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MARK V FFAG EXPANDED EQUATIONS OF MOTION*

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Abstract

The expansions of the equations of motion of betatron oscillations in the Mark V (spiral sector) F. F. A. G. accelerator in powers of the deviation from a circle and from an equilibrium orbit are discussed. It is found that in the case of large machines where $k \gg 1$ and $\frac{1}{w} \gg 1$ that comparatively simple equations for the combined radial and axial motion are quite accurate, while in the case of small machines, where these conditions do not hold, no such simple equations appear to exist. Coefficients in these equations are derived, and their magnitudes estimated and numerical examples given.

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Introduction

It is desirable to expand the equations of motion of betatron oscillations in the spiral sector (Mark V) FFAG in order to apply the methods of Moser(1) and Sturrock(2) and to investigate the possibilities of reducing the troublesome effects on non-linearities by judicious choice of the field shape.

Laslett(3) has discussed this expansion with emphasis on the linear terms and Judd(4) has discussed the non-linearities for motion in the median plane. The present report reviews this work and extends the results to the coupled radial and axial motion.

I. Exact Equations of Motion and Development of the Vector Potential

Consideration of the spiral sector accelerator began with the median plane field

$$\begin{cases} B_r = B_\theta = 0 \\ B_z = -B_0\left(\frac{r}{r_0}\right)^k\left\{1 + f\sin\psi\right\} \end{cases} \tag{1.1}$$

with

$$\psi = \frac{1}{w}\ln\left(\frac{r}{r_0}\right) - N\theta \tag{1.2}$$

in cylindrical coordinates. This specifies the magnetic field everywhere by Maxwell's Equations. The equations of motion of the betatron oscillations can be derived from the Lagrangian

$$L = m_0 c \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2} + \frac{e}{c}\underline{v}\cdot\underline{A} \tag{1.3}$$

$$= m_0 c \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2} + \frac{e}{c}\left[\dot{r}A_r + r\dot{\theta}A_\theta + \dot{z}A_z\right] \tag{1.4}$$

The equations of motion are simpler if θ is used as independent variable instead of t. Thus

$$S = \int L\,dt = \int \mathcal{L}\,d\theta$$

and $\mathcal{L} = L\frac{dt}{d\theta}$. Now $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = r'\dot{\theta}$

so that

$$\mathcal{L} = m_0 c^2 \sqrt{r'^2 + r^2 + z'^2} + \frac{e}{c}\left[r'A_r + rA_\theta + z'A_z\right]$$

It is customary to use the dimensionless variables x and y defined by

$$r = r_0(1+x)$$
$$z = r_0 y \tag{1.5}$$

and

$$\mathcal{L} = m_0 r_0 c\sqrt{(1+x)^2 + x'^2 + y'^2} + \frac{er_0}{c}\left[(1+x)A_\theta + x'A_x + y'A_y\right]$$

We use units such that

$$\frac{pc}{e} = r_0 = B_0 = 1 \tag{1.6}$$

and since $\theta' = 1$,

$$\mathcal{L} = \sqrt{(1+x)^2 + x'^2 + y'^2} + (1+x) A_\theta + x' A_x + y' A_y \qquad (1.7)$$

The equations of motion follow from the Euler-Lagrange equations

$$\begin{cases} \dfrac{d}{d\theta}\left(\dfrac{\partial \mathcal{L}}{\partial x'}\right) - \dfrac{\partial \mathcal{L}}{\partial x} = 0 \\ \dfrac{d}{d\theta}\left(\dfrac{\partial \mathcal{L}}{\partial y'}\right) - \dfrac{\partial \mathcal{L}}{\partial y} = 0 \end{cases} \qquad (1.8)$$

In the case of motion in the median plane ($y \equiv 0$), we have

$$\frac{d}{d\theta}\left(\frac{x'}{\sqrt{(1+x)^2 + x'^2}}\right) = \frac{1+x}{\sqrt{(1+x)^2 + x'^2}} - (1+x)^{k+1}\left[1 + f \sin \psi\right] \qquad (1.9)$$

which can be derived from the Hamiltonian

$$\mathcal{H} = -(1+x)\sqrt{1 - p_x^2} + \frac{(1+x)^{k+2}}{k+2}\left\{1 + \frac{f w^2 (k+2)}{1 + w^2 (k+2)^2}\left[\sin\psi - \frac{\cos\psi}{w(k+2)}\right]\right\} \qquad (1.10)$$

The first order canonical equations derived from (1.10) are the Ridge Runner equations integrated numerically by Illiac.

The vector potential components may be developed in many forms related to one another by gauge transformations. Laslett(3) has chosen a form with $\nabla \cdot \underline{A} = 0$ and Powell(5) a form with $A_z = 0$.

Following Laslett, we write

$$\frac{e}{p}A_\theta = D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4 \tag{1.11}$$

in the median plane, where, with $\xi = N\,\theta$,

$$\left\{\begin{aligned}
D_1 &= -1 + f\sin\xi \\
D_2 &= -\frac{k-1}{2} + f\left(-\frac{1}{2w}\cos\xi + \frac{k-1}{2}\sin\xi\right) \\
D_3 &= -\frac{k^2-2k+3}{6} + f\left[-\frac{k-1}{3w}\cos\xi + \left(-\frac{1}{6w^2} + \frac{k^2-2k+3}{6}\right)\sin\xi\right] \\
D_4 &= -\frac{k^3-4k^2+7k-12}{24} + f\left[\frac{\frac{1}{w^2} - 3k^2 + 8k - 7}{24w}\cos\xi + \right. \\
&\qquad \left. + \frac{1}{24}\left\{-\frac{(3k+4)}{w^2} + k^3 - 4k^2 + 7k - 12\right\}\sin\xi\right]
\end{aligned}\right. \tag{1.12}$$

where the coefficient of $\sin\xi$ in D_4 has been corrected from Laslett's first edition. Off the median plane we assume the form for the vector potential components.

$$\begin{cases} \frac{e}{p} A_\theta = \sum\limits_{m+n\leq 4} \theta_{mn} x^m y^n \\ \frac{e}{p} A_r = \sum\limits_{m+n\leq 4} X_{mn} x^m y^n \\ \frac{e}{p} A_y = \sum\limits_{m+n\leq 4} Y_{mn} x^m y^n \end{cases} \tag{1.13}$$

By using Maxwell's equations, we find

$$\left\{\begin{array}{ll} \theta_{10} = D_1 & \theta_{02} = -\frac{1}{2}(D_1 + 2D_2) \\ \theta_{20} = D_2 & \theta_{12} = -\frac{1}{2}(-2D_1 + 2D_2 + 6D_3 + D_1'') \\ \theta_{30} = D_3 & \theta_{22} = -\frac{1}{2}(3D_1 - 3D_2 + 3D_3 + 12D_4 - 2D_1'' + D_2'') \\ \theta_{40} = D_4 & \theta_{04} = \frac{1}{24}(3D_1 - 6D_2 + 12D_3 + 24D_4 - 2D_1'' + 4D_2'') \\ & \\ X_{12} = D_1' & Y_{11} = -D_1' \\ X_{22} = -2D_1' + D_2' & Y_{21} = D_1' - D_2' \\ X_{04} = \frac{1}{6}(D_1' - 2D_2') & Y_{31} = -(D_1' - D_2' + D_3') \\ Y_{03} = \frac{1}{6}(-D_1' + 2D_2') & Y_{13} = \frac{1}{6}(3D_1' - 4D_2' + 6D_3' + D_1''') \end{array}\right.$$

(1.14)

and all other coefficients are zero. (θ_{22} and θ_{04} have been corrected from Laslett's first edition).

II. Expansion About a Circle

Laslett and the present writer have carried through the expansion of the equations of motion in powers of x and y about the circle x = y = 0. After solving for x" and y", one has

$$
\begin{aligned}
x'' + \left\{ k+1 + f\left[\frac{1}{w}\cos\xi - (k+2)\sin\xi\right]\right\} &= \\
= f\sin\xi - \Big\{(k+1)(k+2) + f\Big[\frac{2k+3}{w}\cos\xi &+ \frac{1-(k+1)(k+2)w^2}{w^2}\sin\xi\Big]\Big\}\frac{x^2}{2} - \\
- \Big\{k(k+1)(k+2) + f\Big[-\frac{1-(3k^2+6k+2)w^2}{w^3}\cos\xi &+ \frac{3(k+1)-k(k+1)(k+2)w^2}{w^2}\sin\xi\Big]\Big\}\frac{x^3}{6} + \\
+ \left\{1+3f\sin\xi\right\}\frac{x'^2}{2} - \left\{(3k+4) + 3f\left[\frac{1}{w}\cos\xi - k\sin\xi\right]\right\}\frac{xx'^2}{2} &- \\
-(Nf\cos\xi)yy' - \left\{1-f\sin\xi\right\}\frac{y'^2}{2} &+ \\
+ \left\{k^2 + f\left[\frac{2k}{w}\cos\xi + \frac{1+(N^2-k^2)w^2}{w^2}\sin\xi\right]\right\}\frac{y^2}{2} &+ \\
+ \left\{k^3 + f\left[-\frac{1+(N^2-3k^2)w^2}{w^3}\cos\xi + \frac{3k+k(N^2-k^2)w^2}{w^2}\sin\xi\right]\right\}\frac{xy^2}{2} &- \\
- Nf\left[k\cos\xi + \frac{1}{w}\sin\xi\right]xyy' &+ \\
+ \left\{k + f\left[\frac{1}{w}\cos\xi - k\sin\xi\right]\right\}x'yy' &- \\
- \left\{k + f\left[\frac{1}{w}\cos\xi - k\sin\xi\right]\right\}\frac{xy'^2}{2} & \qquad (2.1)
\end{aligned}
$$

