MURA 100

MURA TO-5

Internal

MARK V WITH SCALLOPED MOTION IN THE AXIAL DIRECTION

Tihiro Ohkawa*

University of Illinois and MURA**

The scalloped motion in Mark V FFAG lies in the median plane and enhances the axial focusing. In the following the effect of a scalloped motion in the axial direction on the focussing properties is discussed.

The magnetic field in the median plane of this type is given by

$$B_{z0} = B_0 \left(\frac{r}{r_0}\right)^k$$
$$B_{r0} = B_0 f \left(\frac{r}{r_0}\right)^k \sin\left(\frac{\ln \frac{r}{r_0}}{w} - N\theta\right) \qquad (1)$$

The field and its vector potential near the median plane can be expanded in power series of z,

$$\begin{cases} B_r = \sum B_{ri} z^i \\ B_\theta = \sum B_{\theta i} z^i \\ B_z = \sum B_{zi} z^i \end{cases} \qquad (2)$$

$$\begin{cases} A_r = \sum A_{ri} z^i \\ A_\theta = \sum A_{\theta i} z^i \\ A_z = \sum A_{zi} z^i \end{cases} \qquad (3)$$

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The field which gives $B_{z0} = B_0\left(\frac{r}{r_0}\right)^k$ in the median plane is well known and we can get the whole field by superposing two fields which are $B_{z0}^{(1)} = 0$, $B_{r0}^{(1)} = B_0 f\left(\frac{r}{r_0}\right)^k \sin\left(\frac{\ln\frac{r}{r_0}}{w} - N\theta\right)$ and $B_{z0}^{(0)} = B_0\left(\frac{r}{r_0}\right)^k$, $B_{r0}^{(0)} = 0$ respectively.

By choosing the conditions for $\vec{A}^{(1)}$ as

$$\operatorname{div} \vec{A}^{(1)} = 0, \quad A_{r0}^{(1)} = 0, \quad A_{\theta 0}^{(1)} = 0 \tag{4}$$

the field $\vec{B}^{(1)}$ is given by

$$B_r^{(1)} = -A_{\theta 1}^{(1)} - 3A_{\theta 3}^{(1)} z^2 + \cdots$$
$$B_\theta^{(1)} = A_{r1}^{(1)} + 3A_{r3}^{(1)} z^2 + \cdots \tag{5}$$
$$B_z^{(1)} = \left\{ \frac{1}{r}\frac{\partial}{\partial r}(rA_{\theta 1}^{(1)}) - \frac{1}{r}\frac{\partial A_{r1}^{(1)}}{\partial \theta} \right\} + \left\{ \frac{1}{r}\frac{\partial}{\partial r}(rA_{\theta 3}^{(1)}) - \frac{1}{r}\frac{\partial A_{r3}}{\partial \theta} \right\} z^3 + \cdots$$

The coefficients of the vector potentials are determined by the Maxwell condition and (4).

$$A_{\theta, 2i}^{(1)} = 0 \qquad A_{r, 2i}^{(1)} = 0$$
$$A_z = 0$$
$$A_{r1}^{(1)} = -\frac{B_0 f N}{1+\alpha}\left(\frac{r}{r_0}\right)^k \left\{ w^2(k+1)\cos\Theta + w\sin\Theta \right\} \tag{6}$$
$$A_{\theta 1}^{(1)} = -B_0 f\left(\frac{r}{r_0}\right)^k \sin\Theta$$
$$A_{r3}^{(1)} = \frac{B_0 f N r^{k-2}}{6 r_0^k} \frac{1}{1+\alpha}\left[a\cos\Theta - b\sin\Theta\right]$$
$$A_{\theta 3}^{(1)} = \frac{B_0 f r^{k-2}}{6 r_0^k} \frac{1}{1+\alpha}\left[C\sin\Theta + d\cos\Theta\right]$$

where

$$\Theta = \frac{1}{w}\ln\frac{r}{r_0} - N\theta, \qquad \alpha = w^2(k+1)^2$$
$$a = (k+1)\left[(1+\alpha) - w^2N^2\right], \qquad b = \frac{1}{w}\left[1+\alpha+w^2N^2\right]$$
$$C = (1+\alpha)\left[(k^2-1) - \frac{1}{w^2}\right] - w^2N^2\left[(k^2-1) + \frac{1}{w^2}\right]$$
$$d = \frac{2}{w}\left[(1+\alpha)k - w^2N^2\right]$$

