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AN APPLICATION OF FFAG PRINCIPLE TO THE PHASE OSCILLATION IN SYNCHROTRON

Tihiro Okhawa*
University of Illinois and Midwestern Universities Research Association**

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It is well known that the A.G. principle can be applied to the phase oscillation in synchrotrons. The restoring force in the phase equation is very weak and the application seems to give no advantage.

However, if the A.G. principle is applied in the sense of FFAG principle, it can be shown that both intervals, rising phase and falling phase of r-f voltage, become stable and give the correct accelerating voltage. This fact is advantageous to avoid the difficulties at "Transition Energy" because the entire accelerating phase is stable.

Similarly to FFAG, there are several actual ways to realize the FFAG focusing. The examples show Mark I type and Mark V type. Mark I type has a large "circumference factor" which means in this case a large cavity voltage compared to the actual energy gain per turn. Mark V requires two frequencies of r-f voltage, accelerating voltage and focusing or "ridge" voltage, which correspond to momentum compaction field $(r/ro)^K$ and ridged field Sin $(\frac{x}{\lambda} - N\theta)$ in Mark V.

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a. Mark I type

The phase equation in synchrotron is given by[1]

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2 K}\right)\frac{d\phi}{dt} + \frac{eV}{2\pi}\sin\phi = \frac{eV}{2\pi}\sin\phi_s \tag{1}$$

where $K = 1 + \frac{n}{1-n}\frac{1}{\beta_s^2}$

E_s : total energy of a particle

ω_s : angular frequency of a particle

neglecting the damping term due to radiation loss. This equation is written as follows:

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2 K}\right)\frac{d\phi}{dt} + \frac{e}{2\pi}\left[\text{Actual voltage on a particle}\right] = \frac{e}{2\pi}\left[\text{required voltage per turn for a particle}\right] \tag{2}$$

For convenience t is replaced by τ

$$t = \frac{2\pi}{\omega_s}\tau \tag{3}$$

Then

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} + 2\pi e\left[\text{Actual Voltage}\right] = 2\pi e\left[\text{Required Voltage}\right]$$

If two cavities with voltage V_1 and V_2 and angular frequency $\omega_1 = \omega_2 = \omega_s$, are applied in such a way that the phase difference between two voltages seen from a particle is π (for instance, two cavities are laid on the opposite side of the circumference of machine and work in phase),

the phase equation becomes

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} + 2\pi e V_1 \sin\phi = 2\pi e V_{01} \qquad m < \tau < m+\tfrac{1}{2} \tag{4}$$

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} + 2\pi e V_2 \sin(\phi+\pi) = 2\pi e V_{02} \qquad m+\tfrac{1}{2} < \tau < m$$

$V_{01} + V_{02}$ = required voltage gain per turn

= V_0

m : integer

Putting $\phi = \phi_s + x$, the equations for x are obtained.

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} + 2\pi e V_1(\sin\phi_s\cos x + \cos\phi_s\sin x) = 2\pi e V_{01}$$

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} - 2\pi e V_2(\sin\phi_s\cos x + \cos\phi_s\sin x) = 2\pi e V_{02} \tag{5}$$

Developing in terms of x, we get

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} + 2\pi e\left[V_1\sin\phi_s - V_{01}\right] + x\left[2\pi e V_1\cos\phi_s\right] - \frac{x^2}{2}\left[2\pi e V_1\sin\phi_s\right] + \cdots = 0$$

(6)

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{d\phi}{d\tau} - 2\pi e\left[V_2\sin\phi_s + V_{02}\right] - x\left[2\pi e V_2\cos\phi_s\right] + \frac{x^2}{2}\left[2\pi e V_2\sin\phi_s\right] + \cdots = 0$$

Now we choose the fixed point ϕ_s similar to r_1 determined by momentum relation in the motion in FFAG.

$$V_1 \sin\phi_s = V_{01}$$
$$V_2 \sin\phi_s = -V_{02} \qquad (7)$$

namely

$$(V_1 - V_2)\sin\phi_s = V_0$$

Hence the circumference factor is given by $\dfrac{V_1}{V_1 - V_2}$

In linear approximation, the equations for x are

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{dx}{d\tau} + 2\pi e V_1 \cos\phi_s \; x = 0 \qquad (8)$$

$$\frac{d}{d\tau}\left(\frac{E_s}{K}\right)\frac{dx}{d\tau} - 2\pi e V_2 \cos\phi_s \; x = 0$$

These are the same equations that appeared in Mark I., if we neglect the slow variation of E_s, K, and ϕ_s with τ. Hence, the circumference factor is about 5 and V_1/V_2 is about 1.3. It can be said analogically that the falling phase corresponds to the r -direction and the rising phase to z-direction except for the absence of the centrifugal term.

b. Mark V type.