and

$$
\begin{aligned}
& y'' - \left\{ k + f\left[\frac{1}{w} \cos\xi - k \sin\xi \right] \right\} y = \\
& = \left\{ k(k+1) + f\left[\frac{2k+1}{w} \cos\xi + \frac{1-k(k+1)w^2}{w^2} \sin\xi \right] \right\} xy + \\
& \quad + \left\{ k^2(k+1) + f\left[-\frac{1-k(3k+2)w^2}{w^3} \cos\xi + \frac{3k+1-k^2(k+1)w^2}{w^2} \sin\xi \right] \right\} \frac{x^2 y}{2} - \\
& \quad - \left\{ k^2(k-2) + f\left[-\frac{1-\{k(3k-4)-N^2\}w^2}{w^3} \cos\xi + \right. \right. \\
& \qquad\qquad \left. \left. + \frac{3k-2+(k-2)(N^2-k^2)w^2}{w^2} \sin\xi \right] \right\} \frac{y^3}{6} + \\
& \quad + \left\{ 1 + f\sin\xi \right\} x'y' + \left(Nf\cos\xi \right) x'y - \\
& \quad - \left\{ (k+2) + f\left[\frac{1}{w} \cos\xi - k\sin\xi \right] xx'y' + \right. \\
& \quad + \left\{ k + f\left[\frac{1}{w} \cos\xi - k\sin\xi \right] \right\} \left(\frac{x'^2 + 3y'^2}{2} \right) y + \\
& \qquad + Nf\left[k\cos\xi + \frac{1}{w}\sin\xi \right] xx'y
\end{aligned}
\tag{2.2}
$$

When $y \equiv 0$, (2.1) reduces to

$$
\begin{aligned}
& x'' + \left\{ (k+1) + f\left[\frac{1}{w}\cos\xi - (k+2)\sin\xi \right] x = \right. \\
& = f\sin\xi - \left\{ (k+1)(k+2) + f\left[\frac{2k+3}{w}\cos\xi + \lambda_1 \sin\xi \right] \right\} \frac{x^2}{2} -
\end{aligned}
$$

$$-\left\{k(k+1)(k+2)+f\left[\lambda_2\cos\xi+\lambda_3\sin\xi\right]\right\}\frac{x^3}{6}+$$

$$+\left\{1+3f\sin\xi\right\}\frac{x'^2}{2}-\left\{3k+4+3f\left[\frac{1}{w}\cos\xi-k\sin\xi\right]\right\}\frac{xx'^2}{2} \qquad (2.3)$$

where

$$\left\{\begin{aligned} \lambda_1 &= \frac{1-(k+1)(k+2)w^2}{w^2} \\ \lambda_2 &= -\frac{1-(3k^2+6k+2)w^2}{w^3} \\ \lambda_3 &= \frac{3(k+1)-k(k+1)(k+2)w^2}{w^2}, \end{aligned}\right. \qquad (2.4)$$

which is Judd's[4] radial equation except for the coefficient of $xx'^2\sin\xi$. The difference arises from our taking the coefficient of x" from the differentiation before expansion to one higher power, since x satisfies an inhomogeneous differential equations and therefore $x'' \sim f$ and not $\sim x$.

Laslett has developed in a Fourier series a large amplitude radial motion found with the Illiac. This is the motion at the fixed point of order three ($\sigma = \frac{2}{3}\pi$) which lies on the boundary of the stable region of the phase plane*. It is interesting to compare numerically the Illiac and expanded solutions and to compare magnitudes of various terms of the expanded

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* A forthcoming report by Laslett and Cole will present Illiac data and discuss stability limits found.

solution. Thus we calculate x" at several different values of θ.

$k = 160$, $1/w = 2302$, $f = 1/4$, $N = 40$.

$N\theta$	0	$3\pi/2$	3π	$9\pi/2$
x'^2 term	$0.120 \cdot 10^{-4}$	$0.696 \cdot 10^{-6}$	$0.385 \cdot 10^{-4}$	$0.174 \cdot 10^{-4}$
$x\,x'^2$ term	$1.079 \cdot 10^{-5}$	$-0.143 \cdot 10^{-6}$	$0.473 \cdot 10^{-5}$	$0.991 \cdot 10^{-8}$
$f \sin \xi$	0	-0.2500	0	0.2500
x term	0.29931	-0.02138	0.040943	-0.00100
x^2 term	-0.01750	0.00713	0.000780	$-0.472 \cdot 10^{-4}$
x^3 term	-0.03338	$-1.28 \cdot 10^{-4}$	-0.000484	$-0.616 \cdot 10^{-7}$
sum = x"	0.24844	-0.26438	0.04132	0.24897
Illiac x"	0.25020	-0.26647	0.04206	0.25181
Error	0.7%	0.8%	1.8%	1.1%

The error in calculating x" from the Illiac data is small compared to these errors except in the case $N\theta = 3\pi$, where it is about 1%.

The "derivative" terms (x'^2 and xx'^2) are always small compared to the linear term. The cubic term can be as large as 11% of the linear term and the quadratic as large as 30%. Note, however, that the cubic term is twice as large as the quadratic at $N\theta = 0$, where $x/w = 0.94$. It would appear from this that the expansion converges slowly for large x/w. However, these numbers are somewhat fortuitous, since the largest part of the quadratic term is zero at this θ.

III. The Equilibrium Orbit

The equilibrium (closed) orbit is that solution of (2.3) which has period $2\pi/N$ in θ. Laslett[3] has found the Fourier coefficients by direct substitution of the Fourier Series and solution of the resulting algebraic equations. Judd[4] has obtained more accurate results by an iteration procedure. Ribe[6] has discussed the problem in terms of a variable α which is similar to the ϕ used by Cole and Kerst[7] in discussing the Mark I equilibrium orbit.

We expand the solution in powers of f, which gives equations similar to Judd's, but having different forcing terms in each order. We assume

$$x(\theta) = \sum_{n=0}^{\infty} f^n x_n(\theta) \qquad (3.1)$$

with

$$x_n\left(\theta + \frac{2\pi}{N}\right) = x_n(\theta) \tag{3.2}$$

Then, substituting in (2.3),

$$x_0'' + (k+1)x_0 = \frac{x_0'^2}{2} - (k+1)(k+2)\frac{x_0^2}{2} - $$

$$- \frac{k(k+1)(k+2)}{6} x_0^3 - \frac{3k+4}{2} x_0 x_0'^2$$

A solution of the x_0 equation which has period $2\pi/N$ is $x_0 \equiv 0$.

Then

$$\left\{\begin{aligned}
x_1'' + (k+1)x_1 &= \sin\xi \\
x_2'' + (k+1)x_2 &= \left[-\frac{1}{w}\cos\xi + (k+2)\sin\xi\right]x_1 + \frac{x_1'^2}{2} - \frac{(k+1)(k+2)}{2}x_1^2 \\
x_3'' + (k+1)x_3 &= \left[-\frac{1}{w}\cos\xi + (k+2)\sin\xi\right]x_2 + x_1'x_2' + \\
&\quad + \frac{3}{2}\sin\xi\, x_1'^2 - (k+1)(k+2)x_1x_2 - \\
&\quad - \left(\frac{2k+3}{w}\cos\xi + \lambda_1\sin\xi\right)\frac{x_1^2}{2} - \frac{k(k+1)(k+2)}{6}x_1^3 - \\
&\quad - (3k+4)\frac{x_1x_1'^2}{2} . \\
x_4'' + (k+1)x_4 &= \left[-\frac{1}{w}\cos\xi + (k+2)\sin\xi\right]x_3 + x_1'x_3' + \frac{x_2'^2}{2} + (3\sin\xi)x_1'x_2' - \\
&\quad - (k+1)(k+2)\left(\frac{x_2^2}{2} + x_1x_3\right) - k(k+1)(k+2)\frac{x_1^2x_2}{2} -
\end{aligned}\right. \tag{3.3}$$

$$\left. \begin{array}{l} -\left(\frac{2k+3}{w}\cos\xi + \lambda_1 \sin\xi\right) x_1 x_2 - \left(\lambda_2 \cos\xi + \lambda_3 \sin\xi\right)\frac{x_1^3}{6} - \\ -\left(3k+4\right)\left(x_2 x_1'^2 + x_1 x_1' x_2'\right) - \frac{3}{2}\left(\frac{1}{w}\cos\xi - k\sin\xi\right) x_1 x_1'^2 \end{array} \right.$$

