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MURA/FTC-4
Internal

EQUILIBRIUM ORBIT IN MARK I FFAG*

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It was shown in MURA/FTC-2 that the equilibrium orbit satisfies the equations

$$\begin{cases} \frac{d\phi}{d\theta} = \frac{R}{\rho_0} F(R,\theta) \sec\phi - 1 \\ \frac{dR}{d\theta} = - R \tan\phi \end{cases} \tag{1}$$

together with the boundary conditions

$$\begin{cases} \phi(\theta + \frac{2\pi}{N}) = \phi(\theta) \\ \phi(0) = 0 \end{cases} \tag{2}$$

where θ is zero at the center of a magnet. $F(R,\theta)$ is defined by giving the field in the median plane as

$$\begin{cases} H_z = H_0 F(R,\theta) \\ p = \frac{e}{c} \rho_0 H_0 \end{cases} \tag{3}$$

In the case of a scaling Mark I machine, the field along any radius varies as R^k, so that

$$F(R,\theta) = R^k f(\theta) \tag{4}$$

Calling $\phi' = \frac{d\phi}{d\theta}$,

etc.,

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$$\phi'' = \frac{k+1}{\rho_0} R^k R' f(\theta)\sec\phi + \frac{R}{\rho_0} R^{k+1} f'(\theta)\sec\phi + \frac{R}{\rho_0} R^{k+1} f(\theta)\sec\phi\tan\phi\,\phi'^{-2}$$

$$= \frac{R^{k+1} f(\theta)\sec\phi}{\rho_0}\left\{[-(k+1)+\phi']\tan\phi + \frac{f'}{f}\right\}$$

$$= (1+\phi')\left\{[-(k+1)+\phi']\tan\phi + \frac{f'}{f}\right\}$$

or

$$\phi'' + k\phi'\tan\phi - \frac{f'}{f}\phi' - \phi'^2\tan\phi + (k+1)\tan\phi = \frac{f'}{f} \qquad (5)$$

In a similar way, θ may be eliminated to give

$$R'' + \frac{R^{k+1}}{\rho_0} f(\theta)[R^2+R'^2]^{3/2} - R - \frac{2R'^2}{R} = 0 \qquad (6)$$

We estimate the order of magnitude of the terms in (5), using parameters appropriate to the Michigan Model. $R \sim 50$, $\rho_0 = 20$, $|F| \sim 1.4$, $k \sim 3$, $N = 8$. From (1),

$$\phi' \sim 2.5(1.4) - 1 \sim 2.5$$

$$\phi \sim \frac{\phi'}{N} \sim 0.3$$

$$\phi'' \sim N\phi' \sim 20$$

$$\frac{f'}{f} \sim N$$

Then in (5) we have

$$\phi'' \sim 20$$

$$k\phi'\tan\phi \sim 2.5$$

$$\frac{f'}{f}\phi' \sim 20$$

$$\phi'^2\tan\phi \sim 2$$

$$(k+1)\tan\phi \sim 1.2$$

$$\frac{f'}{f} \sim 8$$

We introduce a parameter ϵ to show the order of magnitude of the terms. ϵ will approach unity at the end of the calculation. We write (5) as

$$\phi'' - \frac{f'}{f}\phi' + \epsilon k\,\phi' \tan\phi - \epsilon\phi'^2 \tan\phi + \epsilon(k+1)\tan\phi = \frac{f'}{f}$$

and assume

$$\phi = \sum_{n=0}^{\infty} \epsilon^n \phi_n$$

Then

$$\begin{cases} \phi_0'' - \frac{f'}{f}\phi_0' = f'/f \\ \phi_1'' - \frac{f'}{f}\phi_1' = \phi_0\phi_0'^2 - k\phi_0\phi_0' - (k+1)\phi_0 \\ \phi_2'' - \frac{f'}{f}\phi_2' = k(\phi_0\phi_1' + \phi_1\phi_0') + \phi_0'^2\phi_1 - (k+1)\phi_1 \\ \text{etc.} \end{cases} \tag{7}$$

The equation

$$\phi_n'' - \frac{f'}{f}\phi_n' = g_n(\theta)$$

has a general solution

$$\phi_n(\theta) = \int_0^{\theta} f(\theta')\left[c_n + \int_0^{\theta'} \frac{g_n(\theta'')}{f(\theta'')}\,d\theta''\right]d\theta' + d_n \tag{8}$$

where c_n and d_n are arbitrary constants. To satisfy $\phi(0) = 0$ we choose $d_n = 0$. c_n is determined by $\phi_n\left(\frac{2\pi}{N}\right) = 0$ from which

$$c_n = -\frac{\int_0^{\frac{2\pi}{N}} f(\theta') \int_0^{\theta'} \frac{g_n(\theta'')}{f(\theta'')}\,d\theta''\,d\theta'}{\int_0^{\frac{2\pi}{N}} f(\theta')\,d\theta'} \tag{9}$$

and it is easy to show that every ϕ_n is periodic with period .