This type has also two accelerating voltages. One has the voltage V_1 and angular frequency $\omega_1 = \omega_s$ resonant to the frequency of a particle, another has the Voltage V_2 angular frequency ω_2 is near but not equal to some higher harmonic of ω_s. The two voltages are <u>superposed</u>.

$$\omega_2 = N\omega_s = (N' + \delta)\omega_s \qquad (9)$$

N': integer (Order of harmonics)

δ : fraction of one

Relative to a fixed phase of the wave V_1, the wave V_2 moves with velocity u per turn in units of ω_s, where

$$u = 2\pi\delta \qquad (10)$$

A particle of phase ϕ relative to the fundamental wave V_1 is under the force of V_2 given by

$$e V_2 \sin\{N(\phi - \phi_0) - 2\pi\delta\tau\} \qquad (11)$$

where ϕ_0 is the initial phase and can be chosen $\phi_0 = 0$ because of presence of $2\pi\delta\tau$ term.

If we neglect the slow variation of E_s, K and ϕ_s with τ, the equation is written as

$$\frac{d^2\phi}{d\tau^2} + A\sin\phi + B\sin(N\phi - 2\pi\delta\tau) = C \qquad (12)$$

where $A = \frac{K\,2\pi e V_1}{E_s}$, $B = \frac{K\,2\pi e V_2}{E_s}$, $C = \frac{K\,2\pi e V_0}{E_s}$,

Putting $\phi = \phi_s + x$, we get

$$\frac{d^2\phi}{d\tau^2} + A\sin\phi_s\cos x + A\cos\phi_s\sin x + B\sin(N\phi_s - a\tau)\cos Nx + B\cos(N\phi_s - a\tau)\sin Nx = C \tag{13}$$

where $a = 2\pi\delta$

Then developing in terms of x

$$\begin{aligned}\frac{d^2\phi}{d\tau^2} &+ A\sin\phi_s - C + B\sin\theta \\ &+ x\left[A\cos\phi_s + NB\cos\theta\right] \\ &- \frac{x^2}{2}\left[A\sin\phi_s - BN^2\sin\theta\right] \\ &- \frac{x^3}{6}\left[A\cos\phi_s + BN^3\cos\theta\right] + \cdots = 0\end{aligned} \tag{14}$$

where $\theta = N\phi_s - a\tau$

is obtained.

Now ϕ_s is determined by

$$A\sin\phi_s = C \quad \text{or} \quad V_1\sin\phi_s = V_0 \tag{15}$$

Similarly to Mark V, the forced motion x_f and oscillating motion ξ are separated by putting $x = x_f + \xi$,

$$\frac{d^2x_f}{d\tau^2} + B\sin\theta + x_f\left[A\cos\phi_s + NB\cos\theta\right] - \frac{x_f^2}{2}\left[A\sin\phi_s - BN^2\sin\theta\right] + \cdots = 0 \tag{16}$$

$$\frac{d^2\xi}{d\tau^2} + \xi\left[A\cos\phi_s + NB\cos\theta - x_f(A\sin\phi_s - BN^2\sin\theta) - \frac{x_f^2}{2}(A\cos\phi_s + BN^3\cos\theta) + \cdots\right] + \cdots = 0 \tag{17}$$