We define

$$F_n = k+1-(nN)^2 \tag{3.4}$$

and find

$$\left\{ \begin{array}{l} x_1 = \frac{1}{F_1}\sin\xi \\ x_2 = \frac{a_{20}}{F_0} + \frac{a_{22}}{F_2}\cos 2\xi + \frac{b_{22}}{F_2}\sin 2\xi \\ x_3 = \frac{a_{31}}{F_1}\cos\xi + \frac{b_{31}}{F_1}\sin\xi + \frac{a_{33}}{F_3}\cos 3\xi + \frac{b_{33}}{F_3}\sin 3\xi \\ \text{etc.} \end{array} \right. \tag{3.5}$$

where

$$\left\{ \begin{array}{l} a_{20} = \frac{1}{2F_1}\left\{\frac{N^2-(k+1)(k+2)}{2F_1} + k+2\right\} = \frac{1}{4F_1^2}\left[(k+1)(k+2) - (2k+3)N^2\right] \\ a_{22} = \frac{1}{2F_1}\left\{\frac{N^2+(k+1)(k+2)}{2F_1} - (k+2)\right\} = \frac{1}{4F_1^2}\left[-(k+1)(k+2)+(2k+5)N^2\right] \\ b_{22} = -\frac{1}{2wF_1} \end{array} \right. \tag{3.6}$$

and

$$
\left\{
\begin{aligned}
a_{31} &= -\frac{a_{20}}{w} - \frac{1}{2}\frac{a_{22}}{wF_2} + \left(\frac{k+2}{2} + \frac{N^2}{F_1} - \frac{(k+1)(k+2)}{F_1}\right)\frac{b_{22}}{F_2} - \frac{2k+3}{8wF_1^2} \\
&= -\frac{1}{8wF_1^2F_2}\left\{(k+1)^2 - (20k+29)N^2 + 8\cdot\frac{2k+3}{k+1}N^4\right\} \\
b_{31} &= \left(1-\frac{k+1}{F_1}\right)(k+2)a_{20} + \left(\frac{(k+1)(k+2)}{F_1} - \frac{N^2}{F_1} - \frac{k+2}{2}\right)\frac{a_{22}}{F_2} - \\
&\quad - \frac{b_{22}}{2wF_2} + \frac{3}{8}\frac{N^2}{F_1^2} - \frac{3k+4}{8F_1^3}N^2 - \frac{3}{4}\left[\frac{\lambda_1}{4F_1^2} + \frac{k(k+1)(k+2)}{6F_1^3}\right] \\
&\cong \frac{1}{16w^2F_1^2F_2}\left[k+1+8N^2\right] + \frac{1}{16F_1^3}\Big[3k(k+1)^2 - (7k^2+19k+16)N^2 + \\
&\qquad + 4\frac{k+2}{k+1}(2k+3)N^4\Big] \\
a_{33} &= -\frac{a_{22}}{2wF_2} + \left(\frac{2N^2+(k+1)(k+2)}{2F_1} - \frac{k+2}{2}\right)\frac{b_{22}}{F_2} + \frac{2k+3}{8wF_1^2} \\
&= -\frac{1}{8wF_1^2F_2}\left\{-(k+1)(3k+5) + (12k+25)N^2\right\} \\
b_{33} &= \left(\frac{k+2}{2} - \frac{N^2}{F_1} - \frac{(k+1)(k+2)}{2F_1}\right)\frac{a_{22}}{F_2} - \frac{b_{22}}{2wF_2} + \frac{3}{8}\frac{N^2}{F_1^2} - \\
&\quad - \frac{3k+4}{8F_1^3}N^2 + \frac{1}{4}\left[\frac{\lambda_1}{2F_1^2} + \frac{k(k+1)(k+2)}{6F_1^3}\right] \\
&\cong \frac{1}{8F_1^2F_2}\left\{\frac{3}{w^2}\left[k+1-2N^2\right] + (k+2)\left[-(k+1)(2k+3) + 3(k+2)N^2\right]\right\}
\end{aligned}
\right.
\qquad (3.7)
$$

where all terms which contribute more than 0.1% of the leading term are included

Then

$$x = \frac{f^2 a_{20}}{F_0} + \left(\frac{f}{F_1} + \frac{f^3 a_{31}}{F_1} \right) \sin \xi + \frac{f^3 a_{31}}{F_1} \cos \xi +$$
$$+ \frac{f^2 a_{22}}{F_2} \cos 2\xi + \frac{f^2 b_{22}}{F_2} \sin 2\xi +$$
$$+ \frac{f^3 a_{33}}{F_3} \cos 3\xi + \frac{f^3 b_{33}}{F_3} \sin 3\xi + \cdots$$
$$= \sum_{n=0}^{\infty} \left(\alpha_n \cos n\xi + \beta_n \sin n\xi \right) \tag{3.8}$$

The results agree with those of Judd, except in the highest order where different terms are included, due to the different approximation used.

Fourier series for the closed orbit have been obtained from Illiac data by Laslett. We compare below values of α_n and β_n calculated from the development above with those of Laslett for two cases, characteristic of large and small machines.

Large Machine

$k = 160,\ 1/w = 2302,\ f = 1/4,\ N = 40.$

	Calculation	Illiac	
α_o	-230	-216	
α_1	- 86	- 78	
β_1	-1612	-1651	
α_2	- 6	- 13	All x 10^{-7}
β_2	- 80	- 73	
α_3	- 0.7	- 1	
β_3	- 5	- 4	

Small Machine

$k = 0.8,\ 1/w = 23.0,\ f = 1/4.\ N = 5.$

	Calculation	Illiac	
α_o	-177.3	-173.9	
α_1	- 43.2	- 40.8	
β_1	-1049.8	-1052.8	
α_2	- 4.7	- 6.9	All x 10^{-5}
β_2	- 31.5	- 30.0	
α_3	- 0.3	- 0.3	
β_3	- 1.1	- 1.3	

The agreement between leading terms is quite good. In the large machine case, the difference in β_1 could be reduced by including ξ^5 terms which ~~are about 1% of the total~~ and in the correct direction.

Numerically, in both cases the closed orbit is dominated by the forcing term ($f \sin \xi$) and the linear term. The quadratic terms (x^2 and x'^2) contribute an amount only 6% of the leading term, while the contribution of the cubic terms is very small in this approximation.

The terms calculated agree numerically with those of Ribe[6].

IV Expansion About the Closed Orbit

If one expands directly the differential equations (2.1) and (2.2) about the closed orbit by substituting

$$x = x_s + u \tag{4.1}$$

where x_s is the closed orbit given by (3.8), spurious first derivative terms are introduced, as found by Judd[4] in the radial motion. This difficulty may be avoided by expanding the Lagrangian or Hamiltonian, rather than the equations of motion. The Lagrangian has the slight advantage that the vector potential does not occur under the square root sign.

We expand the Lagrangian (1.7)

$$\mathcal{L} = \mathcal{L}(x, x', y, y', \theta) = \sqrt{(1+x)^2 + x'^2 + y'^2} + (1+x)A_\theta + x'A_r + y'A_z \tag{4.2}$$

in a Taylor's series about $x = x_s$, $x' = x_s$, $y = 0$, $y' = 0$, using (4.1). The term independent of the variables gives no contribution to the equations of motion. The terms linear in u and u' cancel, since to this order

$$\mathcal{L} = \left.\frac{\partial \mathcal{L}}{\partial x}\right| u + \left.\frac{\partial \mathcal{L}}{\partial x'}\right| u' + \cdots$$

where $|$ means that the term is evaluated at the closed orbit.

Then

$$\frac{\partial \mathcal{L}}{\partial u} = \left.\frac{\partial \mathcal{L}}{\partial x}\right|$$

$$\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial u'}\right) = \frac{d}{d\theta}\left(\left.\frac{\partial \mathcal{L}}{\partial x'}\right|\right)$$

and

$$\frac{d}{d\theta}\left(\left.\frac{\partial \mathcal{L}}{\partial x'}\right|\right) - \left.\frac{\partial \mathcal{L}}{\partial x}\right| = 0$$

since the equilibrium orbit satisfies the equations of motion.

All terms of the form $y^m\, y'^n\, x^p\, x'^q$ vanish when $m+n$ is odd because of symmetry about the plane of the equilibrium orbit.

