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Box 21

March 22, 1956.

Memorandum to: Jim Snyder

From: Andrew Sessler

I have been studying two dimensional algebraic transformations which correspond to Hamiltonian Systems. If one assumes the linear motion is uncoupled, and furthermore that the transformation is of the following form:

$$Q_1 = \alpha_1 q_1 + \beta_1 p_1 + f_1(\xi, \eta)$$

$$P_1 = \gamma_1 q_1 + \delta_1 p_1 + F_1(\xi, \eta)$$

$$Q_2 = \alpha_2 q_2 + \beta_2 p_2 + f_2(\xi, \eta)$$

$$P_2 = \gamma_2 q_2 + \delta_2 p_2 + F_2(\xi, \eta)$$

where $\xi = q_1 + s_1 p_1$

$\eta = q_2 + s_2 p_2$

and s_1, s_2 are constants

f_1, F_1, f_2, F_2 arbitrary functions

then it is very easy to satisfy the Poisson Bracket relations. This implies that:

$$F_1 = \lambda_1 f_1$$

$$F_2 = \lambda_2 f_2$$

where: $\lambda_1 = \dfrac{\gamma_1 s_1 - \delta_1}{\alpha_1 s_1 - \beta_1}$

$$\lambda_2 = \frac{\gamma_2 s_2 - \delta_2}{\alpha_2 s_2 - \beta_2}$$

and furthermore that: $\dfrac{\partial f_2}{\partial \xi} = \mu \dfrac{\partial f_1}{\partial \eta}$

where: $\mu = \dfrac{\alpha_2 \ S_2 - \beta_2}{\alpha_1 \ S_1 - \beta_1}$

Thus if one now takes f_1 & f_2 as power series in ξ and η one immediately obtains a general transformation.

I have not yet been able to establish that the most general non-linear Hamiltonian Transformation is a function of only ξ and ζ. In fact this may not be true, except in one dimension where the proof is trivial ---- thus readily proving Powell's Form to be the most general form.

Nevertheless this assumed form has 11 arbitrary parameters to describe the non-linear motion alone, when terms through order three in ξ and ζ are included. This seems sufficiently general to cause us to seriously think of programming the Illiac for the transformation, especially when we remember how useful this will be to study stability limits in two dimensions, as well as long range dynamical stability.

I think then we should have available the following routine, to be calculated roughly in this order:

$$\begin{cases} \xi = q_1 + S_1 p_1 \\ \zeta = q_2 + S_2 p_2 \end{cases}$$

$$\begin{cases} A(\xi, \eta) = k_1 \xi^2 + k_2 \eta^2 + k_3 \xi \eta + k_4 \xi^3 + k_5 \eta^3 + k_6 \xi^2 \eta + k_7 \xi \zeta^2 \\ B(\xi, \eta) = k_1' \xi^2 + k_2' \zeta^2 + k_3' \xi \zeta + k_4' \xi^3 + k_5' \zeta^3 + k_6' \xi^2 \eta + k_7' \xi \zeta^2 \end{cases}$$

$$T_1 \begin{cases} Q_1 = \alpha_1 q_1 + \beta_1 p_1 + A(\xi, \eta) \\ P_1 = \gamma_1 q_1 + \delta_1 p_1 + \lambda_1 A(\xi, \eta) \\ Q_2 = \alpha_2 q_2 + \beta_2 p_2 + B(\xi, \eta) \\ P_2 = \gamma_2 q_2 + \delta_2 p_2 + \lambda_2 B(\xi, \eta) \end{cases}$$

where: s_1, s_2, λ_1, λ_2, α_1, β_1, γ_1, δ_1, α_2, β_2, γ_2, δ_2, $k_1 \ldots k_7$, $k_1^1 \ldots k_7^1$ are constants to be specified.

(These of course are related, but program them as if they were not.)

We will want to be able to evaluate:

$$\left[T_1^n \quad T_2^m \quad T_3^r \quad \ldots . \right]^s$$

where T_1, T_2, T_3 .... are transformations of the same form, but with different constants.

We should be able to have at least T_1 & T_2, and probably three or four would be nice.

n, m, r $\approx$ zero to about 100.

s should not be limited $\approx 10^7$.

It will be important to have double accuracy, as well as speed. In this last connection, can the program be made so that if some constants are zero then the operations involving them are not performed and so some speed is gained? Is this not worth the effort?

We should like to run <u>n</u> sectors and then print out for <u>N_p</u> consecutive transformations, and then run <u>n</u> again, etc. Better include <u>N_T</u>, an end constant.

As for range of parameters, we have for <u>large machines</u>:

$$q_1 , q_2 < 4 \times 10^{-3}$$

$$p_1 , p_2 < 2 \times 10^{-2}$$

$$\alpha, \beta, \gamma, \delta < 50.0$$

$$\left.\begin{matrix} k_1 , k_2 , k_3 \\ k_1^1 , k_2^1 , k_3^1 \end{matrix}\right\} < 10^6$$

$$\left.\begin{matrix} k_4 , k_5 , k_6 , k_7 \\ k_4^1 , k_5^1 , k_6^1 , k_7^1 \end{matrix}\right\} <^* 5 \times 10^9$$

$$\lambda_1 , \lambda_2 < 10^2$$

$$s_1 , s_2 < 10^2$$

For small machines:

$$q_1 , q_2 < 0.4$$

$$p_1 , p_2 < 2.0$$

$$\alpha, \beta, \gamma, \delta < 50.0$$

$$k_1 \ldots k_3^1 < 2 \times 10^2$$

$$k_4 \ldots k_7^1 < 5 \times 10^3$$

$$\lambda_1 , \lambda_2 < 10^2$$

$$s_1 , s_2 < 10^2$$

These estimates have been very liberal, and many of the non-linear terms will probably be smaller than the maxima quoted by a couple of orders of magnitude. I hesitate to restrict things more, at this point. The estimates for λ_i and S_i are obtained from free space and may well be very wrong.

Book 21

103

March 22, 1956.

MEMO ON 2-Dimensional ALGEBRAIC TRANSFORMATIONS WHICH CORRESPOND TO HAMILTONIAN SYSTEMS

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$$T_1 \begin{cases} q_1 = \alpha_1 q_1 + \beta_1 p_1 + A(\xi, \eta) \\ r_1 = \gamma_1 q_1 + \delta_1 p_1 + \lambda_1 A(\xi, \eta) \\ q_2 = \alpha_2 q_2 + \beta_2 p_2 + B(\xi, \eta) \\ r_2 = \gamma_2 q_2 + \delta_2 p_2 + \lambda_2 B(\xi, \eta) \end{cases}$$

[illegible] λ α β_1 γ_1 [illegible] δ_2

[illegible] are constants to be specified.

[illegible] they were not.)

[illegible]

$T_1 \; T_2 \; T_3 \ldots$ [illegible]

where T_1, T_2, T_3 .... are transformations of the same form, but with [illegible]

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