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CHARACTER OF PARTICLE MOTION IN THE MARK V FFAG ACCELERATOR

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I. INTRODUCTION

The Mark V or "spiral ridge" FFAG accelerator is a version, originally proposed by Kerst,[1,2] of the fixed-field class of A-G machines. In this design the general r^k increase of field with radius is modified, to produce alternate-gradient focusing with no marked increase of circumference, by introducing a spatial ripple into the guide field so that the particles encounter regions in which the local "n" and restoring forces alternate. This is achieved by constructing a field which, in comparison with the average field at a given radius, is alternately higher and lower along oblique curves which all particles must cross. In practice such a field would be attained by the use of spiral ridges on the pole surfaces, supplemented, when required, by similarly disposed current-carrying conductors.

It is the purpose of this report to derive analytically information concerning the particle motion in the Mark V accelerator and, in Appendices, to record some techniques useful for further study of the motion by aid of the ILLIAC digital computer.

II. THE MAGNETIC FIELD

A. Form Assumed in the Median Plane:

Without the use of poles excessively close to the median plane, the type of variation of magnetic field which is most readily realizable is sinusoidal. To obtain a field which would subject the particles to alternate focusing forces, it was originally

conceived that the field prescribed in the median plane be of the form

$$B_{zo} = -B_{o}(r/r_0)^k \left\{1 + f \sin\left[\frac{r-r_0}{\lambda} - N\phi\right]\right\} .$$

In order that the field <u>scale</u>, however, in such a way that the essential features of its effect on all particles be the same,[3] it appears desirable to make the quantitatively minor modification of adopting the form

$$B_{zo} = -B_{o}(r/r_0)^k \left\{1 + f \sin\left[\frac{\ln(r/r_0)}{w} - N\phi\right]\right\} ,$$

with w constant. This revised form for the median plane field will be the basis for the remainder of this report. The momentum compaction is then clearly given by $(\Delta r/r)/(\Delta p/p) = \frac{1}{k+1}$.

From these expressions it is seen that N is the number of spiralling ridges passed over by a particle in going around the machine once in the θ direction. f is the fractional flutter, in the magnetic field, due to the ridges. Finally, if the radial width of the annulus is small in comparison to the outer radius, r_0, $\lambda = 2\pi\lambda \doteq 2\pi r_0 w$ is substantially the radial separation of the ridges. The angle by which the ridges spiral out from a reference circle is of the order Nw and in practice will be quite small. The exponent k is taken to be positive.

It will be convenient in what follows to work with dimensionless quantities defined as follows:

$$\left(\frac{r_1}{r_0}\right)^{k+1} \equiv \frac{p}{e B_{o} r_0}$$

$$x \equiv \frac{r - r_1}{r_1}$$

$$y \equiv \frac{z}{r_1} ;$$

the median plane field may then be written

$$B_{zo} = -\frac{p_1}{er_1}(1+x)^k \left\{1 + f \sin\left[\frac{1}{w}\ln(1+x) - N\theta\right]\right\},$$

where $N\phi = N\theta + \ln(r_1/r_o)$.

B. Development of Vector Potential:

To obtain the differential equations governing the particle motion it is desirable to characterize the magnetic field by a vector potential, which should be at least approximately compatible with the prescribed median plane field and with Maxwell's equations, in order that the resulting equations be rigorously Hamiltonian and the solutions thus satisfy Liouville's theorem. In attempting to write suitable expansions for components of the field and vector potential, one may be guided by the consideration that x and y will themselves be quite small but that x/w and y/w may, in cases of practical interest, be comparable with unity. In the work described in the body of this report terms involving powers of these latter quantities will be retained so far as practicable, but no more than quite limited accuracy may be expected for values of x or y nearly as large as w. kx and ky, however, will be typically rather small (≤ 0.1). Also N_x and N_y are normally less than kx and ky.

We undertake an expansion of the median plane field, through cubic terms in x, to obtain

$$B_{zo} = -\frac{p_1}{er_1}(1+x)^k\left[1 + f \sin\frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}{w} - N\theta)\right]$$

$$= -\frac{p_1}{er_1}\left[A_0 + A_1 x + A_2 x^2 + A_3 x^3\right],$$

where $A_0 = 1 - f \sin N\theta$

$$A_1 = k + f\cdot\left(\frac{1}{w}\cos N\theta - k \sin N\theta\right)$$

$$A_2 = \frac{k(k-1)}{2} + f\left[\frac{k-\frac{1}{2}}{w}\cos N\theta + \left(\frac{1}{2w^2} - \frac{k(k-1)}{2}\right)\sin N\theta\right.$$

$$A_3 = \frac{k(k-1)(k-2)}{6} + f\left\{\frac{-\frac{1}{w^2} + 3k(k-2) + 2}{6w}\cos N\theta + \left[\frac{k-1}{2w^2} - \frac{k(k-1)(k-2)}{6}\right]\sin N\theta\right\}.$$

Likewise, for use in what follows,

$$(1+x)B_{z0} = -\frac{p_1}{er_1}\left[B_0 + B_1 x + B_2 x^2 + B_3 x^3\right.,$$

where $B_0 = 1 - f \sin N\theta$

$$B_1 = k + 1 + f\left(\frac{1}{w}\cos N\theta - (k+1)\sin N\theta\right)$$

$$B_2 = \frac{k(k+1)}{2} + f\left[\frac{k+\frac{1}{2}}{w}\cos N\theta + \left(\frac{1}{2w^2} - \frac{k(k+1)}{2}\right)\sin N\theta\right]$$

$$B_3 = \frac{(k+1)k(k-1)}{6} + f\left\{\frac{-\frac{1}{w^2} + 3k^2 - 1}{6w}\cos N\theta + \left[\frac{k}{2w^2} - \frac{(k+1)k(k-1)}{6}\right]\sin N\theta\right\}.$$