The Lagrangian expanded through terms of fourth order is then

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}a_1 u^2 + a_2 u u' + \tfrac{1}{2}a_3 u'^2 + \tfrac{1}{2}a_4 y^2 + a_5 y y' + \tfrac{1}{2}a_6 y'^2 + \\ & + \tfrac{1}{6}b_1 u^3 + \tfrac{1}{6}b_2 u'^3 + \tfrac{1}{2}b_3 u^2 u' + \tfrac{1}{2}b_4 u'^2 u + \tfrac{1}{2}b_5 u y^2 + \\ & + \tfrac{1}{2}b_6 u' y^2 + \tfrac{1}{2}b_7 u y'^2 + \tfrac{1}{2}b_8 u' y'^2 + b_9 u y y' + \\ & + \tfrac{1}{24}c_1 u^4 + \tfrac{1}{24}c_2 u'^4 + \tfrac{1}{24}c_3 y^4 + \tfrac{1}{24}c_4 y'^4 + \tfrac{1}{2}c_5 u u' y^2 + \\ & + \tfrac{1}{2}c_6 u^2 y y' + \tfrac{1}{6}c_7 u^3 u' + \tfrac{1}{4}c_8 u^2 u'^2 + \tfrac{1}{6}c_9 u u'^3 + \\ & + \tfrac{1}{4}c_{10} u^2 y^2 + \tfrac{1}{4}c_{11} u^2 y'^2 + \tfrac{1}{4}c_{12} u'^2 y'^2 + \\ & \qquad + \tfrac{1}{6}c_{13} y^3 y' + \tfrac{1}{12}c_{14} u u' y^2\end{aligned} \tag{4.3}$$

Define

$$L_0 = \sqrt{(1+x_s)^2 + x_s'^2} \tag{4.4}$$

Then the coefficients of the Lagrangian (4.3) are

$$\left\{\begin{aligned} a_1 &= \frac{\partial^2 \mathcal{L}}{\partial x^2}\bigg| = \frac{x_s'^2}{L_0^3} + 2\left[\theta_{10} + 2\theta_{20}x_s + 3\theta_{30}x_s^2 + 4\theta_{40}x_s^3\right] + \\ &\qquad + (1+x_s)\left[2\theta_{20} + 6\theta_{30}x_s + 12\theta_{40}x_s^2\right] \\ a_2 &= \frac{\partial^2 \mathcal{L}}{\partial x \partial x'}\bigg| = -\frac{(1+x_s)x_s'}{L_0^3} \\ a_3 &= \frac{\partial^2 \mathcal{L}}{\partial x'^2}\bigg| = \frac{(1+x_s)^2}{L_0^3}\end{aligned}\right.$$

$$
\left\{
\begin{aligned}
a_4 &= \left.\frac{\partial^2 \mathcal{L}}{\partial y^2}\right| = (1+x_s)\left[2\theta_{02} + 2\theta_{12} x_s + 2\theta_{22} x_s^2\right] + x_s'\left(2X_{12} x_s + 2X_{22} x_s^2\right) \\
a_5 &= \left.\frac{\partial^2 \mathcal{L}}{\partial y \partial y'}\right| = Y_{11} x_s + Y_{21} x_s^2 + Y_{31} x_s^3 \\
a_6 &= \left.\frac{\partial^2 \mathcal{L}}{\partial y'^2}\right| = \frac{1}{L_0} \\
b_1 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x^3}\right| = -3\frac{x_s'^2(1+x_s)}{L_0^5} + 3\left[2\theta_{20} + 6\theta_{30} x_s + 12\theta_{40} x_s^2\right] + (1+x_s)\left[6\theta_{30} + 24\theta_{40} x_s\right] \\
b_2 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x'^3}\right| = -3x_s'\frac{(1+x_s)^2}{L_0^5} \\
b_3 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x^2 \partial x'}\right| = \frac{x_s'}{L_0^5}\left[2(1+x_s)^2 - x_s'^2\right] \\
b_4 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x \partial x'^2}\right| = \frac{(1+x_s)}{L_0^5}\left[2x_s'^2 - (1+x_s)^2\right] \\
b_5 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x \partial y^2}\right| = 2\theta_{02} + 2\theta_{12} x_s + 2\theta_{22} x_s^2 + (1+x_s)\left(2\theta_{12} + 4\theta_{22} x_s\right) + x_s'\left(2X_{12} + 4X_{22} x_s\right) \\
b_6 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x' \partial y^2}\right| = 2X_{12} x_s + 4X_{22} x_s^2 \\
b_7 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x \partial y'^2}\right| = -\frac{(1+x_s)}{L_0^3} \\
b_8 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x' \partial y'^2}\right| = -\frac{x_s'}{L_0^3} \\
b_9 &= \left.\frac{\partial^3 \mathcal{L}}{\partial x \partial y \partial y'}\right| = Y_{11} + 2Y_{21} x_s + 3Y_{31} x_s^2 \\
c_1 &= \left.\frac{\partial^4 \mathcal{L}}{\partial x^4}\right| = -\frac{3x_s'^2}{L_0^5} + 15x_s'^2\frac{(1+x_s)^2}{L_0^7} + 4\left[6\theta_{30} + 24\theta_{40} x_s\right] + (1+x_s)\left[24\theta_{40}\right] \\
c_2 &= \left.\frac{\partial^4 \mathcal{L}}{\partial x'^4}\right| = -3\frac{(1+x_s)^2}{L_0^5} + \frac{15x_s'^2(1+x_s)^2}{L_0^7} \\
c_3 &= \left.\frac{\partial^4 \mathcal{L}}{\partial y^4}\right| = (1+x_s)\,24\theta_{04} + x_s'\cdot 24X_{04} \\
c_4 &= \left.\frac{\partial^4 \mathcal{L}}{\partial y'^4}\right| = -\,3/L_0^3
\end{aligned}
\right.
$$

$$
\left.\begin{aligned}
c_5 &= \frac{\partial^4 \mathcal{L}}{\partial x \partial x' \partial y^2}\Big| = 2X_{12} + 4X_{22}x_s \\
c_6 &= \frac{\partial^4 \mathcal{L}}{\partial x^2 \partial y \partial y'}\Big| = 2Y_{21} + 6Y_{31}x_s \\
c_7 &= \frac{\partial^4 \mathcal{L}}{\partial x^3 \partial x'}\Big| = -\frac{6x_s'(1+x_s)}{L_0^5} + \frac{15x_s'^3(1+x_s)}{L_0^7} \\
c_8 &= \frac{\partial^4 \mathcal{L}}{\partial x^2 \partial x'^2}\Big| = \frac{2}{L_0^3} - \frac{15x_s'^2(1+x_s)^2}{L_0^7} \\
c_9 &= \frac{\partial^4 \mathcal{L}}{\partial x \partial x'^3}\Big| = -\frac{6x_s'(1+x_s)}{L_0^5} + \frac{15(1+x_s)^3 x_s'}{L_0^7} \\
c_{10} &= \frac{\partial^4 \mathcal{L}}{\partial x^2 \partial y^2}\Big| = 2\left[2\Theta_{12} + 4\Theta_{22}x_s\right] + (1+x_s)4\Theta_{22} + 4x_s'X_{22} \\
c_{11} &= \frac{\partial^4 \mathcal{L}}{\partial x^2 \partial y'^2}\Big| = \frac{2}{L_0^3} - \frac{3x_s'^2}{L_0^5}\ ; \quad c_{13} = \frac{\partial^4 \mathcal{L}}{\partial y^3 \partial y'}\Big| = 6Y_{03} + 6Y_{13}x_s \\
c_{12} &= \frac{\partial^4 \mathcal{L}}{\partial x'^2 \partial y'^2}\Big| = -\frac{1}{L_0^3} + \frac{3x_s'^2}{L_0^5}\ ; \quad c_{14} = \frac{\partial^4 \mathcal{L}}{\partial x \partial x' \partial y^2}\Big| = \frac{3(1+x_s)x_s'}{L_0^5}
\end{aligned}\right\} \quad (4.5)
$$

Using the Euler-La Grange equations, one derives the two second order differential equations

$$
\begin{aligned}
&u''\left[a_3 + b_2 u' + b_4 u + \tfrac{1}{2}c_2 u'^2 + \tfrac{1}{2}c_8 u^2 + c_9 u u' + \tfrac{1}{2}c_{12} y'^2\right] + \\
&\qquad + y''\left[b_8 y' + c_{12} u' y' + c_{14} u y'\right] = \\
&= -a_3' u' + (a_1 - a_2')u + \tfrac{1}{2}\left[b_1 - b_3'\right]u^2 - \tfrac{1}{2}(b_4 + b_2')u'^2 - \\
&\quad - b_4 u u' + \tfrac{1}{2}(b_5 - b_6')y^2 + \tfrac{1}{2}(b_7 - b_8')y'^2 + \\
&\quad + (b_9 - b_6)y y' + \tfrac{1}{6}(c_1 - c_7')u^3 - \tfrac{1}{6}(2c_9 + c_2')u'^3 + \\
&\quad - \tfrac{1}{2}c_8' u^2 u' - (c_8 + c_9')u u'^2 + \tfrac{1}{2}(c_{10} - c_5')u y^2 + \\
&\quad + (c_6 - c_5)u y y' + \tfrac{1}{2}(c_{11} - c_{14}')u y'^2 - \tfrac{1}{2}c_{12}' u' y'^2
\end{aligned}
$$

