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MURA 115

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Memorandum to: Jim Snyder

Date: May 17, 1956

Subject: A Proposed Possible Extension of the Digital Computer Program for Studying Foils and Simple R. F. Gaps; to Include Cavities which are Inclined at an Angle to a Particle's Trajectory, and which have Electric Fields whose Magnitudes are a Function of Radius.

Derivation: Because the formula to be presented in this memorandum is unlike the formulas of the previous proposal in that it is not obvious, it appears best if a derivation is included. We are interested here in the effect on particles of a gap which is not radial, but rather placed along the open spiral of a separated sector accelerator. One might consider first, the case where the electric field along this cavity is independent of position. Also, we are interested in radial cavities which either have electric fields which vary with radius, or more extremely, cavities which terminate. It proves convenient to consider all of these situations simultaneously.

In order to carry out this analysis we must have some idea of the electric and magnetic fields in typical cavities. In this regard, Cavity II of The Proposal seems both of sufficient interest, and sufficient simplicity to be amenable to analysis. It consists of an accelerating section whose length is short compared to a wave-length, and consequently $\vec{E}$ is large and independent of radius while $\vec{B}$ is negligible. This then, is the kind of cavity included in the last memorandum. Beyond this section there is a portion for the return of the bias flux which is so constructed that the $\vec{B}$ field is almost constant over the region, and thus $\vec{E}$ drops off linearly with the radius. By studying this region we will have included both situations: a cavity where E varies slowly with radius, and a cavity which terminates.

Introduce the coordinate systems in the median plane of the accelerator as shown in Fig. 1, where following Symon (and not Jones) ξ is the angle from the radius vector to the spiral.

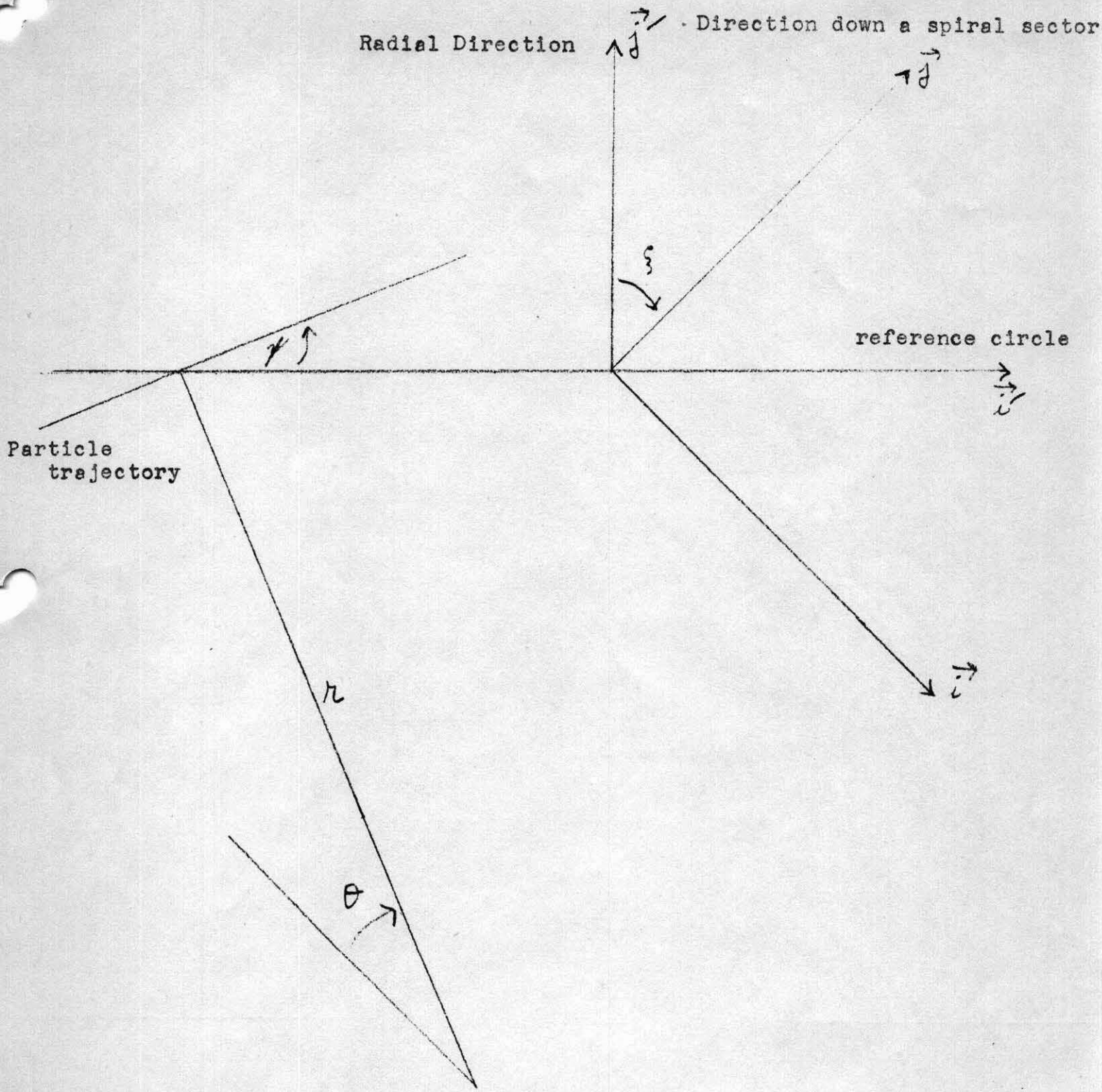

Fig 1: Coordinate Systems in the Median Plane.