We now seek a vector potential such that A_r and A_z vanish at $z = 0$ (in general, the components A_θ and A_r will be even functions of z or y with A_r involving only y^2 and higher even powers of y, while A_z will be an odd function). Then in the median plane A_θ must satisfy[4]

$$\frac{1}{r_1}\frac{\partial}{\partial x}\left[(1+x)A_{\theta 0}\right] = (1+x)B_{z0},$$

leading to the possible solution

$$\frac{e}{p_1}(1+x)A_{\theta 0} = C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4, \text{ or}$$

$$\frac{e}{p_1}A_{\theta 0} = D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4,$$

where $C_1 = -B_0 = -1 + f \sin N\theta$

$$C_2 = -\frac{B_1}{2} = -\frac{k+1}{2} + f\left(-\frac{1}{2w}\cos N\theta + \frac{k+1}{2}\sin N\theta\right)$$

$$C_3 = -\frac{B_2}{3} = -\frac{k(k+1)}{6} + f\left[-\frac{2k+1}{6w}\cos N\theta + \left(-\frac{1}{6w^2} + \frac{k(k+1)}{6}\right)\sin N\theta\right]$$

$$C_4 = -\frac{B_3}{4} = -\frac{(k+1)k(k-1)}{24}$$

$$+ f\left\{\frac{\frac{1}{w^2} - 3k^2 + 1}{24w}\cos N\theta + \frac{1}{24}\left[-\frac{3k}{w^2} + (k+1)k(k-1)\right]\sin N\theta\right\},$$

$$D_1 = C_1 = -1 + f\sin N\theta$$

$$D_2 = C_2 - C_1 = -\frac{k-1}{2} + f\left(-\frac{1}{2w}\cos N\theta + \frac{k-1}{2}\sin N\theta\right)$$

$$D_3 = C_3 - C_2 + C_1$$

$$= -\frac{k^2-2k+3}{6} + f\left[-\frac{k-1}{3w}\cos N\theta + \left(-\frac{1}{6w^2} + \frac{k^2-2k+3}{6}\right)\sin N\theta\right.$$

$$D_4 = C_4 - C_3 + C_2 - C_1$$

$$= -\frac{k^3 - 4k^2 + 7k - 12}{24}$$

$$+ f\left\{\frac{\frac{1}{w^2} - 3k^2 + 8k - 7}{24w}\cos N\theta + \frac{1}{24}\left[-\frac{3k-4}{w^2} + k^3 - 4k^2 + 7k - 12\right]\sin N\theta\right\}.$$

To develop the vector potential for points not in the median plane we employ a gauge in which div $\overline{A} = 0$ and note that, in the notation of E.S. Akeley[4]

$$\overline{A}_t \doteq \left\{1 - \frac{z^2\Delta_t}{2} + \frac{z^4\Delta_t\Delta_t}{24}\right\}[A_{\theta_0}\,\hat{e}_\theta]$$

$$= A_{\theta_0}\hat{e}_\theta - \frac{z^2}{2}\left\{\left[(\nabla_t^2 A_{\theta_0}) - \frac{A_{\theta_0}}{r^2}\right]\hat{e}_\theta - \frac{2}{r^2}\frac{\partial A_{\theta_0}}{\partial\theta}\hat{e}_r\right\}$$

$$+ \frac{z^4}{24}\left\{\begin{bmatrix} \nabla_t^2\left(\nabla_t^2 A_{\theta_0} - \frac{A_{\theta_0}}{r^2}\right) - \frac{4}{r^4}\frac{\partial^2 A_{\theta_0}}{\partial\theta^2} \\ \\ -\dfrac{\nabla_t^2 A_{\theta_0} - \frac{A_{\theta_0}}{r^2}}{r^2} \end{bmatrix}\hat{e}_\theta + \begin{bmatrix} -\frac{2}{r^2}\frac{\partial}{\partial\theta}\left(\nabla_t^2 A_{\theta_0} - \frac{A_{\theta_0}}{r^2}\right) \\ \\ -2\nabla_t^2\left(\frac{1}{r^2}\frac{\partial A_{\theta_0}}{\partial\theta}\right) + \frac{2}{r^4}\frac{\partial A_{\theta_0}}{\partial\theta} \end{bmatrix}\hat{e}_r\right\}.$$

Likewise

$$A_z \doteq -\left\{z - \frac{z^3}{6}\Delta_t\right\} \nabla_t \cdot \bar{A}_{\theta_0} \quad .$$

In this way we find

$$\frac{e}{p_1} A_\theta = D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4$$
$$- \frac{y^2}{2}\Big[(D_1 + 2D_2) + (-2D_1 + 2D_2 + 6D_3 + D_1'')x$$
$$+ (2D_1 - 3D_2 + 3D_3 + 12D_4 - 2D_1'' + D_2'')x^2\Big]$$
$$+ \frac{y^4}{24}\left[D_1 - 6D_2 + 12D_3 + 24D_4 - 2D_1'' + 4D_2''\right],$$

$$\frac{e}{p_1} A_r = \left[D_1' x + (-2D_1' + D_2')x^2\right] y^2 + \frac{D_1' - 2D_2'}{6} y^4 \text{, and}$$

$$\frac{e}{p_1} A_z = -y\left[D_1' x + (-D_1' + D_2')x^2 + (D_1' - D_2' + D_3')x^3\right]$$
$$+ \frac{y^3}{6}\left[(-D_1' + 2D_2') + (3D_1' - 4D_2' + 6D_3' + D_1''')x\right],$$

primes denoting differentiation with respect to θ. These components of the vector potential represent expansions through fourth order in x or y and, as a check, can be verified to satisfy

$$r_1 \nabla \cdot \bar{A} \equiv \frac{\partial A_z}{\partial y} + \frac{\partial A_r}{\partial x} + \frac{A_r}{1+x} + \frac{\partial A_\theta / \partial \theta}{1+x} = 0$$

through third order.

III. THE EQUATIONS OF MOTION

A. "Lagrangian" for Use in Principle of Least Action:

The differential equations governing the particle trajectories in the aforementioned magnetic field may be conveniently obtained from the principle of least action by use of the "space Lagrangian"

$$
\begin{aligned}
\mathcal{L}(x,y;x',y';\theta) &= p r_1 \sqrt{(1+x)^2 + x'^2 + y'^2} + e r_1 \left[(1+x)A_\theta + x'A_r + y'A_z\right] \\
&\approx 1 + y + \frac{1}{2}\,\frac{x'^2+y'^2}{1+x} - \frac{1}{8}(x'^2+y'^2)^2 \\
&\quad + \frac{e}{p}\left[(1+x)A_\theta + x'A_r + y'A_z\right] \\
&\cong 1 + \frac{1}{2}\,\frac{x'^2+y'^2}{1+x} - \frac{1}{8}(x'^2+y'^2)^2 + x \\
&\quad + \frac{p_1}{p}\Big\{D_1' x' x y^2 + \{[-D_1' x + (D_1' - D_2')x^2]y + [-D_1' + 2D_2']\frac{y^3}{6}\}y' \\
&\quad + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 \\
&\quad - \frac{y^2}{2}\Big[(D_1 + 2D_2) + (-D_1 + 4D_2 + 6D_3 + D_1'')x \\
&\quad + (-D_2 + 9D_3 + 12D_4 - D_1'' + D_2'')x^2\Big] \\
&\quad + \frac{y^4}{24}\left[D_1 - 6D_2 + 12D_3 + 24D_4 - 2D_1'' + 4D_2''\right]\Big\},
\end{aligned}
$$

in which we have treated x' and y' as of the same order as x and y despite the fact that these derivatives may be expected to be some N times greater than the dependent variables themselves.

