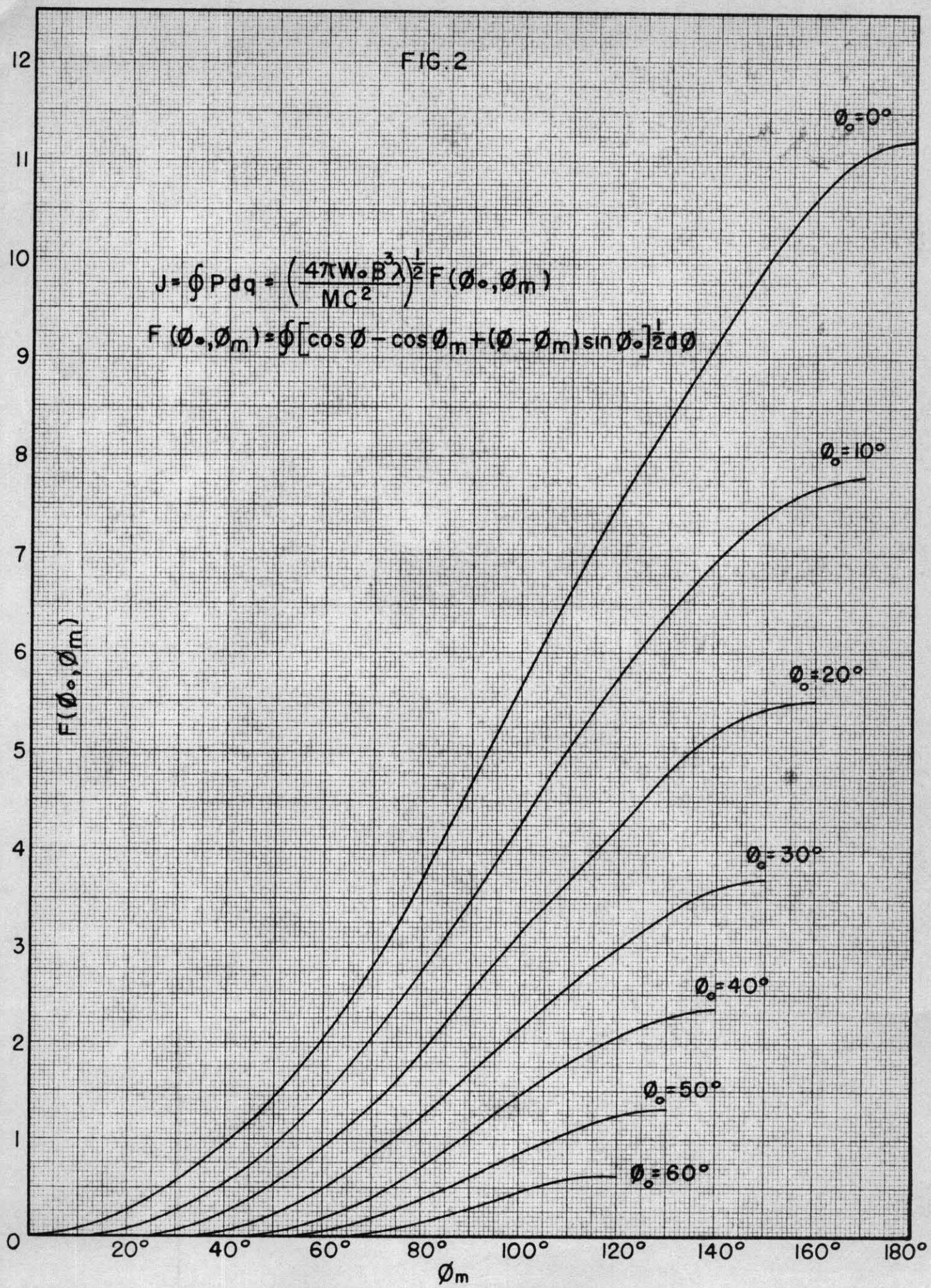
FIG. 2



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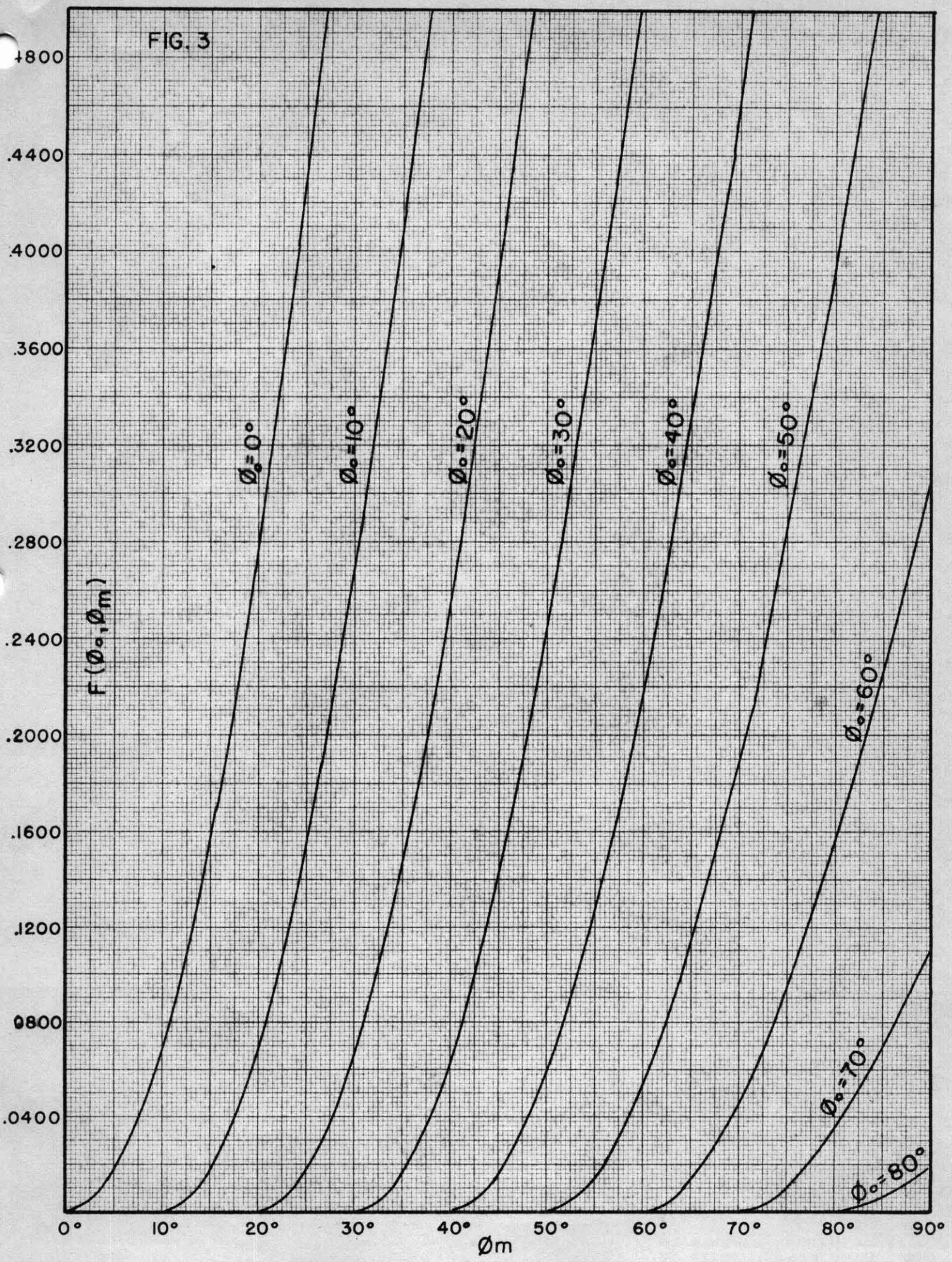
FIG. 3
.4800
.4400
.4000
.3600
.3200
.2800
.2400
.2000
.1600
.1200
.0800
.0400
F(Ø₀, Øm)
Ø₀=0°