This equation shows there exists a "scalloped" effect. The forced motions are solved as follows

$$x_f = -\frac{B}{a^2}\left(1 - \frac{A}{a^2}\cos\phi_s\right)\sin\theta - \frac{NB^2}{8a^4}\sin 2\theta - \frac{1}{16}\frac{B^2}{a^4}\cos 2\theta + \cdots \tag{18}$$

with a slight modification of the relation of ϕ_s and V_0

$$V_1 \sin\phi_s\left(1 - \frac{1}{4}\frac{B^2}{a^4}\right) = V_0 \tag{19}$$

Inserting this expression of x_f into the equation (17)

$$\begin{aligned}\frac{d^2\xi}{d\tau^2} + \xi\Big[&\left\{A\cos\phi_s + \frac{B^2}{2a^2}\left(1 - \frac{A}{a^2}\right)\left(N^2 + \frac{A}{2}\sin\phi_s\right)\right\} \\ &+ \cos\theta \cdot NB \\ &+ \sin\theta \cdot \frac{BA}{a^2}\sin\phi_s\left(1 - \frac{A}{a^2}\right) \\ &+ \cos 2\theta\left\{-A\sin\phi_s - \frac{B}{a^2}\left(1 - \frac{A}{a^2}\right)\left(N^2 + \frac{A}{2}\sin\phi_s\right)\right\} \\ &+ \sin 2\theta\left\{-A\sin\phi_s\frac{NB^2}{8a^4}\right\} + \cdots\Big] + \cdots = 0\end{aligned} \tag{20}$$

This is the same as Hill's equation which describes the motion in spirally ridged magnetic field. In the ordinary phase equation the restoring force is weak and A and B is small. Then in rough approximation the sin terms in (20) are neglected

$$\frac{d^2\xi}{d\tau^2} + \xi\left[A\cos\phi_s + \frac{N^2B^2}{(2\pi\delta)^2} + NB\cos(N\phi_s - 2\pi\delta\tau) - \cos 2(N\phi_s - 2\pi\delta)\cdot\left(A\sin\phi_s + \frac{N^2B^2}{(2\pi\delta)^2}\right)\right] \sim 0$$

If we put $N\phi_s - 2\pi\delta\tau = 2y$ the standard form is obtained,

$$\frac{d^2\xi}{dy^2} + \xi\left[\lambda_0 + \lambda_2\cos 2y + \lambda_4\cos 4y\right] = 0$$

$$\lambda_0 = \frac{1}{(\pi\delta)^2}\left(A\cos\phi_s + \frac{N^2B^2}{(2\pi\delta)^2}\right) \tag{21}$$

$$\lambda_2 = \frac{1}{(\pi\delta)^2}NB$$

$$\lambda_4 = \frac{1}{(\pi\delta)^2}\left(A\sin\phi_s + \frac{N^2B^2}{(2\pi\delta)^2}\right)$$

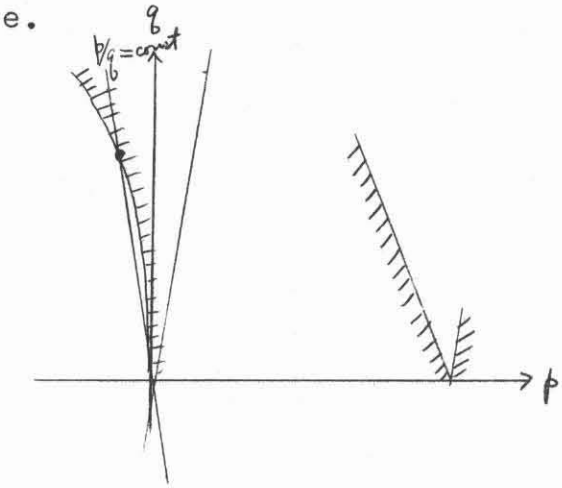
The required voltage gain per turn and the cavity voltages determine the equilibrium phase by (19) and there are two possible values of $\cos\phi_s$, namely ϕ_s and $\pi - \phi_s$.

Both equilibrium phases can be made stable at the same time by choosing the adequate values of parameters.

Now both phases, falling and rising, are stable and the width of phases accepting particles at injection increases by factor 2. Also it may be true that particles

can go through the transition energy without appreciable loss. Across the transition energy, K in the phase equation changes its sign.