The Euler-Lagrange equations, if applied to the Lagrangian of the preceding paragraph, lead to differential equations for the motion which might be susceptible to solution by digital computation[5], but which are not in a form most suitable for analytic study. The equation for the radial motion, in particular, is marked by the presence of a forcing term $f \cdot \sin N\theta$ derived from the term $(1+C_1)x$ in the Lagrangian. It can, in fact, be shown that the magnitude of the (periodic) response to this forcing term is sufficient ($\cong -f/N^2$) that non-linear terms in the differential equations affect significantly the character of small amplitude betatron oscillations.[6,7] It is desirable, therefore, to undertake a change of dependent variable such that the forcing term is suppressed and the resulting equations, if then linearized, may be used to provide an analytic basis for determining the character of small-amplitude free oscillations.

The Lagrangian as written is in a form somewhat inconvenient for the analytical work to follow because of the presence of terms arising from centrifugal effects. Since the first derivative terms which result in the differential equations are in practice small for excursions of the order of the forced motion (at least in the case of "full-scale" high-energy accelerators), it is expedient to simplify the Lagrangian in such a way that the troublesome terms are removed but with the remaining terms of the differential equation modified only slightly. We accordingly continue by use of the following Lagrangian, which yields

differential equations free from terms involving first derivatives of the dependent variables and, in the remaining terms of the equations, modifies only slightly the original terms involving y^2, xy, xy^2, x^2y, xy^3, and x^4:

$$\mathcal{L} \cong \frac{x'^2 + y'^2}{2} + x + \frac{x^2}{2} + \frac{p_1}{p}\left[E_1 x + E_2 x^2 + E_3 x^3 + E_4 x^4 + F_0 y^2 + F_1 x y^2 + F_2 x^2 y^2 + G y^4\right],$$

with

$$E_1 = C_1 \qquad F_0 = -\frac{D_1}{2} - D_2$$

$$E_2 = C_2 + \frac{C_1}{2} \qquad F_1 \doteq \frac{D_1}{4} - \frac{5}{2}D_2 - 3D_3 - \frac{D_1''}{4}$$

$$E_3 = C_3 + \frac{2}{3}C_2 \qquad F_2 \doteq \frac{3}{8}D_1 - D_2 - \frac{27}{4}D_3 - 6D_4 - \frac{D_1''}{8} - \frac{D_2''}{4}$$

$$E_4 = C_4 + \frac{3}{4}C_3 \qquad G = \frac{D_1}{24} - \frac{D_2}{4} + \frac{D_3}{2} + D_4 - \frac{D_1''}{24} + \frac{D_2''}{12} .$$

B. <u>The Forced Motion</u>:

With the aim of separating out the major effect of the forced oscillations we now introduce the new dependent variable u by the substitution

$$x = K_1 \sin N\theta + K_2 \cos N\theta + u ,$$

a numerical integration for a particular example having suggested that the forced motion is in fact close to sinusoidal. The resulting Lagrangian (after subtracting a term which is a function only of θ) is:

$$
\begin{aligned}
\mathcal{L} \doteq \frac{u'^2+y'^2}{2} &+ N(K_1\cos N\theta - K_2\sin N\theta)\,u' \\
&+[1+K_1\sin N\theta + K_2\cos N\theta]u + \frac{u^2}{2} \\
&+\frac{p_1}{p}\Big\{\big[E_1+2E_2(K_1\sin N\theta+K_2\cos N\theta) \\
&\quad +3E_3(K_1\sin N\theta+K_2\cos N\theta)^2+4E_4(K_1\sin N\theta+K_2\cos N\theta)^3\big]u \\
&\quad +\big[E_2+3E_3(K_1\sin N\theta+K_2\cos N\theta)+6E_4(K_1\sin N\theta+K_2\cos N\theta)^2\big]u^2 \\
&\quad +\big[E_3+4E_4(K_1\sin N\theta+K_2\cos N\theta)\big]u^3+E_4u^4 \\
&\quad +\big[F_0+F_1(K_1\sin N\theta+K_2\cos N\theta)+F_2(K_1\sin N\theta+K_2\cos N\theta)^2\big]y^2 \\
&\quad +\big[F_1+2F_2(K_1\sin N\theta+K_2\cos N\theta)\big]u\,y^2 \\
&\quad +F_2u^2y^2+Gy^4\Big\},
\end{aligned}
$$

of which we shall be chiefly interested in terms of second or lower order in the variables u,y.

This Lagrangian leads to a residual forcing term in the equation for the u-motion given by

$$
\begin{aligned}
&1+(N^2+1)(K_1\sin N\theta+K_2\cos N\theta) \\
&+\frac{p_1}{p}\big[E_1+2E_2(K_1\sin N\theta+K_2\cos N\theta)+3E_3(K_1\sin N\theta+K_2\cos N\theta)^2 \\
&\qquad +4E_4(K_1\sin N\theta+K_2\cos N\theta)^3\big]
\end{aligned}
$$

$$\cong 1+(N^2+1)(K_1 \sin N\theta + K_2 \cos N\theta)$$

$$+\frac{p_1}{p}\Big\{-1+f\sin N\theta+\Big[-(k+2)+f\Big(-\frac{1}{w}\cos N\theta+(k+2)\sin N\theta\Big)\Big](K_1\sin N\theta+K_2\cos N\theta)$$

$$+\Big[-\frac{k^2}{2}+f\Big(-\frac{k}{w}\cos N\theta+\Big(-\frac{1}{2w^2}+\frac{k^2}{2}\Big)\sin N\theta\Big)\Big](K_1\sin N\theta+K_2\cos N\theta)^2$$

$$+\Big[-\frac{k^3}{6}+f\Big(\Big(-\frac{k^2}{2w}+\frac{1}{6w^3}\Big)\cos N\theta+\Big(-\frac{k}{2w^2}+\frac{k^3}{6}\Big)\sin N\theta\Big)\Big](K_1\sin N\theta+K_2\cos N\theta)^3$$

and is to be suppressed by suitable choice of the constants p_1/p, K_1, and K_2. It appears from this development that a measure of the adequacy of the analysis is afforded by the degree to which the values found for K_1/w and K_2/w are small in comparison to unity.

The forcing term contains the following Fourier-components, which may be made to vanish:

Constant Term: $1+\frac{p_1}{p}\Big[-1+\frac{1}{2}f\beta K_2-\frac{1}{2}f\alpha K_1+\frac{a}{2}(K_1^2+K_2^2)$

$$+\frac{3}{8}fBK_2(K_1^2+K_2^2)+\frac{3}{8}fCK_1(K_1^2+K_2^2)\Big],$$

Coeff. of sin $N\theta$: $(N^2+1)K_1+\frac{p_1}{p}\Big[f+\alpha K_1+\frac{fc}{4}(3K_1^2+K_2^2)+\frac{fb}{2}K_1K_2+\frac{3}{4}AK_1(K_1^2+K_2^2)\Big],$

Coeff. of cos $N\theta$: $(N^2+1)K_2+\frac{p_1}{p}\Big[\alpha K_2+\frac{fb}{4}(K_1^2+3K_2^2)+\frac{fc}{2}K_1K_2+\frac{3}{4}AK_2(K_1^2+K_2^2)\Big],$

where $\alpha=-(k+2)$ $\quad\beta=-\frac{1}{w}$.

