MURA 90

MURA AMS-5
Internal

HALF-SECTOR PHASE PLANE TRANSFORMATIONS

FOR AN A.G. SYNCHROTRON*

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I. Introduction

The study of displaced and misaligned sectors would be facilitated by the existance of half-sector transformations. Previous work of the MURA Group has been limited by the use of only full sector transformations, which do not correspond to displacements which are physically realistic. In this report the methods outlined previously (MURA AMS 2,3,4 -- hereafter called I, II, III) are used to generate half-sector one dimensional transformations for a non-linear C.L.S. machine. These are for each half-sector of Powell's form (MURA RW JLP-5). It is felt that Birkhoff Variables are best suited for the study of bumps, but because the Illiac is already programed for Powell's transformation, the results of this report may be of immediate value.

II. Construction of the Transformations

We shall take as the form of the transformation through a half-sector:

$$x = \alpha_i x_o + \beta_i y_o + \beta_i k_i \left\{(1 + \alpha_i)x_o + \beta_i y_o\right\}^3$$
$$y = \gamma_i x_o + \alpha_i y_o + (1 + \alpha_i) k_i \left\{(1 + \alpha_i)x_o + \beta_i y_o\right\}^3 \qquad (1)$$

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where i = 1 for a positive half-sector, and i = 2 for a negative half-sector. We must determine the two parameters k_i, and for this purpose the methods of I and II may be used.

First consider only the transformation through a positive half-sector. We can transform it into Birkhoff Variables and thus determine the betatron wavelength in a machine of only positive sectors. Upon comparison with perturbation theory k_1 is determined. Similarly we may work with the negative half-sectors and compare imaginary betatron wavelengths to fix k_2. However, there are other much simpler methods available which yield the same result. These are based on the theory of Birkhoff, but avoid the use of the complicated formalism developed in I and II.

For each half sector we may readily obtain the dependence of σ on amplitude, by the method of harmonic balance. Starting with:

$$\ddot{x} + w_o^2 x + \frac{1}{3} e x^3 = 0 \qquad (2)$$

we demand a solution of the form x = A sin w t, and equating the coefficients of the lowest harmonic (sin w t) yields:

$$w = w_o \left[1 + \frac{1}{8} \frac{e A^2}{w_o^2} \right] \qquad (3)$$

Similarly for:

$$\ddot{x} - w_o^2 x - \frac{1}{3} e x^3 = 0 \qquad (4)$$

one obtains the same result for w, by a similar procedure and solution of the form x = A sinh w t.

Now one reuslt of Birkhoff's work, is that for small amplitudes the transformation at a point of symmetry (<u>any</u> point for a constant gradient machine of positive half-sectors) may be written as:

$$x = x_o \cos \sigma_1 + y_o \sin \sigma_1$$
$$y = - x_o \sin \sigma_1 + y_o \cos \sigma_1 \qquad (5)$$

$$\text{where } \sigma_1 = \sigma_{o_1} + \Delta \sigma_1 \left[x_o^2 + \beta_o^2 y_o^2\right]$$

This is true because to first approximation the x - y variables are equivalent to Birkhoff u - v Variables except for a change of scale in the v-variable. Thus if we take $y_o = 0$, then:

$$R_1 \equiv \frac{x_1}{x_o} = \cos \sigma_1 \qquad (6)$$

The procedure is to calculate R_1 using a transformation of the form of Equation 1 and thus determine k_1 by comparison with Equation 3. From Equation 1:

$$R_1 \equiv \frac{x_1}{x_o} = \alpha_1 + k_1 \beta_1 (1 + \alpha_1)^3 x_o^2 \qquad (7)$$

which solving for $\Delta \sigma_1$ yields:

$$\Delta \sigma_1 = - \frac{k_1 \beta_1 (1 + \alpha_1)^3}{\sqrt{1 - \alpha_1^2}} \qquad (8)$$

Comparison with Equation 3, which implies:

$$\Delta \sigma_1 = \frac{e_1 t_1}{8 w_{o_1}} \qquad (9)$$

yields a value for k_1.

Similarly for negative half-sectors one obtains:

$$R_2 \equiv \frac{x_2}{x_o} = \cosh \sigma_2 \tag{10}$$

$$R_2 = \alpha_2 + k_2 \beta_2 (1 + \alpha_2)^3 x_o^2$$

and thus

$$\sigma_2 = \frac{k_2 \quad _2 (1 + \quad _2)^3}{_2^{\;2} - 1} \tag{11}$$

which on comparison with

$$\Delta \sigma_2 = \frac{e_2 t_2}{8 \, w_{o_2}} \tag{12}$$

yields a value for k_2.

As an example consider the special case of a machine specified by:

$$w_{o_1} = - w_{o_2} = 4$$

$$t_1 = t_2 = 1/4 \tag{13}$$

$$e_1 = -e_2 = 1000$$

This implies:

$$\sigma_{o_1} = \sigma_{o_2} = 1$$

$$\sigma_o = .5847 \tag{14}$$

$\alpha_1 = .5404$ $\qquad$ $\alpha_2 = 1.543$

$\beta_1 = .2104$ $\qquad$ $\beta_2 = .2938$

$\gamma_1 = -3.3649$ $\qquad$ $\gamma_2 = 4.7010$

One readily obtains: $\Delta\sigma_1 = -.9140\ k_1 = 7.8125$

$$\Delta\sigma_2 = 4.112\ k_2 = 7.8125 \qquad (15)$$

and thus:

$$k_1 = -8.55$$
$$k_2 = 1.90 \qquad (16)$$

III. Calculation of Sigma

We have thus obtained area preserving half-sector trnasformations which can easily be itterated using an existing Illiac program. The question may be raised as to how the σ for the combined full sector transformation so obtained, compares with the correct σ. This may be readily answered by reducing to Birkhoff Variables and so obtaining σ, the transformation obtained by combining algebraicly two transformations of the form of Equation 1. The result of this is clearly that to first order, Birkhoff Variables are the same as $x - y$ variables. Thus it follows that to first order, the reduction of the combined transformations to Birkhoff Variables yields the same result as reducing each transformation to Birkhoff Variables and the combining. Hence to this order, the procedure of determining k_1 and k_2 as above, will yield σ correctly. That is, σ is correct through terms depending on x_o^2, y_o^2, and $x_o\, y_o$.

It will be noted that the results of I may be obtained very easily, by the methods of this report.

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