These vector potentials are expanded about the reference circle $r = r_1$ by putting

$$r = r_1(1+x) \quad , \quad z = r_1 y$$

$$A_\theta = \frac{p_1 f}{e}\Big[\, y \sin N\theta + xy\left(k \sin N\theta - \frac{1}{w}\cos N\theta\right) + \frac{y^3}{6}\frac{1}{1+\alpha}\left(d\cos N\theta - c \sin N\theta\right)$$

$$- \frac{x^2 y}{2}\left\{\left(-k(k-1) + \frac{1}{w^2}\right)\sin N\theta + \frac{2k-1}{w}\cos N\theta\right\} - \frac{x^3 y}{6}\Big\{\left(-k(k-1)(k-2) + \frac{3(k-1)}{w^2}\right)\sin N\theta$$

$$+ \left(\frac{3k(k-1)}{w} - \frac{3k-2}{w} - \frac{1}{w^3}\right)\cos N\theta \Big\} + \frac{xy^3}{6}\frac{1}{1+\alpha}\left\{\left(\frac{d}{w} - c(k-2)\right)\sin N\theta + \left((k-2)d + \frac{c}{w}\right)\cos N\theta\right\}\Big]$$

$$A_r = \frac{p_1 f}{e}\left[\frac{N}{1+\alpha}\right]\Big[-y\left\{w^2(k+1)\cos N\theta - w \sin N\theta\right\} - xy\left\{w \sin N\theta + (w^2 k(k+1) - 1)\cos N\theta\right\}$$

$$+ \frac{y^3}{6}\left\{a\cos N\theta + b \sin N\theta\right\} - \frac{3x^2 y}{6}\left\{\left((3k^2-1)w - \frac{1}{w}\right)\sin N\theta + \left(wk(k^2-1) - 3k\right)\cos N\theta\right\}$$

$$- \frac{x^3 y}{6}\left\{\left(\frac{2(k-1)}{w} + w(2k^2 - 6k + 2)\right)\sin N\theta + \left(w^2 k(k-1)(k-2)(k+1) - 6k + 5 - \frac{1}{w^2}\right)\cos N\theta\right\}$$

$$+ \frac{xy^3}{6}\left\{\left(\frac{a}{w} + b(k-2)\right)\sin N\theta + \left((k-2)a - \frac{b}{w}\right)\cos N\theta\right\}\Big]$$

On the other hand, the vector potentials of the momentum compaction field $B^{(0)}$ are obtained simply by putting $f = 0$ in the ordinary field. From Laslett's expression of the vector potentials, we get

$$(1+x)A_\theta^{(0)} = \frac{p_1}{e}\Big[-x - \frac{k+1}{2}x^2 - \frac{k(k+1)}{6}x^3 - \frac{k(k+1)(k-1)}{24}x^4$$

$$- \frac{y^2}{2}\left\{-k - (k^2 - 2k)x + \frac{k}{2}(-k^2 + 5k - 4)x^2\right\} + \frac{y^4}{24}(-k^3 + 2k^2)\Big] \qquad (8)$$

$$A_r^{(0)} = A_z^{(0)} = 0$$

Finally the superposed vector potentials are

$$(1+x)A_\theta = \frac{p_1}{e}\Big[\, C_1 x + J_1 y + C_2\frac{x^2}{2} + (J_1 + J_2)\,xy + C_5\frac{y^2}{2}$$

$$+ C_3\frac{x^3}{6} + \frac{xy^2}{2}C_6 + \frac{xy^2}{2}(J_4 + 2J_2) + \frac{y^3}{6}J_3 \qquad (9)$$

$$+ C_4\frac{x^4}{24} + \frac{x^3 y}{6}(J_5 + 3J_4) + \frac{x^2y^2}{4}C_7 + \frac{xy^3}{6}(J_6 + J_3) + \frac{y^4}{24}C_8\Big]$$

$$A_r = \frac{p_1}{e}\left[L_1 y + L_2 xy + \frac{y^3}{6}L_3 + \frac{3x^2 y}{6}L_4 + \frac{x^3 y}{6}L_5 + \frac{xy^3}{6}L_6\right]$$

$$A_z = 0$$

where

$$C_1 = -1, \quad C_2 = -(k+1), \quad C_3 = -k(k+1), \quad C_4 = -(k+1)k(k-1)$$

$$C_5 = k \quad C_6 = k^2 \quad C_7 = -2(k^3+3k^2+2k), \quad C_8 = -k^3+2k^2$$

$$J_1 = f \sin N\theta, \quad J_2 = f\left(k \sin N\theta - \frac{1}{w}\cos N\theta\right)$$

$$J_3 = \frac{f}{1+\alpha}\left(-c \sin N\theta + d\cos N\theta\right), \quad J_4 = f\left[\left(k(k-1) - \frac{1}{w^2}\right)\sin N\theta - \frac{2k-1}{w}\cos N\theta\right]$$