$a=-\frac{k^2}{2}$ $\quad b=-\frac{k}{w}$ $\quad c=-\frac{1}{2w^2}+\frac{k^2}{2}$

$A=-\frac{k^3}{6}$ $\quad B=-\frac{k^2}{2w}+\frac{1}{6w^3}$ $\quad C=-\frac{k}{2w^2}+\frac{k^3}{6}$.

We have attempted to find solutions which make these coefficients vanish when the machine parameters lie within what may be considered the normal range of values. In this way we find:

$$K_1 \doteq -\frac{f}{N^2-(k+1)+\frac{3}{8}\left(\frac{f}{wN}\right)^2} \doteq -\frac{f}{N^2-(k+1)},$$

$$K_2 \doteq \frac{f^3 k}{4wN^6}$$ or very nearly zero, and

$$\frac{p_1}{p} \doteq 1-\frac{k+2}{2}\left(\frac{f}{N}\right)^2.$$

The forced motion is thus represented approximately by:

$$x_{\text{forced}} \doteq -\frac{1}{2}\frac{k+2}{k+1}\left(\frac{f}{N}\right)^2 - \frac{f}{N^2-(k+1)}\left[\sin N\theta - \frac{k}{4w}\left(\frac{f}{N^2}\right)^2 \cos N\theta\right]$$

$$\cong -\frac{1}{2}\left(\frac{f}{N}\right)^2 - \frac{f}{N^2-(k+1)}\sin N\theta,$$

$$x'_{\text{forced}} \doteq -\frac{Nf}{N^2-(k+1)}\left[\cos N\theta + \frac{k}{4w}\left(\frac{f}{N^2}\right)^2 \sin N\theta\right]$$

$$\cong -\frac{Nf}{N^2-(k+1)}\cos N\theta;$$

accordingly, at $\theta = 0$, the "fixed points" are given by

$$x_{\text{fixed}} \cong -\frac{1}{2}\left(\frac{f}{N}\right)^2$$

$$x'_{\text{fixed}} \cong -\frac{Nf}{N^2-(k+1)},$$

and the amplitude of the forced motion is given approximately by the magnitude of the coefficient $-\frac{f}{N^2-(k+1)}$.
The validity of these results is expected, as noted previously, to be measured by the degree to which K_1/w or $\frac{f/w}{N^2-(k+1)}$ is small in comparison to unity.

C. Character of Small-Amplitude Betatron Oscillations:

For small-amplitude oscillations about the equilibrium orbit, the governing differential equations will be of the form

$$u'' + F_u u = 0$$
$$y'' + F_y y = 0 .$$

On the basis of the Lagrangian of the previous sub-section, the spring factors which determine the frequencies of the oscillations are respectively (neglecting $\frac{p_1 - p}{p}$, K_2, and powers of K_1 above the first):

$$F_u = -1 - 2E_2 - 6K_1E_3 \sin N\theta$$
$$\cong k+1+\frac{f}{w}\cos N\theta - \frac{(f/w)^2}{N^2-(k+1)}\sin^2 N\theta$$
$$= k+1-\frac{1}{2}\frac{(f/w)^2}{N^2-(k+1)} + \frac{f}{w}\cos N\theta + \frac{1}{2}\frac{(f/w)^2}{N^2-(k+1)}\cos 2N\theta ;$$

$$F_y = -2F_0 - 2K_1F_1 \sin N\theta$$
$$\cong -k - \frac{f}{w}\cos N\theta + \frac{(f/w)^2}{N^2-(k+1)}\sin^2 N\theta$$
$$= -k + \frac{1}{2}\frac{(f/w)^2}{N^2-(k+1)} - \frac{f}{w}\cos N\theta - \frac{1}{2}\frac{(f/w)^2}{N^2-(k+1)}\cos 2N\theta .$$

The linearized equations representing small amplitude betatron oscillations are seen to be of the Hill type. Some aids for the solution of these equations — especially for the determination of stability boundaries and the characteristic exponents (σ_u and σ_y) of the motion — are noted in Appendix III. As Kerst has pointed out[8], useful orientation is readily provided, however, by application of the "smooth approximation" technique introduced by Symon[9]. If the normally-small contributions from the cos $2N\theta$ terms are ignored and if $k+1$ is neglected in comparison to N^2, the smooth approximation leads to differential equations of the form

$$u'' + \nu_u^2 u = 0$$

$$y'' + \nu_y^2 y = 0,$$

where $\nu_u^2 \doteq k+1-\frac{1}{2}\left(\frac{f}{wN}\right)^2+\frac{1}{2}\left(\frac{f}{wN}\right)^2$
$= k+1$ and

$$\nu_y^2 \doteq -k+\frac{1}{2}\left(\frac{f}{wN}\right)^2+\frac{1}{2}\left(\frac{f}{wN}\right)^2$$
$$=\left(\frac{f}{wN}\right)^2-k\ .$$

It is thus seen that the frequency of the free radial oscillations is substantially determined by the exponent characterizing the radial increase of average field strength, while axial stability may be obtained concurrently if $(\frac{f}{w\ N})^2$ is sufficiently large to dominate $-k$.

It will be noted that these features of the betatron motion differ markedly from the performance which would be expected on the basis of an expansion about a circular reference orbit while ignoring the presence of the forced oscillations. This situation can be understood physically[6,7] by reference to a diagram on which are drawn contours of constant magnetic field strength in the median plane, with the expected equilibrium orbit superposed (Fig. 1). One notes that the field gradient is in a sense to favor radial focusing over a smaller interval of θ if one examines the gradient in the neighborhood of the scalloped curve than if one merely examined it along a line of constant radius.