In particular $g_0(\theta) = \dfrac{f'(\theta)}{f(\theta)}$ so that

$$c_0 = \frac{2\pi}{N}\left[\int_0^{\frac{2\pi}{N}} f(\theta')\,d\theta'\right]^{-1} \tag{10}$$

and (9) may be written

$$c_n = -c_0 \frac{N}{2\pi} \int_0^{\frac{2\pi}{N}} f(\theta') \int_0^{\theta'} \frac{g_n(\theta'')}{f(\theta'')} d\theta'' d\theta' \tag{11}$$

$f(\theta)$ is of course an even function, so we may expand it as

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos nN\theta$$

with

$$a_0 = \frac{N}{2\pi} \int_0^{\frac{2\pi}{N}} f(\theta) d\theta \tag{12}$$

Then $c_0 = \frac{1}{a_0}$ and

$$\phi_0 = c \int_0^{\theta} f(\theta) d\theta - \theta = \frac{1}{N} \sum_n{}' \frac{\alpha_n}{n} \sin nN\theta \tag{13}$$

where the primed summation <u>omits</u> the n = 0 term and $\alpha_n = \frac{a_n}{a_0}$. The introduction of α_n means that ϕ_0 is independent of the normalization of $f(\theta)$. It is easy to see from (7) and (8) that this is true in every order.

Now

$$g_1(\theta) = \phi_0 \left\{ (\phi_0' - k)\phi_0' - (k+1) \right\}$$

and, calling $\xi = N\theta$

$$\frac{g_1(\theta)}{f(\theta)} = \frac{1}{a_0 N} \left\{ \frac{\sum_n{}' \frac{\alpha_n}{n} \sin n\xi}{\sum_n \alpha_n \cos n\xi} \left[\left(\sum_n{}' \alpha_n \cos n\xi - k \right) \sum_n{}' \alpha_n \cos n\xi - (k+1) \right] \right\}$$

and by using twice the relation

$$1 + \sum_n{}' \alpha_n \cos n\xi = \sum \alpha_n \cos n\xi ,$$

$$\frac{g_1(\theta)}{f(\theta)} = \frac{1}{a_0 N} \left\{ \sum_n{}' \frac{\alpha_n}{n} \sin n\xi \left[\sum_n{}' \alpha_n \cos n\xi - (k+1) \right] \right\}$$

and

$$Q(\theta)=\int_0^\theta \frac{g_1(\theta')}{f(\theta')}\,d\theta' = -\frac{k+1}{a_0 N^2}\sum_n{}' \frac{\alpha_n}{n^2}\cos n\xi -$$

$$-\frac{1}{2a_0 N^2}\Bigg[\sum_{n,m}{}' \frac{\alpha_n\alpha_m}{n(n+m)}\cos(n+m)\xi +$$

$$+\sum_{n\neq m}{}' \frac{\alpha_n\alpha_m}{n(n-m)}\cos(n-m)\xi\Bigg] + \Gamma_1$$

where

$$\Gamma_1 = \frac{1}{2N^2}\left\{2(k+1)\sum_n{}'\frac{\alpha_n}{n^2} + \sum_{n,m}{}'\frac{\alpha_n\alpha_m}{n(n+m)} + \sum_{n\neq m}{}'\frac{\alpha_n\alpha_m}{n(n-m)}\right\}$$

$$= \frac{1}{2N^2}\left\{2(k+1)\sum_n{}'\frac{\alpha_n}{n^2} + \frac{1}{2}\sum_n{}'\frac{\alpha_n^2}{n^2} + 2\sum_{n\neq m}{}'\frac{\alpha_n\alpha_m}{n^2-m^2}\right\} \tag{14}$$

Then

$$\psi(\theta) = \int_0^\theta f(\theta')\,Q(\theta')\,d\theta' = \frac{\Gamma_1}{N}\sum_p{}'\frac{\alpha_p}{p}\sin n\xi + \Gamma_1\theta -$$

$$-\frac{k+1}{N^2}\sum_n{}'\frac{\alpha_n}{n^3}\sin n\xi - \frac{1}{2N^3}\left\{\sum_{n,m}{}'\frac{\alpha_n\alpha_m}{n(n+m)^2}\sin(n+m)\xi + \sum_{n\neq m}{}'\frac{\alpha_n\alpha_m}{n(n-m)^2}\sin(n-m)\xi\right\} -$$

$$-\frac{k+1}{2N^3}\left\{\sum_{n,p}{}'\frac{\alpha_n\alpha_p}{n^2(n+p)}\sin(n+p)\xi + \sum_{n\neq p}{}'\frac{\alpha_n\alpha_p}{n^2(n-p)}\sin(n-p)\xi\right\} -$$