Ø₀=10°
Ø₀=20°
Ø₀=30°
Ø₀=40°
Ø₀=50°
Ø₀=60°
Ø₀=70°
Ø₀=80°
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
Øm
10 X 10 TO THE CM. 359-14
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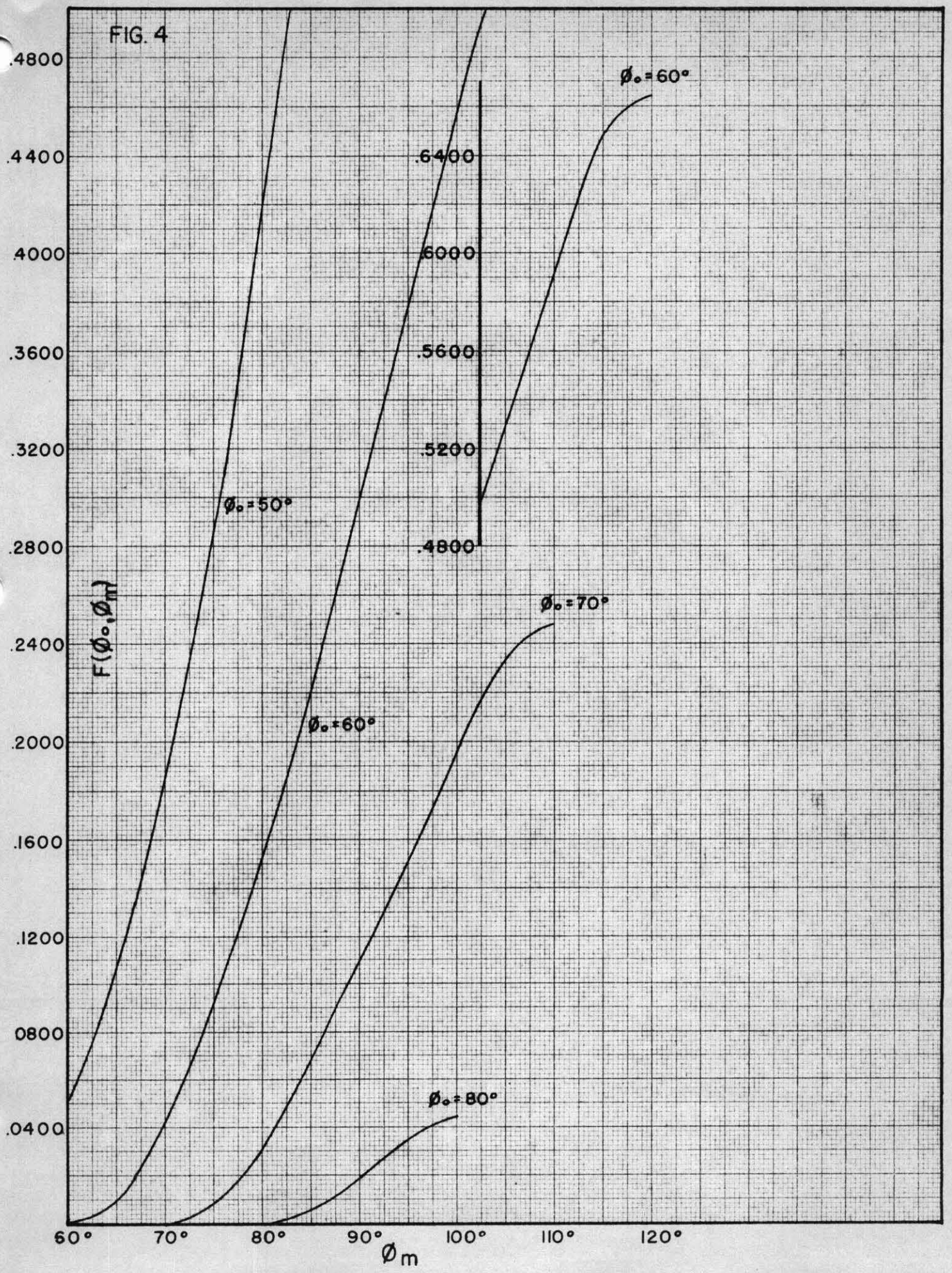