IV. ILLIAC STUDIES OF THE PARTICLE MOTION

Although the results of the foregoing analytical work are believed to decsribe reasonably well the general character of particle motion in typical Mark V machines, it is clearly desirable to study the motion in representative structures of this type by means of digital computation. Such a program not only would provide a useful check on the analytical results and provide information concerning structures for which the approximations which we have introduced are invalid, but can take account of the inherently non-linear character of the dynamical equations and provide accurate information concerning stability regions. Work directed toward these ends is listed below:

(i) Exact differential equations governing the motion in the median plane have been prepared for use with the ILLIAC

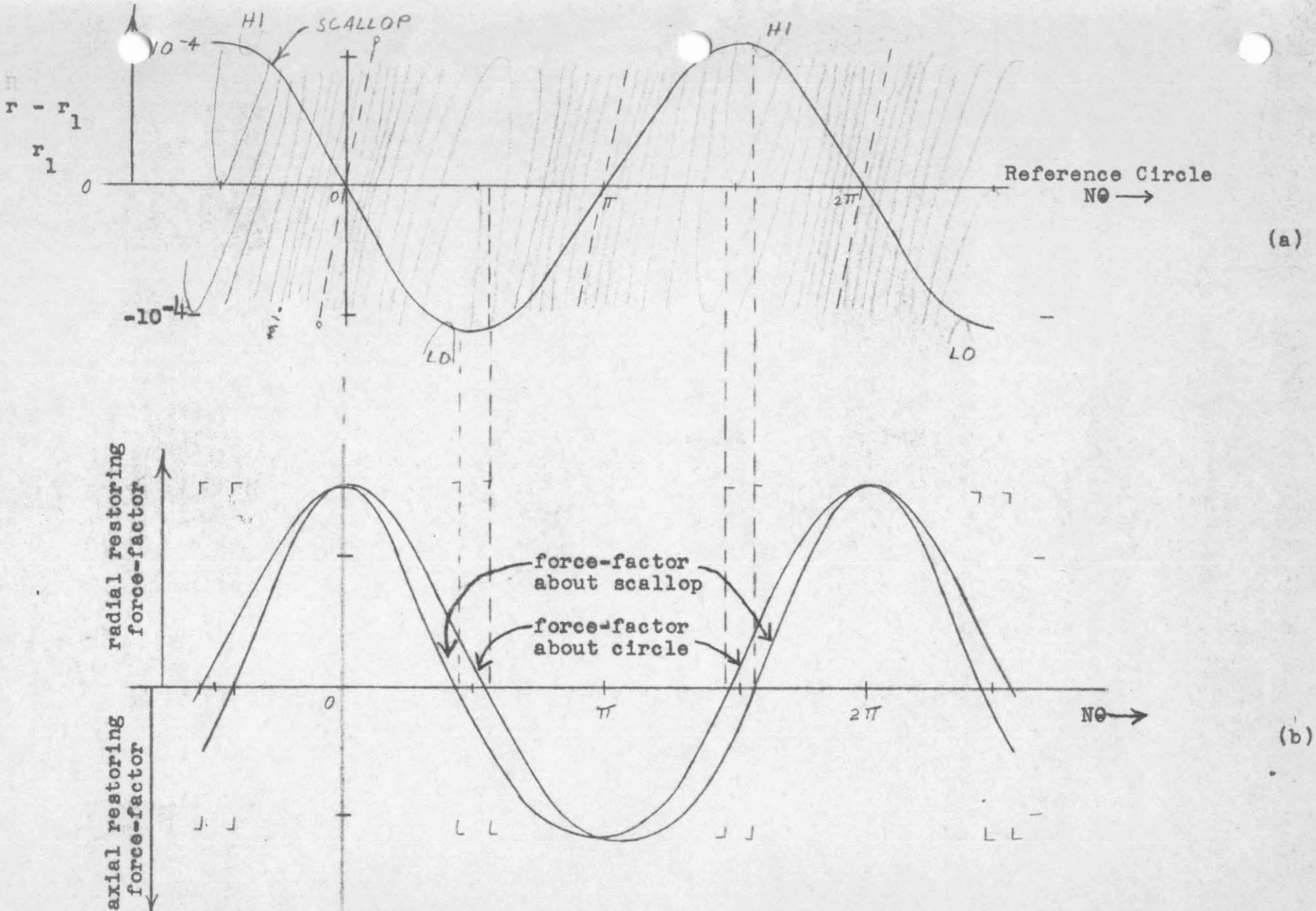

Fig. 1. Diagram Illustrating Effect of Scalloped Equilibrium Orbit on Stability

(a). Contour lines of Magnetic Field

(b). Force-factors for Motion about Circular and Scalloped Reference Curve

("Ridge Runner" program).

(ii) Relatively simple, approximate differential equations for the three-dimensional motion have been prepared, attempting to take account of the fact that x/w and y/w may be large (comparable with unity), but supposing that the variables x and y themselves will be small("Feckless Five" program).

(iii) More accurate, but somewhat more elaborate, differential equations for the three-dimensional motion have also been set up by Vogt-Nilsen, based on recent vector-potential developments[4] of E.S. Akeley ("Feckful Five" program).[10] These computer programs are being directed toward a comprehensive study of the particle dynamics in Mark V machines, chiefly through the efforts of the Illinois group.*

In Appendices I and II to follow we outline the development of the equations listed as (i) and (ii) above. In Appendix III we describe some techniques which have been applied for obtaining information concerning solutions to the Hill equation developed in Section IV of this report. In Appendix IV we make some numerical comparisons, in certain examples, between results obtained from the analytic theory and from the ILLIAC computer. As Appendix V we present a stability diagram computed from the analytic theory.

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* Note added in proof: See also the similar development proposed by Powell [MURA-JLP-6].

V. NOTES AND REFERENCES

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† Institute for Atomic Research and Ames Laboratory of the U.S. Atomic Energy Commission.

‡ Assisted by the National Science Foundation.

1 D. W. Kerst, et al, MURA-DWK/KMT/LWJ/KRS-3 (November 12, 1954); D. W. Kerst, MURA-DWK-7 (November 17, 1954); Revision MURA-DWK-7 (March 2, 1955); D. W. Kerst, et al, Bull. Amer. Phys. Soc. 30, No. 1, Paper D5 (January 27, 1955).

2 Subsequent relevant reports include D. W. Kerst, MURA/DWK-10 (May 10, 1955); MURA-DWK-11 Internal (June 27, 1955).

3 The failure of the originally proposed field to scale the essential features exactly would, on the basis of linear theory applicable to small amplitude betatron oscillations, result in the traversal of a couple of resonances (integral or half-integral) by the operating point of some typical machines.

4 While this work was in progress a report by E.S. Akeley [ESA(MURA-1), dated January 26, 1955] appeared which systematizes this procedure. Subsequent reports, ESA(MURA-2) and ESA (MURA-3) have extended this technique.

5 A more accurate formulation for use with digital computation is presented, however, in Appendix II.

[6] This point has been brought out previously in informal communications: letters to D. W. Kerst, dated 21, 23, and 25 January 1955; 1955 Sesquimonthly MURA technical meetings at Northwestern University, Indiana University, and the University of Minnesota.

[7] L. J. Laslett, Bull. Amer. Phys. Soc. 30, No. 3, Paper T8 (April 30, 1955).

[8] D. W. Kerst, private telephone conversation, 25 January 1955.

[9] K.R. Symon, KRS(MURS)-1, -4 (July 1 and August 10, 1954).