$$J_5 = -f\left[\left\{-k(k-1)(k-2) + \frac{3}{w^2}(k-1)\right\}\sin N\theta + \frac{-\frac{1}{w^2} + 3k(k-2)+2}{w}\cos N\theta\right]$$

$$J_6 = \frac{f}{1+\alpha}\left[\left(-kc + \frac{d}{w}\right)\sin N\theta + \left(kd + \frac{c}{w}\right)\cos N\theta\right]$$

$$L_1 = \frac{fN}{1+\alpha}\left(w \sin N\theta - w^2(k+1)\cos N\theta\right), \quad L_2 = -\frac{fN}{1+\alpha}\left(w\sin N\theta + (1+w^2k(k+1))\cos N\theta\right)$$

$$L_3 = \frac{fN}{1+\alpha}\left(b \sin N\theta + a\cos N\theta\right)$$

$$L_4 = -\frac{fN}{1+\alpha}\left[\left(w(k^2+2k-1) + \frac{1}{w}\right)\sin N\theta + \left(w^2k(k-1)(k+1) + k - 2\right)\cos N\theta\right]$$

The Lagrangian which determines the trajectories of the particles is given by [MURA LJL-5]

$$\begin{aligned}
\mathcal{L} &= p r_1\sqrt{(1+x)^2 + x'^2 + y'^2} + e r_1\left[(1+x)A_\theta + x'A_r + y'A_z\right] \\
&\simeq 1 + x + \frac{1}{2}\,\frac{x'^2+y'^2}{1+x} - \frac{1}{8}(x'^2+y'^2) \\
&\quad + \frac{e}{p}\left[(1+x)A_\theta + x'A_r + y'A_z\right] \\
&\simeq 1 + \frac{1}{2}(x'^2+y'^2) - \frac{x}{2}(x'^2+y'^2) + \frac{x^2}{2}(x'^2+y'^2) - \frac{1}{8}(x'^2+y'^2)^2 \\
&\quad + x + \frac{p_1}{p}\Big[-x + J_1 y - \frac{x^2}{2}(k+1) + xy(J_1+J_2) + \frac{k}{2}y^2 \\
&\quad - \frac{x^3}{6}k(k+1) + \frac{xy^2}{2}k^2 + \frac{y^3}{6}J_3 + \frac{x^2y}{2}(J_4+2J_2) \\
&\quad - \frac{(k+1)k(k-1)}{24}x^4 + \frac{x^3y}{6}(J_5+3J_4) + \frac{x^2y^2}{4}(-2k^3-6k^2-6k) + \frac{xy^3}{6}(J_6+J_3) \\
&\quad + \frac{y^4}{24}(-k^3+2k^2) + x'\frac{N}{1+\alpha}\left\{L_1 y + L_2 xy + \frac{y^3}{6}L_3 + \frac{3x^2y}{6}L_4\right\}
\end{aligned} \tag{10}$$

The above Lagrangian contains a term $J_1 y$ representing the forced motion in the y-direction; forced motion in the x-direction is excited by this y-forced-motion through the xy ($J_1 \;\; J_2$) term.

By putting

$$
\begin{aligned}
y &= y_f + \eta = K_1 \sin N\theta + K_2 \cos N\theta + \eta \\
x &= x_f + \xi = M_0 + M_1 \sin N\theta + M_2 \cos N\theta + \xi
\end{aligned}
\qquad (11)
$$

we can seperate the forced motion

K's and M's are determined by comparing the coefficients of the Fourier components of $N\theta$ in the equation of motion and they are approximately

$$
\begin{aligned}
K_1 &\doteqdot \frac{-f}{N^2 + k + \dfrac{f^2 c}{4(1+\alpha)(N^2+k)}} \sim \frac{-f}{N^2+k} \\
K_2 &\doteqdot \frac{-f^3}{4(N^2+k)^3} \frac{\alpha}{1+\alpha} \sim 0 \\
M_0 &\doteqdot \frac{-f^2}{2(N^2+k)} \left\{ 1 - \frac{1}{1 - \dfrac{f^2}{2N^2 W^2 (k+1)}} \right\} \\
M_1 &\doteqdot \frac{f^2}{\{N^2 - (k+1)\}(N^2+k)} \frac{N w}{1+\alpha} \sim 0 \\
M_2 &\doteqdot \frac{-f^2}{\{N^2 - (k+1)\}(N^2+k)} \frac{W^2(k+1)}{1+\alpha} \sim 0
\end{aligned}
\qquad (12)
$$