(4.6)

and

$$y''\left[a_6 + b_7 u + b_8 u' + \tfrac{1}{2}c_4 y'^2 + \tfrac{1}{2}c_{11}u^2 + \tfrac{1}{2}c_{12}u'^2 + c_{14}uu'\right] +$$
$$+ u''\left[b_8 y' + c_{12}u'y' + c_{14}uy'\right] =$$
$$= -a_6'y' + (a_4 - a_5')y + (b_5 - b_9')uy + (b_6 - b_9)u'y -$$
$$- b_7'uy' - (b_7 + b_8')u'y' +$$
$$+ \tfrac{1}{6}(c_3 - c_{13}')y^3 - \tfrac{1}{6}c_4'y'^3 + \tfrac{1}{2}(c_{10} - c_6')u^2y +$$
$$+ (c_5 - c_6)uu'y - \tfrac{1}{2}c_{11}'(u^2y') - (c_{11} + c_{14}')uu'y' -$$
$$- \tfrac{1}{2}(2c_{14} + c_{12}')u'^2y'$$

(4.7)

These equations are of the form

$$\begin{cases} f_1 u'' + g y'' = F(u, u', y, y', \theta) \\ g u'' + f_2 y'' = G(u, u', y, y', \theta) \end{cases} \qquad (4.8)$$

and have "solutions"

$$\begin{cases} u'' = \dfrac{f_2 F - g G}{f_1 f_2 - g^2} \\[2ex] y'' = \dfrac{f_1 G - g F}{f_1 f_2 - g^2} \end{cases} \qquad (4.9)$$

We expand these quantities, keeping terms through third order, and derive the differential equations.

$$
\begin{aligned}
a_3 u'' + a_3' u' - (a_1 - a_2')u = {} & \tfrac{1}{2}\Big[(b_1 - b_3') - 2d_1(a_1 - a_2') + \\
& + \tfrac{2b_7}{a_6}(a_1 - a_2')\Big] u^2 + \\
& + \tfrac{1}{2}\Big[-(b_4 + b_2') + 2a_3' d_2 - \tfrac{2b_8 a_3'}{a_6}\Big] u'^2 + \\
& + \Big\{-b_4' + a_3' d_1 - d_2(a_1 - a_2') + \tfrac{1}{a_6}\big[-a_3' b_7 + b_8(a_1 - a_2')\big]\Big\} uu' + \\
& + \tfrac{1}{2}\big[b_5 - b_6'\big] y^2 + \tfrac{1}{2}\Big[b_7 - b_8' + \tfrac{a_6' b_8}{a_6}\Big] y'^2 + \\
& + \Big[b_9 - b_6 - \tfrac{b_8}{a_6}(a_4 - a_5')\Big] yy' + \\
& + \tfrac{1}{6}\Big\{c_1 - c_7' - 3d_1(b_1 - b_3') - 3d_3(a_1 - a_2') + \tfrac{3}{a_6}\big[b_7(b_1 - b_3') + \\
& \qquad + c_{11}(a_1 - a_2') - 2b_7 d_1(a_1 - a_2')\big]\Big\} u^3 + \\
& + \tfrac{1}{6}\Big\{-c_2' - 2c_9 + 3d_2\Big(b_4 + b_2' + \tfrac{2b_8 a_3'}{a_6}\Big) + 3d_4 a_3' + \tfrac{1}{a_6}\big[3b_8(-b_4 - b_2') - a_3' c_{12}\big]\Big\} u'^3 + \\
& + \tfrac{1}{2}\Big\{-c_8' + 2d_1\Big[b_4' - \tfrac{1}{a_6}\big(-b_7 a_3' + b_8(a_1 - a_2')\big)\Big] - d_2\Big[b_1 - b_3' + \tfrac{2b_7}{a_6}(a_1 - a_2')\Big] + d_3 a_3' - \\
& \qquad - 2d_5(a_1 - a_2') + \tfrac{1}{a_6}\big[-2b_7 b_4' + b_8(b_1 - b_3') - c_{11} a_3' + 2c_{14}(a_1 - a_2')\big]\Big\} u^2 u' + \\
& + \tfrac{1}{2}\Big\{-c_8 - c_9' + d_1\Big(b_4 + b_2' + \tfrac{2b_8 a_3'}{a_6}\Big) + 2d_2\Big[b_4' - \tfrac{1}{a_6}\big(-b_7 a_3' + b_8(a_1 - a_2')\big)\Big] - d_4(a_1 - a_2') + \\
& \qquad + 2d_5 a_3' + \tfrac{1}{a_6}\big[-b_7(b_4 + b_2') - 2b_8 b_4' + c_{12}(a_1 - a_2') - 2c_{14} a_3'\big]\Big\} uu'^2 + \\
& + \tfrac{1}{2}\Big\{c_{10} - c_5' + \tfrac{1}{a_6}\big[b_7(b_5 - b_6')\big] - d_1(b_5 - b_6')\Big\} uy^2 +
\end{aligned}
$$

$$+\left\{c_6-c_5-d_1(b_9-b_6)+\frac{d_1 b_8}{a_6}(a_4-a_5')+\frac{1}{a_6}\left[b_7(b_9-b_6)-b_8(b_5-b_9')-c_{14}(a_4-a_5')\right]\right\}uyy'+$$
$$+\frac{1}{2}\left\{c_{11}-c_{14}'-d_1\left(b_7-b_8'+\frac{2a_6'}{a_6}b_8\right)-d_6(a_1-a_2')+\frac{1}{a_6}\left[b_7(b_7-b_8')+c_4(a_1-a_2')+\right.\right.$$
$$\left.\left.+2b_8b_7'+2c_{14}a_6'\right]\right\}uy'^2+$$
$$+\frac{1}{2}\left\{-c_{12}'-d_2\left(b_7-b_8'+\frac{2a_6'b_8}{a_6}\right)+d_6a_3'+\frac{1}{a_6}\left[3b_8(b_7-b_8')-c_4a_3'+2c_{12}a_6'\right]\right\}u'y'^2+$$
$$+\frac{1}{2}\left\{-d_2(b_5-b_6')+\frac{b_8}{a_6}(b_5-b_6')\right\}u'y^2+$$
$$+\left\{-d_2\left[b_9-b_6-\frac{b_8}{a_6}(a_4-a_5')\right]+\frac{1}{a_6}\left[2b_8(b_9-b_6)-c_{12}(a_4-a_5')\right]\right\}u'yy' \tag{4.10}$$

and

$$a_6y''+a_6'y'-(a_4-a_5')y=\left\{b_5-b_9'+\frac{b_4}{a_3}(a_4-a_5')-d_1(a_4-a_5')\right\}uy+$$
$$+\left\{b_6-b_9+\frac{b_2}{a_3}(a_4-a_5')-d_2(a_4-a_5')\right\}u'y+$$
$$+\left\{-b_7'+d_1a_6'+\frac{1}{a_3}\left[-b_4a_6'-b_8(a_1-a_2')\right]\right\}uy'+\left\{-b_7-b_8'+\right.$$
$$\left.+\frac{1}{a_3}\left[-b_2a_6'+b_8a_3'\right]+d_2a_6'\right\}u'y'+\frac{1}{6}\left\{c_3-c_{13}'\right\}y^3+$$
$$+\frac{1}{6}\left\{-c_4'+\frac{1}{a_3}\left[-3c_{12}a_6'-3b_8(b_7-b_8')+3d_6a_6'\right]\right\}y'^3+$$
$$+\frac{1}{2}\left\{c_{10}-c_6'+\frac{1}{a_3}\left[2b_4(b_5-b_9')+c_8(a_4-a_5')\right]-2d_1\left[b_5-b_9'+\frac{b_4}{a_3}(a_4-a_5')\right]-\right.$$
$$\left.-d_3(a_4-a_5')\right\}u^2y+$$
$$+\left\{c_5-c_6+\frac{1}{a_3}\left[b_2(b_5-b_9')+b_4(b_6-b_9)+c_9(a_4-a_5')\right]-d_1\left[(b_6-b_9)+\right.\right.$$
$$\left.\left.+\frac{b_2}{a_3}(a_4-a_5')\right]-d_2\left[b_5-b_9'+\frac{b_4}{a_3}(a_4-a_5')\right]-d_5(a_4-a_5')\right\}uu'y$$
$$+$$

$$+\frac{1}{2}\Big\{-c_{11}'+\frac{1}{a_3}\Big[-2b_4b_7'-c_8a_6'-b_8(b_1-b_3')-2c_{14}(a_1-a_2')\Big]-$$
$$-2d_1\Big[-b_7'+\frac{1}{a_3}(-b_4a_6'-b_8(a_1-a_2'))\Big]+d_3a_6'\Big\}\;u^2y'+$$