$$-\frac{k+1}{2N^2}\sum_p{}'\frac{\alpha_p^2}{p^2}\theta - \frac{1}{4N^3}\left\{\sum_{n,m,p}{}'\frac{\alpha_n\alpha_m\alpha_p}{n(n+m)(n+m+p)}\sin(n+m+p)\xi +\right.$$

$$\left.+\sum_{n+m-p\neq 0}{}'\frac{\alpha_n\alpha_m\alpha_p}{n(n+m)(n+m-p)}\sin(n+m-p)\xi\right\} -$$

$$-\frac{1}{4N^3}\sum_{n,m}{}'\frac{\alpha_n\alpha_m\alpha_{n+m}}{n(n+m)}\theta - \frac{1}{4N^3}\left\{\sum_{\substack{n-m+p\neq 0\\ n-m\neq 0}}{}'\frac{\alpha_n\alpha_m\alpha_p}{n(n-m)(n-m+p)}\sin(n-m+p)\xi +\right.$$

$$\left.+\sum_{\substack{n-m-p\neq 0\\ n-m\neq 0}}{}'\frac{\alpha_n\alpha_m\alpha_p}{n(n-m)(n-m-p)}\sin(n-m-p)\xi\right\} - \tag{15}$$

$$-\frac{1}{4N^2}\left\{\sum_{n-m\neq 0}{}'\frac{\alpha_n\alpha_m\alpha_{m-n}}{n(n-m)} + \sum_{n-m\neq 0}{}'\frac{\alpha_n\alpha_m\alpha_{n-m}}{n(n-m)}\right\}\theta$$

From (10), $c_n = -\frac{1}{a_0}\frac{N}{2\pi}\psi\left(\frac{2\pi}{N}\right)$

$$= -\frac{1}{a_0}\left\{ \Gamma_1 - \frac{k+1}{2N^2}\sum_p{}'\left(\frac{\alpha_p}{p}\right)^2 - \frac{1}{4N^2}\sum_{n,m}{}'\frac{\alpha_n\alpha_m\alpha_{m+n}}{n(n+m)}\right\} \tag{16}$$

and

$$\phi_1(\theta) = c_n\int_0^\theta f(\theta')d\theta' + \psi(\theta) \tag{17}$$

We combine the first two approximations (13) and (17) to give

$$\phi(\theta) = \sum_n{}'\beta_n \sin nN\theta \tag{18}$$

with

$$\begin{aligned}\beta_n = \frac{\alpha_n}{nN} + \frac{1}{nN^3}\Bigg\{ & \left(\Gamma N^2 - \frac{k+1}{n^2}\right)\alpha_n - \\ & - \frac{1}{2n}\left[\sum_{m+p=n}{}'\frac{\alpha_m\alpha_p}{m} + \sum_{m-p=n}{}'\frac{\alpha_m\alpha_p}{m}\right] \\ & - \frac{k+1}{2}\left[\sum_{m+p=n}{}'\frac{\alpha_m\alpha_p}{m^2} + \sum_{m-p=n}{}'\frac{\alpha_m\alpha_p}{m^2}\right] - \\ & - \frac{1}{4}\left[\sum_{m+p+q=n}{}'\frac{\alpha_m\alpha_p\alpha_q}{m(m+p)} + \sum_{m+p-q=n}{}'\frac{\alpha_m\alpha_p\alpha_q}{m(m+p)}\right] - \\ & - \frac{1}{4}\left[\sum_{\substack{m-p+q=n\\ m\neq p}}{}'\frac{\alpha_m\alpha_p\alpha_q}{m(m-p)} + \sum_{\substack{m-p-q=n\\ m\neq p}}{}'\frac{\alpha_m\alpha_p\alpha_q}{m(m-p)}\right]\Bigg\}\end{aligned} \tag{19}$$

It is easy to find R(θ) from (1). Clearly

$$R = R_0 e^{-\int_0^\theta \tan\phi \, d\theta}$$

$$\cong R_0 \left\{ 1 - \int_0^\theta \phi(\theta) d\theta + \frac{1}{2}\left(\int_0^\theta \phi(\theta) d\theta\right)^2 + \dots \right\} \tag{20}$$

$$\cong R_0 \left\{ 1 + \frac{1}{N} \sum_n{}' \frac{\beta_n}{n} \cos nN\theta + \right.$$

$$\left. + \frac{1}{2N^2} \sum_{m,n}{}' \frac{\beta_m \beta_n}{2mn} \left[\cos(m+n)N\theta + \cos(m-n)N\theta \right] + \dots \right\} \tag{21}$$

Each approximations to ϕ appears to be less than 10% of the preceding one, so that the convergence is quite rapid.