K+E 10 X 10 TO THE CM. 359-14
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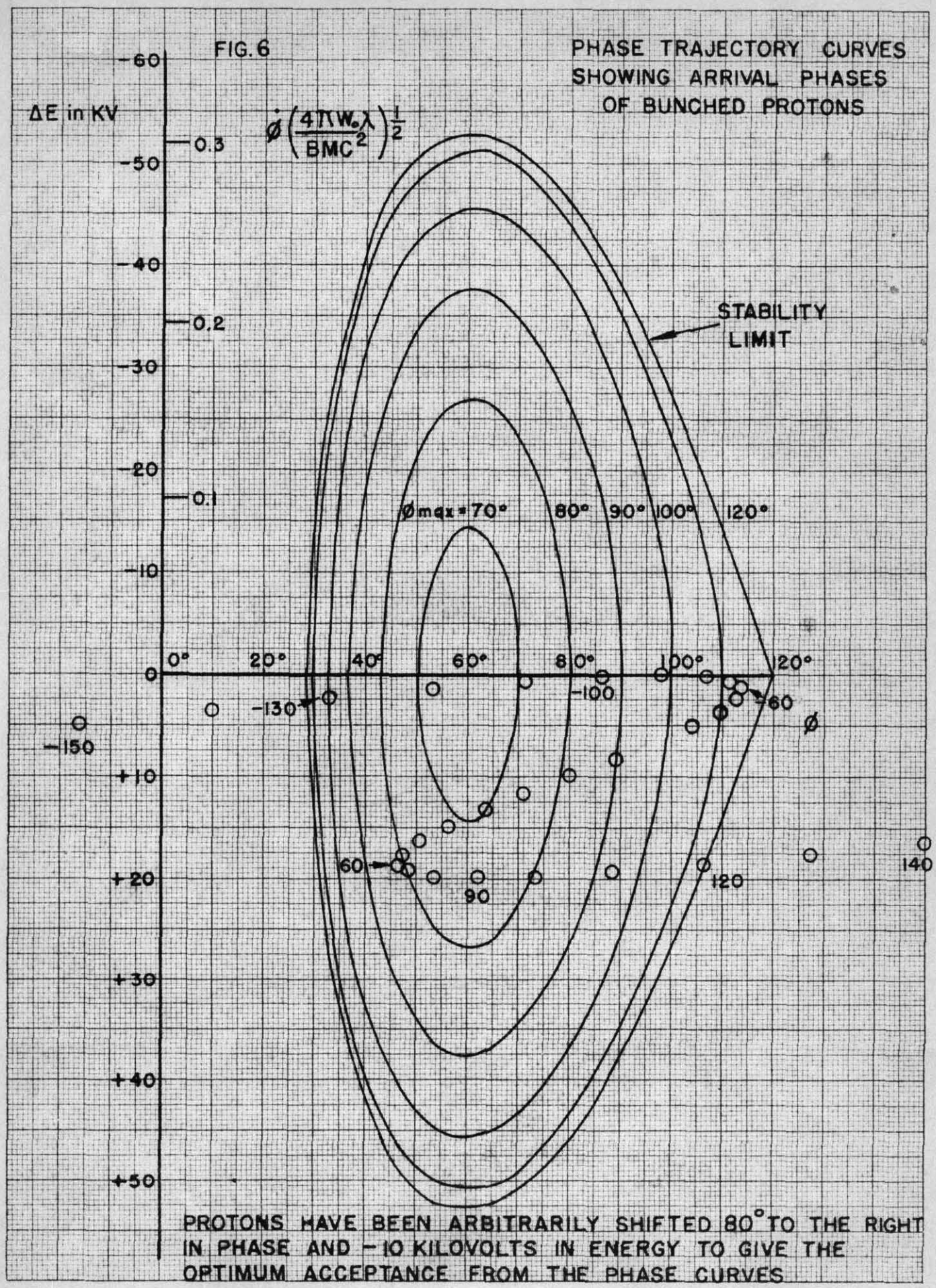
FIG. 6
PHASE TRAJECTORY CURVES SHOWING ARRIVAL PHASES OF BUNCHED PROTONS
ΔE in KV
$\dot{\phi}\left(\frac{4\pi W_0 \lambda}{BMC^2}\right)^{\frac{1}{2}}$