$$+\Big\{-c_{11}-c_{14}'+\frac{1}{a_3}\Big[-b_2b_7'-b_4(b_7+b_8')-c_9a_6'+b_8b_4'-c_{12}(a_1-a_2')+$$
$$+a_3'c_{14}\Big]-d_1\Big[-b_7-b_8'+\frac{1}{a_3}(-b_2a_6'+b_8a_3')\Big]-$$
$$-d_2\Big[-b_7'+\frac{1}{a_3}(-b_4a_6'-b_8(a_1-a_2')\Big]+d_5a_6'\Big\}\;uu'y'+$$

$$+\frac{1}{2}\Big\{-c_{12}'-2c_{14}'+\frac{1}{a_3}\Big[-2b_2(b_7+b_8')-c_2a_6'+b_8(b_4+b_2')+2c_{12}a_3'\Big]-$$
$$-2d_2\Big[-b_7-b_8'+\frac{1}{a_3}(-b_2a_6'+b_8a_3')\Big]+d_4a_6'\Big\}\;u'^2y'+$$

$$+\frac{1}{2}\Big\{\frac{1}{a_3}\Big[2b_2(b_6-b_9)+c_2(a_4-a_5')\Big]-2d_2\Big[b_6-b_9+\frac{b_2}{a_3}(a_4-a_5')\Big]$$
$$-d_4(a_4-a_5')\Big\}\;u'^2y\;+$$

$$+\frac{1}{2}\Big\{\frac{1}{a_3}\Big[c_{12}(a_4-a_5')-2b_8(b_9-b_6)-d_6(a_4-a_5')\Big\}\;yy'^2+$$

$$+\frac{1}{2}\Big\{-\frac{b_8}{a_3}(b_5-b_6')\Big\}\;y^2y'$$

(4.11)

where

$$\left\{\begin{aligned}
d_1 &= \frac{a_6 b_4 + a_3 b_7}{a_3 a_6} \\
d_2 &= \frac{a_6 b_2 + a_3 b_8}{a_3 a_6} \\
d_3 &= \frac{a_6 c_8 + 2 b_4 b_7 + a_3 c_{11}}{a_3 a_6} - 2 d_1^2 \\
d_4 &= \frac{a_6 c_2 + 2 b_2 b_8 + a_3 c_{12}}{a_3 a_6} - 2 d_2^2 \\
d_5 &= \frac{a_6 c_4 + b_4 b_8 + b_2 b_7 + a_3 c_{14}}{a_3 a_6} - 2 d_1 d_2 \\
d_6 &= \frac{a_6 c_{12} - b_8 a_3 a_6 + a_3 c_4}{a_3 a_6}
\end{aligned}\right. \tag{4.12}$$

It may be noted that a_3 and a_6 are always different from zero, since $|x_s| \ll 1$ and $|x_s'| \ll 1$, so that the denominators offer no difficulty.

Equations (4.10) and (4.11) are obviously much too complex to be useful. In the next section we estimate the sizes of the terms and find that most are negligibly small.

V. Approximate Expanded Equations

To estimate the sizes of the terms occurring in (4.10) and (4.11), we must estimate the sizes of the vector potential components. We see from (1.14) that the Θ_{mn}, X_{mn} and Y_{mn} depend on the D_n of (1.12), which are the form

$$\mathcal{D}_n = A_n(k) + B_n(k,w)\cos\xi + C_n(k,w)\sin\xi \tag{5.1}$$

so that

$$|\mathcal{D}_n| \leq |A_n| + \sqrt{B_n^2 + C_n^2} \tag{5.2}$$

and

$$|\mathcal{D}_n'| \leq N\,|\mathcal{D}_n| \tag{5.3}$$

From Section III, we see that the closed orbit is of order of magnitude

$$\begin{cases} |x_s| \sim \dfrac{f}{N^2} \\ |x_s'| \sim N|x_s| \sim \dfrac{f}{N^2} \end{cases} \tag{5.4}$$

We carry through the estimates for three different sets of paramets

A. $k \sim 150$, $1/w \sim 2.10^3$, $f = 1/4$, $N \sim 40$

B. $k \sim 100$, $1/w \sim 10^3$, $f = 1/4$, $N \sim 40$.

C. $k \sim 1$, $1/w \sim 20$, $f = 1/4$, $N \sim 5$

A is an example of a proposed full scale machine design, while B is a full scale machine with more conservative parameters. C is an example of a model sized machine.

We give orders of magnitude below

	D_1	D_2	D_3	D_4
A	1	$3 \cdot 10^2$	10^5	10^8
B	1	$1.2 \cdot 10^2$	$4 \cdot 10^4$	10^7
C	1	2.5	20	80

	θ_{10}	θ_{20}	θ_{30}	θ_{40}	θ_{02}	θ_{12}	θ_{22}	θ_{04}
A	1	$3 \cdot 10^2$	10^5	10^8	$3 \cdot 10^2$	$3 \cdot 10^5$	$5 \cdot 10^8$	10^8
B	1	10^2	$4 \cdot 10^4$	10^7	10^2	10^5	10^8	10^7
C	1	3	20	80	3.5	60	500	80

	X_{12}	X_{22}	X_{04}	Y_{11}	Y_{21}	Y_{31}	Y_{03}	Y_{13}
A	40	10^4	$1/3 \cdot 10^4$	40	10^4	$4 \cdot 10^6$	$1/3 \cdot 10^4$	$4 \cdot 10^6$
B	40	$6 \cdot 10^3$	$2 \cdot 10^3$	40	$6 \cdot 10^3$	$1.6 \cdot 10^5$	$2 \cdot 10^3$	$2 \cdot 10^6$
C	5	25	6	5	20	100	6	150

	a_1	a_2	a_3	a_4	a_5	a_6
A	$7\cdot10^2$	$6\cdot10^{-3}$	1	$7\cdot10^2$	$6\cdot10^{-3}$	1
B	$2.5\cdot10^2$	$6\cdot10^{-3}$	1	$2.5\cdot10^2$	$6\cdot10^{-3}$	1
C	10	0.06	1	10	$5\cdot10^{-2}$	1

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
A	10^6	$2\cdot10^{-2}$	$2\cdot10^{-2}$	1	10^6	10^{-2}	1	$6\cdot10^{-3}$	40
B	$3\cdot10^5$	$2\cdot10^{-2}$	$2\cdot10^{-2}$	1	$2.5\cdot10^5$	10^{-2}	1	$6\cdot10^{-3}$	40
C	150	0.2	0.06	1	150	0.11	1	0.06	5

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
A	$2\cdot10^9$	10	$2\cdot10^9$	3	80	$2\cdot10^4$	$4\cdot10^{-2}$
B	$2\cdot10^8$	10	$2\cdot10^8$	3	80	$2\cdot10^4$	$4\cdot10^{-2}$
C	$2\cdot10^3$	10	$2\cdot10^3$	3	10	50	0.3

	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
A	10	$6\cdot10^{-3}$	$2\cdot10^9$	1	1	$2\cdot10^4$	$2\cdot10^{-2}$
B	10	10^{-2}	$4\cdot10^8$	1	1	10^4	$2\cdot10^{-2}$
C	10	0.06	$2\cdot10^3$	1	1	40	0.15

	d_1	d_2	d_3	d_4	d_5	d_6
A	1	$2\cdot10^{-2}$	10	10	$5\cdot10^{-2}$	3
B	1	$2\cdot10^{-2}$	10	10	$5\cdot10^{-2}$	3
C	1	0.2	10	10	1	3

We estimate also the following derivatives which appear in the equations of motion.

	a_2'	a_3'	a_5'	a_6'	b_2'	b_3'	b_4'	b_6'
A	1	$6\cdot10^{-3}$	0.25	$6\cdot10^{-3}$	3	2	$1.2\cdot10^{-2}$	0.5
B	1	$6\cdot10^{-3}$	0.25	$6\cdot10^{-3}$	3	2	$1.2\cdot10^{-2}$	0.5
C	1	0.06	0.25	0.06	3	2	0.12	0.5

	b_7'	b_8'	b_9'	c_2'	c_4'	c_5'	c_6'	c_7'	c_8'
A	$6\cdot10^{-3}$	1	$2\cdot10^{3}$	$6\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{3}$	$8\cdot10^{5}$	6	$6\cdot10^{-2}$
B	$6\cdot10^{-3}$	1	$2\cdot10^{3}$	$6\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{3}$	$8\cdot10^{5}$	6	$6\cdot10^{-2}$
C	0.06	1	25	0.6	0.3	50	250	6	0.6

	c_9'	c_{11}'	c_{12}'	c_{13}'	c_{14}'
A	6	$6\cdot10^{-3}$	$2\cdot10^{-2}$	$8\cdot10^{5}$	3
B	6	$6\cdot10^{-3}$	$2\cdot10^{-2}$	$4\cdot10^{5}$	3
C	6	0.06	0.25	200	3