[10] Cf. Nils Vogt-Nilsen, MURA/NVN/2 (June, 1955), where a more general complex field is treated.

[11] L.J. Laslett, LJL(MAC)-4 (March-April, 1954), eq. (4), p.2.

[12] I am indebted to Dr. J.N. Snyder for advice on this point.

[13] Cf. L. J. Laslett, MURA Notes (1 February 1955).

[14] E. Courant and H. Snyder, B.N.L. Report EDC-15 (September 2, 1954).

[15] Laslett, Snyder, and Hutchinson, MURA Notes (20 April 1955).

[16] L. J. Laslett, supplement to MURA Notes cited[15] (31 May 1955).

[17] Letter to J. N. Snyder (11 July 1955).

APPENDIX I

EXACT DIFFERENTIAL EQUATIONS FOR MOTION IN THE MEDIAN PLANE

For the accurate exploration of the character of particle motion in the median plane of the Mark V accelerator, and for aid in checking results obtained by other methods, exact differential equations governing this motion were prepared in a form suitable for ILLIAC computation. It is clear that this is possible, since the field -- and hence the nature of the forces -- is prescribed in the median plane. The resultant program has been termed the "Ridge Runner".

For Z identically zero, the equation of motion is[11]

$$\frac{d}{d\phi}\left(\frac{r'}{\sqrt{r^2+r'^2}}\right)=\frac{r}{\sqrt{r^2+r'^2}}+\frac{e}{p}rB_z\ .$$

With $r = r_1\ (1+x)$,

$$\frac{d}{d\phi}\left(\frac{x'}{\sqrt{(1+x)^2+x'^2}}\right)=\frac{1+x}{\sqrt{(1+x)^2+x'^2}}+\frac{er_1}{p}(1+x)B_z\ .$$

We let $p_x=\dfrac{x'}{\sqrt{(1+x)^2+x'^2}}$, or $x'=(1+x)\dfrac{p_x}{\sqrt{1-p_x^2}}$,

and put $eB_0(r_1/r_0)^k r_1 = p_1$, $N\theta = N\phi + \ln(r_0/r_1)$

to obtain the simultaneous first order differential equations

$$p_x' = \sqrt{1-p_x^2} - \frac{p_1}{p}(1+x)^{k+1}\left\{1+f\sin\left[\frac{1}{w}\ln(1+x)-N\theta\right]\right\}$$

$$x' = (1+x)\frac{p_x}{\sqrt{1-p_x^2}}\ .$$

[These equations are clearly in Hamiltonian form, since $\partial x'/\partial x = -\partial p_x'/\partial p_x$, the "Hamiltonian" being

$$\mathcal{H} \propto -(1+x)\sqrt{1-p_x^2} - \frac{e r_1}{p}\int^x (1+x) B_z \, dx$$

$$= -(1+x)\sqrt{1-p_x^2} + \frac{p_1}{p}\int^x (1+x)^{k+1}\left\{1+f\sin\left[\frac{1}{w}\ln(1+x) - N\phi\right]\right\} dx,$$

in which the second term represents the contribution $-\frac{e}{p}\frac{r}{r_1}A_\phi$ from the vector potential.] For automatic digital computation, r_1 may be taken so that $p_1 = p$ (for convenience).

APPENDIX II

APPROXIMATE DIFFERENTIAL EQUATIONS OF MOTION

In the attempt to permit relatively simple exploration of three-dimensional Mark V motion with the ILLIAC, relatively simple differential equations of motion have been formulated. The intention was to retain the dominant influence of the quantity x/w, which is not necessarily small in comparison to unity, but to make approximations consistent with the supposition that x and kx will be small in most cases of interest. The resulting program is termed the "Feckless Five".

We employ the notation

$$x = \frac{r-r_1}{r_1} \qquad y = \frac{z}{r_1} \qquad (r_1/r_0)^k = \frac{p_1}{e B_0 r_1}$$

$$\delta = \tan^{-1}[(k+1)w] \qquad \sec\delta = [1+(k+1)^2 w^2]^{\frac{1}{2}}$$

$$\alpha - i\beta = \left[1-(k^2+k+1-N^2)w^2 - i(2k+1)w\right]^{\frac{1}{2}}$$
$$\doteq \left[1-(k^2-N^2)w^2 - 2ikw\right]^{\frac{1}{2}}$$
$$N\theta = N\phi + \ln(r_0/r_1).$$

The field in the median plane is taken to be

$$B_{z0} = -\frac{p_1}{er_1}(1+x)^k\left\{1+f\sin\left[\frac{1}{w}\ln(1+x) - N\theta\right]\right\}$$
$$\cong -\frac{p_1}{er_1}\left\{(1+x)^k + f\frac{e^{(k+1)x}}{1+x}\left(1-\frac{k+1}{2}x^2\right)\left[\sin\left(\frac{x}{w}-N\theta\right)-\frac{x^2}{2w}\cos\left(\frac{x}{w}-N\theta\right)\right]\right\}$$
$$\cong -\frac{p_1}{er_1}\left\{(1+x)^k + f\frac{e^{(k+1)x}}{1+x}\sin\left(\frac{x}{w}-N\theta\right) - \frac{f\sec\delta}{1+x}\frac{x^2}{2w}\cos(N\theta+\delta)\right\},$$

in which we regard the last term as a small correction.

If the vector potential in the median plane is taken to have a θ-component only, we employ the relation

$$(1+x)B_{z0} = \frac{\partial}{\partial x}\left[\frac{1+x}{r_1}A_{\theta 0}\right]$$

to obtain

$$-\frac{e}{p_1}(1+x)A_{\theta 0} \doteq \frac{(1+x)^{k+2}}{k+2} - \frac{fw\,e^{(k+1)x}}{\sec\delta}\cos\left(\frac{x}{w}-N\theta+\delta\right)$$
$$-f\sec\delta\frac{x^3}{6w}\cos(N\theta+\delta)$$

or

$$-\frac{e}{p_1}A_{\theta 0} \doteq \frac{(1+x)^{k+1}}{k+2} - \frac{fw\,e^{kx}}{\sec\delta}\cos\left(\frac{x}{w}-N\theta+\delta\right)$$
$$-f\sec\delta\frac{x^3}{6w}\cos(N\theta+\delta).$$