The electric and magnetic fields of the cavity may be simple described in the $\vec{i}, \vec{j}$ coordinate system:

$$\vec{E} = \begin{cases} \mathcal{E}_0 \vec{i} & S < S_1 \\ \mathcal{E}_0 \left(\frac{S_2 - S}{S_2 - S_1} \right) \vec{i} & S_1 < S < S_2 \\ 0 & S_2 < S \end{cases} \qquad (1)$$

$$\vec{B} = \begin{cases} 0 & S < S_1 \\ \frac{\mathcal{E}_0 i}{W(S_2 - S_1)} \vec{k} & S_1 < S < S_2 \\ 0 & S_2 < S \end{cases}$$

where S_i and S are distances measured along the spiral sector, $i = \sqrt{-1}$, and W equals the angular frequency of the cavity. We can concentrate our attention on the region $S_1 < S < S_2$.

We shall describe a particle, as before, namely if r = radial distance of the particle from the center of the accelerator, by x and p_x where:

$$r = R_e \ (E) \ (1 + x)$$

$$p_x = \frac{dx}{d\theta} \qquad (2)$$

Let ψ equal the angle between the particles trajectory and the reference circle.

Then:

$$\psi = \frac{1}{r} \frac{dr}{d\theta} \qquad (3)$$

And employing:

$$\frac{dr}{d\theta} = R_e(E) \frac{dx}{d\theta} \qquad (4)$$

$$\psi = \frac{R_e(E)}{r} p_x$$

Let the particle have initial momentum $\vec{p}$ (not to be confused with p_x - the dimensionless "momentum" associated with betatron oscillations).

$$\vec{p_0} = p_{i'} \vec{i'} + p_{j'} \vec{j'} \qquad (5)$$

We now employ the orthogonal transformations.

$$\left.\begin{aligned} j' &= j\cos\xi - i\sin\xi \\ i' &= j\sin\xi + i\cos\xi \end{aligned}\right\} \qquad \left\{\begin{aligned} j &= j'\cos\xi + i'\sin\xi \\ i &= -j'\sin\xi + i'\cos\xi \end{aligned}\right. \quad (6)$$

to obtain $\vec{P_0}$ in the system $\vec{i} - \vec{j}$ where the effects of the cavity may be easily calculated, and then to return to the $\vec{i}' - \vec{j}'$ system.

$$\vec{p_0} = (p_i'\cos\xi - p_j'\sin\xi)\vec{i} + (p_i'\sin\xi + p_j'\cos\xi)\vec{j} \quad (7)$$

after going through the cavity $\vec{p_0}$ is changed into $\vec{p_1}$:

$$\vec{p_1} = (p_i + \Delta p_i)\vec{i} + (p_j + \Delta p_j)\vec{j}$$

or:

$$\vec{p_1} = (p_i' + \Delta p_i\cos\xi + \Delta p_j\sin\xi)\vec{i}' + (p_j' - \Delta p_i\sin\xi + \Delta p_j\cos\xi)\vec{j}' \quad (8)$$

Noting that

$$\tan\psi = \frac{p_j'}{p_i'} \quad (9)$$

and remembering that ψ is a small quantity, we have:

$$\psi \approx \frac{p_j'}{p_i'} \quad (10)$$

so that ψ_1 , the value of ψ after traversing the gap is related to ψ_0 by:

$$\psi_1 = \frac{\psi_0 \left[1 - \frac{\Delta p_i}{\psi_0 p_i'} \sin \xi + \frac{\Delta p_j}{\psi_0 p_i'} \cos \xi \right]}{\left[1 + \frac{\Delta p_i}{p_i'} \cos \xi + \frac{\Delta p_j}{p_i'} \sin \xi \right]} \qquad (11)$$

Employing the facts that,

$$p_j' \approx p_0 \psi_0$$

$$p_i' \approx p_0$$

$$\psi = \frac{R_e(E)}{R_e} p_x = \frac{1}{(1+x)} p_x \approx p_x \qquad (12)$$

to first order

we obtain:

$$\psi_1 = \psi_0 \left\{ 1 - \frac{\Delta p_i}{p_0 p_{x_0}} \sin \xi + \frac{\Delta p_j}{p_0 p_{x_0}} \cos \xi - \frac{\Delta p_i}{p_0} \cos \xi - \frac{\Delta p_j}{p_0} \sin \xi \right\} \qquad (13)$$

or, since we wish to assume the gap is very narrow so that r is not changed on traversal:

$$p_{x_1} = \frac{R_e(E)}{R_e(E+\Delta E)} p_{x_0} \left\{ 1 - \frac{\Delta p_i}{p_0 p_{x_0}} \sin \xi + \frac{\Delta p_j}{p_0 p_{x_0}} \cos \xi - \frac{\Delta p_i}{p_0} \cos \xi - \frac{\Delta p_j}{p_0} \sin \xi \right\} \qquad (14)$$