When we substitute these estimates for case A, the differential equations (4.10) and (4.11) have the orders of magnitude

$$
\begin{aligned}
a_3 u'' + a_3' u' + [7\cdot 10^2] u = {} & \tfrac{1}{2}[10^6] u^2 + \tfrac{1}{2}[4] u'^2 + [20] u u' + \\
& + \tfrac{1}{2}[10^6] y^2 + \tfrac{1}{2}[2] y'^2 + [50] y y' + \\
& + \tfrac{1}{6}[2\cdot 10^9] u^3 + \tfrac{1}{6}[1] u'^3 + \\
& + \tfrac{1}{2}[2\cdot 10^4] u^2 u' + \tfrac{1}{2}[7\cdot 10^3] u u'^2 + \\
& + \tfrac{1}{2}[2\cdot 10^9] u y^2 + [2\cdot 10^4] u y y' + \\
& + \tfrac{1}{2}[4\cdot 10^3] u y'^2 + \tfrac{1}{2}[10^{-1}] u' y'^2 + \\
& \tfrac{1}{2}[2\cdot 10^4] u' y^2 + [7\cdot 10^2] u' y y'
\end{aligned}
$$

$$
\begin{aligned}
a_6 y'' + a_6' y' + [7\cdot 10^2] y = {} & [10^6] u y + [50] u' y + [4] u y' + \\
& + [2] u' y' + \tfrac{1}{6}[2\cdot 10^9] y^3 + \\
& + \tfrac{1}{6}[10^{-1}] y'^3 + \tfrac{1}{2}[2\cdot 10^9] u^2 y + \\
& + [2\cdot 10^4] u u' y + \tfrac{1}{2}[6\cdot 10^3] u^2 y' + \\
& + [7\cdot 10^2] u u' y' + \tfrac{1}{2}[6] u'^2 y' + \\
& + \tfrac{1}{2}[7\cdot 10^3] u'^2 y + \tfrac{1}{2}[2\cdot 10^3] y y'^2 + \\
& + \tfrac{1}{2}[6\cdot 10^3] y^2 y' .
\end{aligned}
$$

If we take $|u'| \sim N|u|$ and $|y'| \sim N|y|$ which is probably an overestimate, then the only terms containing derivatives which are as large as 1% of the leading term of the same power are the uu'^2 and uy'^2 terms of the u equation and the u'^2y and yy'^2 terms of the y equation. Closer examination shows that the two large terms in each of these coefficients cancel exactly, so that the overestimate is gross. Thus no "derivative" term is larger than 0.1% of the leading term of the same power.

Though the numbers are changed in case B, the same conclusion holds as in case A.

In cases A and B, all terms greater than 0.1% of the leading term of the same power are included in the approximate equations

$$\begin{cases} a_3 u'' + a_3' u' - a_1 u = \frac{1}{2} b_1 u^2 + \frac{1}{2} b_5 y^2 + \frac{1}{6} c_1 u^3 + \frac{1}{2} c_{10} u y^2 \\ a_6 y'' + a_6' y' - a_4 y = b_5 u y + \frac{1}{6} c_3 y^3 + \frac{1}{2} c_{10} u^2 y \end{cases} \tag{5.5}$$

which may be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2} a_3 u'^2 + \frac{1}{2} a_6 y'^2 + \frac{1}{2} a_1 u^2 + \frac{1}{2} a_4 y^2 + \frac{1}{6} b_1 u^3 + \frac{1}{2} b_5 u y^2 + \frac{1}{24} c_1 u^4 + \frac{1}{24} c_3 y^4 + \frac{1}{4} c_{10} u^2 y^2 \tag{5.6}$$

In case C the situation does not appear from the same estimates to be so favorable. (4.10) and (4.11) have the following orders of magnitude in this case:

$$
\begin{aligned}
a_3 u'' + a_3' u' + [10]u = {} & \tfrac{1}{2}[150]u^2 + \tfrac{1}{2}[4]u'^2 + [3]uu' + \\
& + \tfrac{1}{2}[150]y^2 + \tfrac{1}{2}[2]y'^2 + [6]yy' + \\
& + \tfrac{1}{6}[2\cdot 10^3]u^3 + \tfrac{1}{6}[3]u'^3 + \\
& + \tfrac{1}{2}[2]u^2u' + \tfrac{1}{2}[20]uu'^2 + \\
& + \tfrac{1}{2}[2\cdot 10^3]uy^2 + [150]uyy' + \\
& + \tfrac{1}{2}[5]uy'^2 + \tfrac{1}{2}[\tfrac{1}{4}]u'y'^2 + \\
& + \tfrac{1}{2}[30]u'y^2 + [12]u'yy'
\end{aligned}
$$

and

$$
\begin{aligned}
a_6 y'' + a_6' y' + [10]y = {} & [150]uy + [5]u'y + [0.6]uy' + \\
& + [2]u'y' + \tfrac{1}{6}[2\cdot 10^3]y^3 + \tfrac{1}{6}[0.3]y'^3 + \\
& + \tfrac{1}{2}[2\cdot 10^3]u^2y + [60]uu'y + \\
& + \tfrac{1}{2}[9]u^2y' + [4]uu'y' + \\
& + \tfrac{1}{2}[6]u'^2y' + \tfrac{1}{2}[0.6]u'^2y + \\
& + \tfrac{1}{2}[0.3]yy'^2 + \tfrac{1}{2}[9]y^2y'
\end{aligned}
$$

The only terms which are less than 1% of the leading terms of the same power (using again $|u'| \sim N|u|$ and $|y'| \sim N|y|$) are the $u^2 u'$ term of the u equation and the $u'^2 y$ and yy'^2 terms of the y equation. Closer examination of the u'^2 term of the u equation as an example shows that there is no cancellation tending to reduce this estimate. There appear to be no simple approximate equations analogous to (5.5) in the case of small (model-sized) machines.

Even in the case of large machines (cases A and B), the validity of the approximate equations (5.5) is open to some question because of the unknown influence of higher terms (fourth andhigher powers) neglected in our original expansion. This is clearly closely connected with the slowness of convergence of the vector potential expansion (1.13).

It appears from the digital computer work that $|u_{max}| \sim w$, though Vogt-Nilsen (unpublished) has found a case where $|u_{max}| = 1.8w$. Then in case A above

$$a_1 u \sim 0.35$$
$$\frac{1}{2} b_1 u^2 \sim 0.125$$
$$\frac{1}{6} c_1 u^3 \sim 0.042$$

and if a fourth power term is included with a coefficient

$$e_1 \sim 4 \cdot 10^{12}$$

then
$$\frac{1}{24} e_1 u^4 \sim 0.01$$

so that the fourth power term would be about 3% of the linear term.

In case B
$$a_1 u \sim 0.25$$
$$\frac{1}{2} b_1 u^2 \sim 0.15$$
$$\frac{1}{6} c_1 u^3 \sim 0.03$$
$$\frac{1}{24} e_1 u^4 \sim 0.008 \qquad (e_1 \sim 2 \cdot 10^{11})$$

so that the quartic term is again about 3% of the linear term.

One may therefore doubt the validity of the whole expansion. It should be pointed out, however, that Moser(1) and Sturrock(2) find that instabilities due to non-linear terms arise only from quadratic and cubic terms. It might be supposed the the same difficulties would enter the determination of the coefficients a_i, b_i and c_i, but here the convergence is much more rapid because $|x_s| \ll 1$.

Expressions for the coefficients appearing in (5.5) in terms of the parameters are given below:

$$a_1 = -\left[(k+1) + \frac{1}{2}\frac{f^2}{k+1-N^2}\left(\frac{1}{w^2} - k^2 + 2k - 3\right)\right] + \cos\xi\left[-\frac{f}{w} + \right.$$

$$\left. + \frac{f^3}{8w(k+1-N^2)^2}\left(\frac{1}{w^2} - 3k^2 + 8k - 7\right)\right] + \sin\xi\left[f(k+1)\right] +$$

$$+ \cos 2\xi\left[\frac{\frac{1}{2}f^2}{k+1-N^2}\left(\frac{1}{w^2} - k^2 + 2k - 3\right)\right] +$$

$$+ \cos 3\xi\left[\frac{f^3}{8w(k+1-N^2)^2}\left(3k^2 - 8k + 7 - \frac{1}{w^2}\right)\right] + \ldots$$

$$a_3 = \frac{(1+x_s)^2}{\left[(1+x_s)^2 + x_s'^2\right]^{3/2}}$$

$$a_4 = \left[k + \frac{1}{2}\frac{f^2}{k+1-N^2}\left(\frac{1}{w^2} - k^2 + k - 4 + N^2\right)\right] +$$

$$+ \cos\xi\left[\frac{f}{w} + \frac{f^3}{2w(k+1-N^2)^2}\left(3k^2 - 6k + 2 - N^2 - \frac{1}{w^2}\right)\right] +$$

$$+ \sin\xi\left[-kf\right] + \cos 2\xi\left[\frac{f^2}{2(k+1-N^2)}\left(k^2 - k + 4 - N^2 - \frac{1}{w^2}\right)\right] +$$

$$+ \cos 3\xi\left[\frac{f^3}{2w(k+1-N^2)^2}\left(\frac{1}{w^2} - 3k^2 + 6k - 2 + N^2\right)\right] + \ldots$$