-60
-50
-40
-30
-20
-10
0
+10
+20
+30
+40
+50
0.3
0.2
0.1
STABILITY LIMIT
ϕmax = 70°
80°
90°
100°
120°
0°
20°
40°
60°
80°
100°
120°
-130
-100
60
-150
60
90
120
140
ϕ
PROTONS HAVE BEEN ARBITRARILY SHIFTED 80° TO THE RIGHT IN PHASE AND −10 KILOVOLTS IN ENERGY TO GIVE THE OPTIMUM ACCEPTANCE FROM THE PHASE CURVES

K+E 10 X 10 TO THE CM. 359-14
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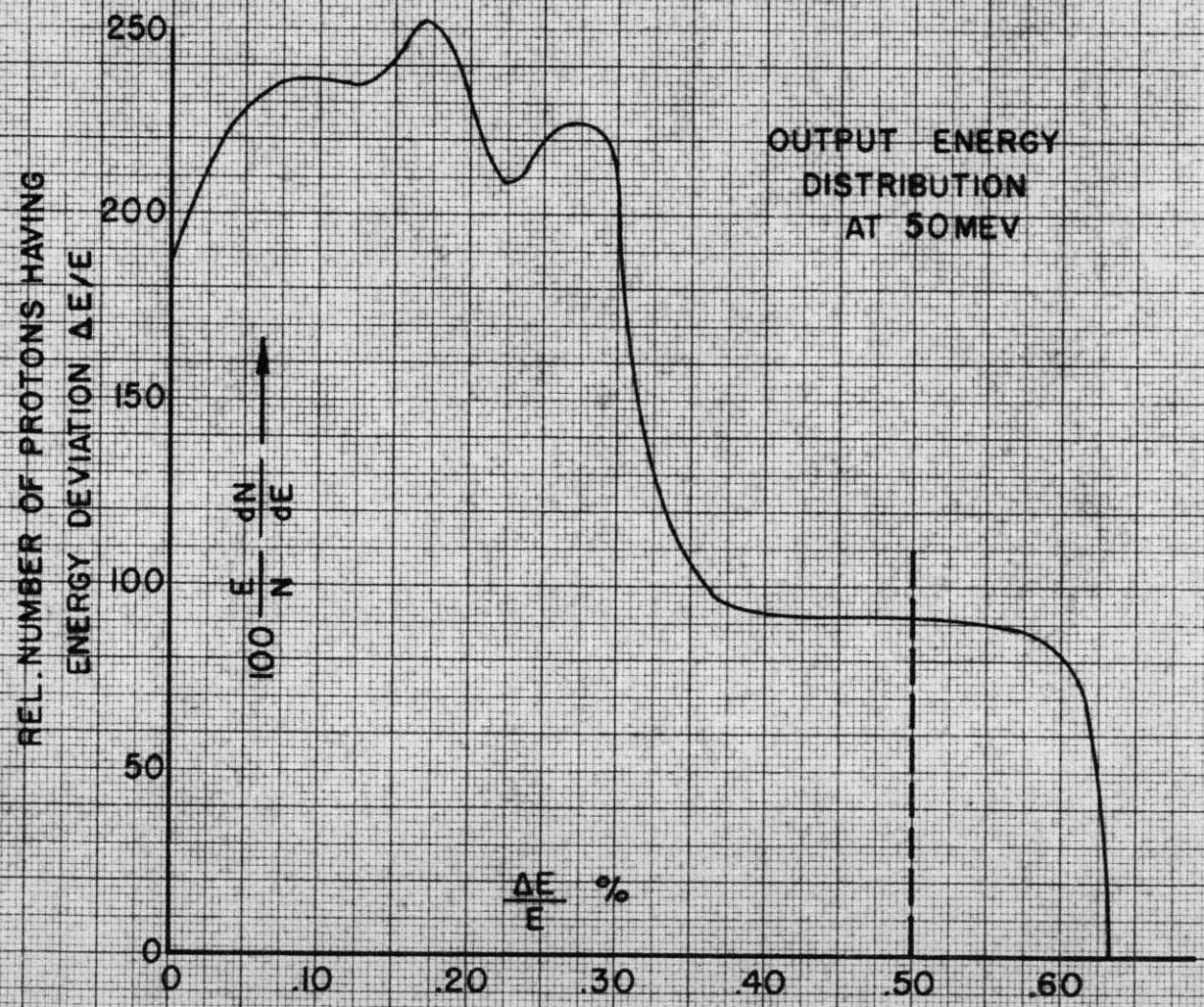
FIG. 7