We must now simply calculate the quantities ΔE, Δp_i, Δp_j:

$$\Delta \vec{p} = \int \vec{F}\, dt \tag{15}$$

Yielding:

$$\Delta p_i = e\,\mathcal{E}_0 \left(\frac{s_2 - s}{s_2 - s_1}\right) \frac{1}{v} \sin\varphi$$

$$\Delta p_j = \frac{e\,\mathcal{E}_0\, v}{W(s_2 - s_1)} \left(\frac{\ell}{\cos\xi}\right) \frac{1}{v} \cos\varphi \tag{16}$$

where we have ignored the higher order corrections in the time the particle remains within the gap, due to $\varphi \neq 0$, and due to the modification in trajectory caused by the energy gain within the gap. Thus our analysis will fail if $\xi > 1$.

Clearly:

$$\Delta E = e\,\mathcal{E}_0\,\ell \left(\frac{s_2 - s}{s_2 - s_1}\right) \sin\varphi \tag{17}$$

And:

$$\frac{\Delta p_i}{p_0} = \frac{\Delta E}{\cos\xi\, v\, p_0} = \frac{E\,\Delta E}{(E^2 - E_0^2)\cos\xi}$$

$$\frac{\Delta p_j}{p_0} = \frac{V(s)\, c \cos\varphi}{W(s_2 - s)\left[E^2 - E_0^2\right]^{1/2} \cos\xi} \tag{18}$$

where:

$$V(s) = V_0 \left(\frac{s_2 - s}{s_2 - s_1}\right)$$

$$\Delta E = V(s) \sin\varphi \tag{19}$$

So that finally:

$$p_{x_1} = \frac{R_e(E)}{R_e(E+\Delta E)}\left\{ p_{x_0}\left[1 - \frac{E\Delta E}{(E^2-E_0^2)} - \frac{V_0 c \cos\varphi \tan\xi}{W(S_2-S_1)(E^2-E_0^2)^{1/2}}\right] - \frac{E\Delta E \tan\xi}{(E^2-E_0^2)} + \frac{V_0 c \cos\varphi}{W(S_2-S_1)(E^2-E_0^2)^{1/2}}\right\} \quad (20)$$

<u>Program Modifications:</u>

We can take the program as outlined in the previous memorandum and modify it at oscillators so that the new p_x (called p previously) is given in the terms of the old by Eqt. 20. Perhaps this can be on a separate master tape, and in fact one might wish to study some of the effects included in Eq. 20 one at a time. In particular the two obvious cases are:

1) $S_2 - S_1 \longrightarrow \infty$

$$p_{x_1} = \frac{R_e(E)}{R_e(E+\Delta E)}\left\{ p_{x_0}\left[1 - \frac{E\Delta E}{(E^2-E_0^2)}\right] - \frac{E\Delta E \tan\xi}{(E^2-E_0^2)}\right\} \quad (21)$$

(This is a gap at an angle to a particle trajectory, but with $\vec{\mathcal{E}}$ independent of radius.)

2) $\xi \rightarrow 0$

$$p_{x_1} = \frac{R_e(E)}{R_e(E+\Delta E)}\left\{ p_{x_0}\left[1 - \frac{E\Delta E}{(E^2-E_0^2)}\right] + \frac{V_0 c \cos\varphi}{W(S_2-S_1)(E^2-E_0^2)^{1/2}}\right\} \quad (22)$$

(This is a radial gap with $\vec{\mathcal{E}}$ a function of radius.)

Other possible modifications which have been suggested are:

A.) The Inclusion of Invariants: If the effect of slaunched gaps and foils is in fact to damp one type of oscillation at the expense of the other, or even to damp or undamp both, as some people have suggested; the inclusion of an invariants routine might be convenient. This would allow one to see, without plotting, the general amount and type of damping. One can envision invariants in x - p, as well as in E - φ , although the latter would only be of interest in the more restricted case of particles within buckets.

B) The Inclusion of Bumps:

It would be interesting to see if bumps cause any new types of resonances with the R. F. also it would be valuable to learn if the energy gain of particles has any effect on their interaction with bumps.

C) Non-Linear Transformations:

With little loss in speed of computation one could probably replace the linear transformations used between foils and gaps, with Powell-Type non-linear transformations including guadratic and cubic terms. This would be interesting in regard to whether or not R. F. knockout now occurs at new resonances, as well as simulating non-linear effects more correctly than proposed previously.

I believe that in regard to these last three modifications there should be considerable discussion before it is decided whether or not to include them in a first formulation of the program.