$$a_6 = \frac{1}{\sqrt{(1+x_s)^2 + x_s'^2}}$$

$$b_1 = -k(k+1) + \frac{1}{2}\frac{f^2}{k+1-N^2}\left[-\frac{3k+4}{w^2} + k^3 - 4k^2 + 7k - 12\right]$$

$$+ \cos\xi\left[\frac{f}{w}(2k+1) + \frac{f^3}{4w(k+1-N^2)^2}\left(\frac{1}{w^2} - 3k^2 + 8k - 7\right)\left(k + 2 - \frac{2k+3}{k+1}N^2\right)\right] +$$

$$+ \sin\xi\left[\frac{f}{w^2}\left(1 - k(k+1)w^2\right)\right] +$$

$$+ \cos 2\xi\left[\frac{1}{2}\frac{f^2}{k+1-N^2}\left(\frac{3k+4}{w^2} - k^3 + 4k^2 - 7k + 12\right)\right] +$$

$$+ \sin 2\xi\left[\frac{f^2}{2w(k+1-N^2)}\left(\frac{1}{w^2} - 3k^2 + 8k - 7\right)\right] + \ldots$$

$$
\begin{aligned}
b_5 = {} & k^2+4-\frac{1}{2}\frac{f^2}{k+1-N^2}\left[k(k-1)(k-2)+(k-5)N^2-\frac{3k+4}{w^2}\right]+ \\
& +\cos\xi\left[\frac{2kf}{w}+\frac{f^3}{4w(k+1-N^2)^2}\left(3k^2-6k+2-N^2-\frac{1}{w^2}\right)\left(k+2-\frac{2k+3}{k+1}N^2\right)\right]+ \\
& +\sin\xi\left[-f\left(k^2+4-N^2-\frac{1}{w^2}\right)\right]+ \\
& +\cos 2\xi\left[\frac{1}{2}\frac{f^2}{k+1-N^2}\left[k(k-1)(k-2)+(k-5)N^2-\frac{3k+4}{w^2}\right]\right]+ \\
& +\sin 2\xi\left[\frac{f^2}{2w(k+1-N^2)}\left(3k^2-6k+2-N^2-\frac{1}{w^2}\right)\right]+\cdots \\
c_1 = {} & -\left\{(k+1)k(k-1)+f\left[\frac{3k^2-1-\frac{1}{w^2}}{w}\cos\xi+\left(\frac{3k+8}{w^2}-(k+1)k(k-1)\right)\sin\xi\right]\right\} \\
c_3 = {} & -k^2(k-2)-f\left[\frac{k(3k-4)-2N^2-\frac{1}{w^2}}{w}\cos\xi+\left((2N^2-k^2)(k-2)+\frac{3(k+2)}{w^2}\right)\sin\xi\right] \\
c_{10} = {} & k^3-k^2+8+f\left\{\frac{3k^2-2k-N^2-\frac{1}{w^2}}{w}\cos\xi+\left(\frac{3k+6}{w^2}-k^3+k^2-8-(k-7)N^2\right)\sin\xi\right\}
\end{aligned}
\tag{5.7}
$$

All terms less than 1% of the leading term of the same power of the variables have been neglected.

The terms linear in the first derivatives u' and y' arise from the variation of inertia due to the scalloping of the equilibrium orbit. These terms may be eliminated by the substitutions

$$\begin{cases} \rho = \sqrt{a_3}\, u \\ \zeta = \sqrt{a_6}\, y \end{cases} \tag{5.8}$$

Then

$$\begin{cases} \rho'' + \left[\frac{1}{4}\left(\frac{a_3'}{a_3}\right)^2 - \frac{1}{2}\frac{a_3''}{a_3} - \frac{a_1}{a_3}\right]\rho = \frac{1}{2}\frac{b_1}{a_3^{3/2}}\rho^2 + \frac{1}{2}\frac{b_5}{a_6\sqrt{a_3}}\zeta^2 + \\ \qquad + \frac{1}{6}\frac{c_1}{a_3^2}\rho^3 + \frac{1}{2}\frac{c_{10}}{a_3 a_6}\rho\zeta^2 \\ \zeta'' + \left[\frac{1}{4}\left(\frac{a_6'}{a_6}\right)^2 - \frac{1}{2}\frac{a_6''}{a_6} - \frac{a_4}{a_6}\right]\zeta = \frac{b_5}{a_6\sqrt{a_3}}\rho\zeta + \frac{1}{6}\frac{c_3}{a_6^2}\zeta^3 + \\ \qquad + \frac{1}{2}\frac{c_{10}}{a_3 a_6}\rho^2\zeta \end{cases} \tag{5.9}$$

Now

$$a_3 = \frac{(1+x_s)^2}{L_0^3} = 1 - x_s + o(x_s^2, x_s'^2)$$

and to better than 0.1%

$$a_3 = 1$$

$$a_3' = \frac{x_s'(1+x_s)}{L_0^5}\left\{2x_s'^2 - (1+x_s)^2 - 3(1+x_s)x_s''\right\}$$

$$= o(x_s')$$

which is negligible.

$$a_3'' = -x_s'' - 3x_s''^2 + o(x_s)$$

and

$$x_s'' = f \sin \xi + o(x_s)$$

so that

$$a_3'' = -f \sin \xi - 3f^2 \sin^2 \xi + o(x_s)$$

$$= -f \sin \xi - \frac{3}{2} f^2 (1 - \cos 2\xi) + o(x_s)$$

This changes a_1 to

$$\overline{a}_1 = a_1 + \frac{1}{2} a_3'' = -\left[k+1+\frac{1}{2}\frac{f^2}{k+1-N^2}\left(\frac{1}{w^2}-k^2+\frac{1}{2}k-\frac{9}{2}+\frac{3}{2}N^2\right)\right] +$$

$$+ \cos \xi \left[-\frac{f}{w} + \frac{f^3}{8w(k+1-N^2)^2}\left(\frac{1}{w^2} - 3k^2 + 8k - 7\right)\right] +$$

$$+ \sin \xi \left[f\left(k + \frac{1}{2}\right)\right] +$$

$$+ \cos 2\xi \left[\frac{\frac{1}{2} f^2}{k+1-N^2}\left(\frac{1}{w^2} - k^2 + \frac{7}{2}k - \frac{3}{2} - N^2\right)\right] +$$

$$+ \cos 3\xi \left[\frac{f^3}{8w(k+1-N^2)^2}\left(3k^2 - 8k + 7 - \frac{1}{w^2}\right)\right]$$

$$+ \dots \qquad (5.10)$$

In the same way,

$$a_6 = \frac{1}{L_0} = 1 - x_s + 0(x_s^2, x_s'^2)$$

$$= 1 \qquad \text{to better than } 0.1\%$$

$$a_6' = -\frac{x_s'}{L_0^3}(1 + x_s + x_s'')$$

$= 0(x_s')$, which may be neglected.

$$a_6'' = -x_s'' - x_s''^2 + 0(x_s)$$
$$= -f \sin\xi - \tfrac{1}{2} f^2 (1 - \cos 2\xi) + 0(x_s),$$

which changes a_4 to

$$\overline{a}_4 = a_4 + \tfrac{1}{2} a_6'' = k + \tfrac{1}{2}\frac{f}{k+1-N^2}\left(\frac{1}{w^2} - k^2 + \tfrac{1}{2}k - \tfrac{9}{2} + 2N^2\right) +$$

$$+ \cos\xi\left[\frac{f}{w} + \frac{f^3}{2w(k+1-N^2)^2}\left(3k^2 - 6k + 2 - N^2 - \frac{1}{w^2}\right)\right] +$$

$$+ \sin\xi\left[-\left(k + \tfrac{1}{2}\right) f\right] +$$

$$+\cos 2\xi\left[\frac{f^2}{2(k+1-N^2)}\left(k^2-\frac{1}{2}k+\frac{9}{2}-2N^2-\frac{1}{w^2}\right)\right]+$$

$$+\cos 3\xi\left[\frac{f^3}{2w(k+1-N^2)^2}\left(\frac{1}{w^2}-3k^2+6k-2+N^2\right)\right]$$

$$+\ldots \tag{5.11}$$

and (5.9) becomes

$$\begin{cases} \rho''-\bar{a}_1\rho = \frac{1}{2}b_1\rho^2+\frac{1}{2}b_5\zeta^2+\frac{1}{6}c_1\rho^3+\frac{1}{2}c_{10}\rho\zeta^2 \\ \zeta''-\bar{a}_4\zeta = b_5\rho\zeta+\frac{1}{6}c_3\zeta^3+\frac{1}{2}c_{10}\rho^2\zeta \end{cases} \tag{5.12}$$

where the coefficients are given by (5.7), (5.10) and (5.11).

As shown above, the quadratic terms in these equations can be as large as 50%, while the cubic terms can be as large as 12%. The effect of the closed orbit is largest on the quadratic term coefficients where it is of the same order of magnitude as the original terms, while the closed orbit has only about 5% effect on the linear terms and a negligible effect on the cubic terms.

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