For developing the vector potential at points not necessarily in the median plane, we note

$$\mathrm{div}_t A_{\theta_0} \doteq N f w \frac{e^{(k+1)x}}{\sec\delta} \sin\left(\frac{x}{w} - N\theta + \delta\right) + \text{Term of Order } \frac{N f x^3}{3w}$$

and apply the methods[4] used previously in Section IIB to obtain

$$-\frac{e}{p_1}(1+x)A_\theta \doteq \frac{(1+x)^{k+2}}{k+2} - k\frac{y^2}{2}(1+x)^k + \frac{k^2(k-2)}{24} y^4 (1+x)^{k-2}$$

$$-\frac{fw}{\sec\delta} e^{(k+1)x} \left[\begin{array}{l} \cos\left(\frac{x}{w} - N\theta + \delta\right) \cosh \frac{\alpha y}{w} \cos \beta y \\ + \sin\left(\frac{x}{w} - N\theta + \delta\right) \sinh \frac{\alpha y}{w} \sin \beta y \end{array} \right]$$

$$-\frac{f \sec\delta}{6w}\left(x^3 - 3xy^2\right)\cos(N\theta + \delta)$$

$$\doteq \frac{1}{k+2} + x + \frac{k+1}{2}x^2 + \frac{k(k+1)}{6}x^3 + \frac{(k+1)k(k-1)}{24}x^4$$

$$-\frac{ky^2}{2} - \frac{k^2}{2}xy^2 - \frac{k^2(k-1)}{4}x^2y^2 + \frac{k^2(k-2)}{24}y^4$$

$$-\frac{fw}{\sec\delta} e^{(k+1)x} \left[\begin{array}{l} \cos\left(\frac{x}{w} - N\theta + \delta\right) \cosh \frac{\alpha y}{w} \cos \beta y \\ + \sin\left(\frac{x}{w} - N\theta + \delta\right) \sinh \frac{\alpha y}{w} \sin \beta y \end{array} \right]$$

$$-\frac{f \sec\delta}{6w}\left(x^3 - 3xy^2\right)\cos(N\theta + \delta),$$

$$-\frac{e}{p_1}A_z \doteq \frac{N f w}{\sec\delta}\, y\, e^{(k-1)x} \sin\left(\frac{x}{w} - N\theta + \delta\right),$$

$$-\frac{e}{p_1}A_r \doteq 0 \quad (\text{being of order } wy^2).$$

The equations of motion are now obtained by use of these vector potential components in the Lagrangian[11]

$$\mathcal{L} = \left[(1+x)^2 + x'^2 + y'^2\right]^{\frac{1}{2}} + \frac{e}{p}\left[(1+x)A_\theta + x'A_r + y'A_z\right]$$

$$\doteq 1 + x + \frac{x'^2 + y'^2}{2(1+x)} + \frac{e}{p}\left[(1+x)A_\theta + y'A_z\right] \text{ (since we take } A_r \doteq 0)$$

or the Hamiltonian

$$\mathcal{H} = -(1+x)\left[1 - p_x^2 - \left(p_y - \frac{e}{p}A_z\right)^2\right]^{\frac{1}{2}} - \frac{e}{p}(1+x)A_\theta$$

$$\doteq -(1+x)\left[1 - \frac{p_x^2 + \left(p_y - \frac{e}{p}A_z\right)^2}{2}\right] - \frac{e}{p}(1+x)A_\theta .$$

One thus obtains, if p_1 is set equal to p:

$$x' = (1+x)\, p_x$$

$$y' = (1+x)\, A$$

$$p_x' = -(k+1)x - \frac{k(k+1)}{2}x^2 - \frac{(k+1)k(k-1)}{6}x^3 + \frac{k^2}{2}y^2 + \frac{k^2(k-1)}{2}xy^2$$

$$-\frac{f}{\sec\delta}e^{(k+1)x}\left\{\begin{array}{l}\left[\sin\left(\frac{x}{w} - N\theta + \delta\right) - (k+1)w\cos\left(\frac{x}{w} - N\theta + \delta\right)\right]\cosh\frac{\alpha y}{w}\cos\beta y \\ -\left[\cos\left(\frac{x}{w} - N\theta + \delta\right) + (k+1)w\sin\left(\frac{x}{w} - N\theta + \delta\right)\right]\sinh\frac{\alpha y}{w}\sin\beta y\end{array}\right\}$$

$$+\frac{f\sec\delta}{2w}(x^2 - y^2)\cos(N\theta + \delta) - \frac{1}{2}p_x^2 - \frac{1}{2}A^2$$

$$-\frac{Nf}{\sec\delta}(1+x)\,y\,e^{(k-1)x}A\left[\cos\left(\frac{x}{w} - N\theta + \delta\right) + (k-1)w\sin\left(\frac{x}{w} - N\theta + \delta\right)\right]$$

$$p_y' = ky + k^2 xy + \frac{k^2(k-1)}{2} x^2 y - \frac{k^2(k-2)}{6} y^3$$

$$+ \frac{f}{\sec\delta} e^{(k+1)x} \left\{ \begin{array}{l} \left[\alpha \sin\left(\frac{x}{w} - N\theta + \delta\right) - \beta w \cos\left(\frac{x}{w} - N\theta + \delta\right)\right] \cosh \frac{\alpha}{w} y \sin \beta y \\ + \left[\alpha \cos\left(\frac{x}{w} - N\theta + \delta\right) + \beta w \sin\left(\frac{x}{w} - N\theta + \delta\right)\right] \sinh \frac{\alpha}{w} y \cos \beta y \end{array} \right\}$$

$$- \frac{f \sec\delta}{w} xy \cos(N\theta + \delta)$$

$$- \frac{Nfw}{\sec\delta} (1+x) e^{(k-1)x} A \sin\left(\frac{x}{w} - N\theta + \delta\right) ,$$

where

$$A \equiv p_y + \frac{Nfw}{\sec\delta} y\, e^{(k-1)x} \sin\left(\frac{x}{w} - N\theta + \delta\right).$$

It is believed that solutions of these equations for certain cases, involving motion in the median plane only, have been in good agreement with solutions of the exact equations of the "Ridge Runner" program. More accurate, and more elaborate, differential equations for the three dimensional motion have been in preparation by N. Vogt-Nilsen,[10] guided by E. S. Akeley's treatment[4] of the vector potential.

APPENDIX III

VARIATIONAL METHOD FOR DETERMINING STABILITY BOUNDARIES AND CHARACTERISTIC EXPONENTS FOR THE HILL EQUATION

By the change of variable $N\theta = 2\tau$, the Hill equation encountered in the body of this report may be put into the standard form:

$$\frac{d^2Y}{d\tau^2} + (A + B\cos 2\tau + C\cos 4\tau)Y = 0.$$

Information relating the coefficients of this equation at the stability boundaries may be obtained conveniently by variational methods, since the equation then has a periodic solution. By considering the "isoperimetric" problem

$$\delta\int_0^{\pi}\frac{1}{2}\left[Y'^2-(B\cos 2\tau+C\cos 4\tau)Y^2\right]d\tau=0$$

$$\int_0^{\pi}\frac{1}{2}Y^2\,d\tau=1\,,$$

with -A playing the role of the Lagrange multiplier, we arrive at the result

$$\Delta=\left[\frac{\int_0^{\pi}\frac{1}{2}\left[Y'^2-(B\cos 2\tau+C\cos 4\tau)Y^2\right]d\tau}{\int_0^{\pi}\frac{1}{2}Y^2\,d\tau}\right]_{min.}$$