The linear parts of the homogeneous equations can be written as follows, by using $y_f \sim K_1 \sin N\theta$ and $x_f = M_0$

$$
\begin{aligned}
\xi'' &= \lambda_1 \xi + \lambda_2 \eta + \lambda_3 \xi' + \lambda_4 \eta' \\
\eta'' &= \mu_1 \xi + \mu_2 \eta + \mu_3 \xi' + \mu_4 \eta'
\end{aligned}
\qquad (13)
$$

with

$$(1-M_0)\lambda_1 = -(k+1)(1+M_0k) + \frac{k_1 f}{2}\left[k(k-1)\cdot(1+M_0(k-2)) - \frac{1}{W^2}(1+3kM_0)\right]$$
$$+ \sin 2N\theta\left(-\frac{k_1 f}{2W}\right)\left[(2k+1) + M_0\left(-\frac{1}{W^2}+3k(k-2)+2+\frac{3d}{1+\alpha}\right)\right]$$
$$+ \cos 2N\theta\left(-\frac{k_1 f}{2}\right)\left[k(k-1)\cdot(1+M_0(k-2)) - \frac{1}{W^2}(1+3kM_0)\right]$$

$$\lambda_1 \sim -(k+1) + \frac{f^2}{2W^2(N^2+k)} - \frac{f^2}{2(N^2+k)W^2}\cos 2N\theta$$

$$(1-M_0)\lambda_2 = f\sin N\theta\left[(k+1) + k_1 k^2 + M_0 f k(k+1) - \frac{M_0^2}{2}k(k-2)(k+1) - \frac{1}{W^2}M_0\right.$$
$$\left. + \frac{3}{2}M_0^2\frac{(k-1)}{W^2}\right] + \frac{f}{W}\cos N\theta\left[-1 - M_0(2k+1) + \frac{M_0^2}{2}\left(-\frac{1}{W^2}\right.\right.$$
$$\left.\left. + 3k(k-2)+2\right)\right]$$

$$\lambda_2 \sim -\frac{f}{W}\cos N\theta$$

$$(1-M_0)\lambda_3 = 0$$

$$(1-M_0)\lambda_4 = \frac{f}{2(N^2+k)} - \frac{f}{2(N^2+k)}\cos 2N\theta$$

$$\lambda_4 \sim 0$$

$$(1-M_0)\mu_1 = \lambda_2 + k_1 N\sin N\theta$$

$$\mu_1 \sim -\frac{f}{W}\cos N\theta$$

$$(1-M_0)\mu_2 = \left[k(1+M_0k) - \frac{f}{1+\alpha}k_1\frac{C}{2} + \frac{M_0^2}{2}(-2k^3-6k^2-6k) + \frac{k_1M_0}{2}\frac{f}{1+\alpha}\left(-kC+\frac{d}{W}\right)\right.$$
$$\left. + \frac{k_1}{12}(-k^3+2k^2)\right] + f\sin 2N\theta\left[\frac{k_1 d}{2(1+\alpha)} + \frac{k_1M_0}{2(1+\alpha)}\left(kd+\frac{C}{W}\right)\right]$$
$$+ f\cos 2N\theta\left[\frac{k_1 C}{2(1+\alpha)} - \frac{k_1M_0}{2(1+\alpha)}\left(-kC+\frac{d}{W}\right) - \frac{-k^3+2k^2}{12}k_1^2\right]$$

$$\mu_2 \sim k - \frac{f^2}{2(N^2+k)W^2} + \frac{f^2}{2W^2(N^2+k)}\cos 2N\theta$$

$$(1-M_0)\mu_3 = \frac{-f}{2(N^2+k)} + \frac{f}{2(N^2+k)}\cos 2N\theta \qquad \sim 0$$

$$(1-M_0)\mu_4 = 0$$

Very rough approximations to (13) are

$$\xi'' \sim \left[-(k+1) + \frac{f^2}{2w^2(N^2+k)} \right] \xi - \frac{f}{w} \cos N\theta \, \eta$$

$$\eta'' \sim - \frac{f}{w} \cos N\theta \, \xi + \left[k - \frac{f^2}{2w^2(N^2+k)} \right] \eta \qquad (14)$$

Using the smooth approximation, we get a rough estimate of ν's, i.e. the number of betatron oscillations per turn.

$$\nu_x^2 \sim k + 1$$

$$\nu_y^2 \sim -k + \frac{f^2}{N^2 w^2}$$

These are the same results as in the ordinary Mark V and scalloped motion in the axial direction still favors the focusing in the axial direction. The only advantage of this type is the smaller circumference factor because of the perpendicular adding of the flutter field.