The above treatments are not correct if the smooth approximation of Symon[2] is valid. If the wave length of the phase oscillation is much larger than the sector length, the smooth approximation teaches us that the A.G. terms are averaged over and the motion is described only by $A\cos\phi_s$. In Mark I type, it means the rising phase is unstable irrespective of A.G. terms. So it is necessary to choose the sector length of comparable order to the wave length of oscillation. To realize this the voltage V_1 is turned on for a period of order of the period of oscillation and turned off, while the voltage V_2 is turned off and on instead. In Mark V type, the large sector length means small δ . If δ is large compared to $A^{1/2}$ the rising phase can not be stable irrespective of the ridge term by the smooth approximation. This is also seen from the developed equation (18). The equation is nearly the Mathieu type $x'' + (p + q\cos 2t)x = 0$. Its stability zone near the q axis is shown in the figure. The boundary line of the stability zone near the origin is almost a parabola. The ratio of p/q is determined by the voltage ratio and the harmonic order and is represented by the straight lines through the origin.



(2) MURA KRS-1

In the region where $p < 0$, the straight lines cross the parabola and below these points the rising phase is unstable. Hence we must choose the value of p and q large enough beyond these points. This is equivalent to choosing a small value of δ guided by the smooth approximation.

Another use of the phase focusing is as a debuncher, which makes it possible to give a stretched pulse at the period of beam out-put, especially useful in the case of synchro-clash. The accelerating voltage is turned off at first and the ridge voltage is changed so that the phase oscillation becomes unstable. Then beams are stretched over the whole circumference of machine.

c. Other possibilities:

As the phase equation (14) shows, there must be a resonant term $\sin\phi$ or $\sin N\phi$ to keep particles accelerated continuously. If an amplitude modulated r-f voltage such as

$$V \cos \omega_m t \sin \omega_1 t$$

ω_m : modulating frequency

$\omega_1 = N\omega_s = (N' + \delta)\omega_s$

is applied, the phase equation becomes

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2 K}\right)\frac{d\phi}{dt} + \frac{eV}{2\pi}\cos\omega_m t \sin(N\phi - \delta\omega_s t) = \frac{eV_0}{2\pi} \tag{22}$$

or

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2 K}\right)\frac{d\phi}{dt} + \frac{eV}{2\pi}\,\frac{1}{2}\Big[\sin(N\phi + (\omega_m - \delta\omega_s)t) + \sin(N\phi - (\omega_s\delta + \omega_m)t)\Big] = \frac{eV_0}{2\pi} \tag{23}$$

Here if we choose

$$\omega_m - \delta\omega_s = 0 \quad \text{or} \quad \omega_m + \delta\omega_s = 0$$

the above equation is equivalent of (12), and we can accelerate the particles by fixed ω_1 , and varying ω_m . In the language of electronics it means use of the side band. Physically, phases of particles, of which the frequency is not resonant to r-f voltage, slip after each turn and, if the voltage is small by modulation while particles lie in the decelerating phases and large during acceleration phase, they can be accelerated. In this case, since A is equal to B, AG focusing is usually not fulfilled and one phase is stable.

Equilibrium phase is determined by

$$\frac{1}{2} V \sin N\phi_s = V_0 \tag{24}$$

and it shows the "circumference factor" 2.

The modulating frequency ω_m is varied slowsly with time and the stable particles around ϕ_s continue to be accelerated. When the r-f voltage with frequency $N\omega_s$ is applied, particles which have the frequency $\omega_s, 2\omega_s, \cdots N\omega_s$

is accelerated by this voltage and the modulating frequency $\omega_m = 0$ and the gaps between these frequencies is covered by amplitude modulation $0 < \omega_m < \omega_s$. However the equilibrium phase is given by (24) and it changes with the change of the harmonic order. To overcome this phase transition between harmonics, it is necessary to keep $\omega_m = 0$ for a while at the transition frequency which is chosen not resonant to the particle frequency. Then the phase oscillation become unstable and particles are averaged in a whole circumference and next we begin to modulate the amplitude with an adequate ω_m to bunch particles again around the new equilibrium phase.

Or, if we choose $\cos\phi_s << 1$, both phases near $\sin\phi_s \sim 1$ can be stable by the ridge term and it is unnecessary to worry about the phase transition. Practically the amplitude modulation requires a low Q cavity and it is a question ~~whether~~ it is preferable to modulate the frequency itself or to modulate the amplitude modulating frequency.

(1) D. Bohm and L. Foldy, Phys. Rev. 70, 249 (1946).