By use of trial solutions

$$Y = 1+2P\cos 2\tau + 2Q\cos 4\tau+\cdots$$

or $Y = \cos\tau + U\cos 3\tau + V\cos 5\tau+\cdots$

the expression to be minimized may be put into an algebraic form appropriate to the $\sigma = 0$ or $\sigma = \pi$ boundaries, respectively. This form is suited to rapid solution by a high-speed digital computer[12] -- by the minor modification of leaving the normalization of the trial functions unspecified, the same general technique may be used to provide simultaneous homogeneous linear equations suitable for solution with a desk computer.[13]

With a bit more algebraic complexity similar methods may be applied to estimate the relation between the parameters of the differential equation and values of σ away from the stability boundaries. For this purpose one notes that on the basis of the Floquet theory, as Courant and Snyder have pointed out,[14] solutions may be written in the "phase-amplitude" form

$$Y(\tau) = w(\tau)e^{\pm i[L\tau + \psi(\tau)]},$$

where, in the stable case, $w(\tau)$ and $\psi(\tau)$ are real periodic functions with the period (π) of the equation and L is a real constant equal to σ/π. One then considers the variational statement

$$\delta\int_0^{\pi}\frac{1}{2}\left[w'^2-(B\cos 2\tau + C\cos 4\tau)w+(L+\psi')^2w^2\right]d\tau=0$$

$$\int_0^{\pi}\frac{1}{2}w^2\,d\tau = 1,$$

to obtain

$$A = \left[\frac{\int_0^{\pi}\frac{1}{2}\left[w'^2-(B\cos 2\tau + C\cos 4\tau)w^2+(L+\psi')^2w^2\right]d\tau}{\int_0^{\pi}\frac{1}{2}w^2\,d\tau}\right]_{min.}$$

By the use of trial functions

$$w = 1 + 2P\cos 2\tau + 2Q\cos 4\tau + \cdots,$$

$$\psi' = 2R\cos 2\tau + 2S\cos 4\tau + \cdots,$$

the expression to be minimized again assumes an algebraic form which, by aid of high-speed computation, can give estimates of the value of A associated with specified values of B, C, and $L \equiv \sigma/\pi$.

The foregoing methods have been used in ILLIAC computations to provide tables[15] giving the estimated values of A for values of the remaining parameters in the range

L: 0 (0.1) 1.0

B: 0 (0.2) 5.0

C: -2.5 (0.5) 2.5,

together with the values found for the coefficients of the trial functions. For convenient use, and because the estimates of A are somewhat inaccurate for values of L close to but less than unity, supplementary graphs[16] have been prepared from these data giving (i) A vs. $\cos\sigma$ for various values of B and C and (ii) A vs. B for various values of C and σ.

As has been remarked, the foregoing methods appear to suffer somewhat in regard to accuracy for values of L near but less than unity, although very close agreement with known values for the stability limits is found in those cases for which comparison can be made. It is believed that close to the $\sigma = \pi$ limit the form assumed for the trial function which represents ψ' is not favorable. It may, therefore, be appropriate to mention a modification[17] of the variational procedure which might be useful if more

accurate results should be desired for other applications. In this modification the single trial function w is employed, use being made of the identity $w^2(L + \psi') = K^2$, a constant. Specifically,

$$\pi L \equiv \sigma = \int_0^{\pi} (L+\psi')\,d\tau = K^2 \int_0^{\pi} \frac{1}{w^2}\,d\tau = K^2 \pi \left\langle \frac{1}{w^2} \right\rangle ,$$

or

$$K^2 = \frac{L}{\left\langle \frac{1}{w^2} \right\rangle} .$$

Since, as has been noted,

$$A = \left[\frac{\langle w'^2 - (B\cos 2\tau + C\cos 4\tau) w^2 + (L+\psi')^2 w^2 \rangle}{\langle w^2 \rangle} \right]_{min.},$$

we obtain the equivalent result

$$A = \left[\frac{\langle w'^2 \rangle - \langle (B\cos 2\tau + C\cos 4\tau) w^2 \rangle + \frac{L^2}{\left\langle \frac{1}{w^2} \right\rangle}}{\langle w^2 \rangle} \right]_{min.}$$

For convenience one may make the chnage of variable

$$v \equiv w^2$$

to obtain

$$A = \left[\frac{\frac{1}{4} \left\langle \frac{v'^2}{v} \right\rangle - \langle (B\cos 2\tau + C\cos 4\tau) v \rangle + \frac{L^2}{\langle 1/v \rangle}}{\langle v \rangle} \right]_{min.}$$

These expressions are conveniently homogeneous of degree zero in their respective trial functions. The trial functions should be non-zero, continuous, have a continuous derivative, be periodic with the period π , and (in the case considered here) be even

about 0 and $\pi/2$. By virtue of the property last mentioned, the averaging need then be taken only over the interval 0 to $\pi/2$. A limited number of hand-computed examples with simple trial functions indicate that this modified procedure will give good results, even for values of L near unity, although in practice some of the integrations associated with the averaging process may have to be performed numerically.

If the trial function v is taken to be of the form

$$v = 1 + 2P_1 \cos 2\tau + 2P_2 \cos 4\tau + \ldots,$$

we thus obtain

$$A \leq \left[\frac{1}{4} \left\langle \frac{(4P_1 \sin 2\tau + 8P_2 \sin 4\tau + \ldots)^2}{1 + 2P_1 \cos 2\tau + 2P_2 \cos 4\tau + \ldots} \right\rangle + \frac{L^2}{\left\langle \frac{1}{1 + 2P_1 \cos 2\tau + 2P_2 \cos 4\tau + \ldots} \right\rangle} - BP_1 - CP_2 \right]_{min.}$$

In tabulating the results of a minimization procedure based on this method, it would be desirable to include the value of $\langle 1/v \rangle$, since an estimate of $1/(L + \psi')^{\frac{1}{2}}$, which equals $\left[\frac{v}{L} \left\langle \frac{1}{v} \right\rangle \right]^{\frac{1}{2}}$, is useful in judging the amplitude resulting from scattering and for determining the displacement of the equilibrium orbit due to misalignments.

APPENDIX IV

NUMERICAL COMPARISON WITH ILLIAC RESULTS

In the table which follows we give comparisons between the results obtained for radial motion with the ILLIAC, using the exact equations of motion, and the corresponding values predicted by the equations of this report.

The theoretical equations used for estimation of the fixed points are

$$x_{fixed} = -\frac{1}{2}(f/N)^2$$

$$x'_{fixed} = -\frac{Nf}{N^2 - (k+1)} \cong p_{x\ fixed}.$$

For comparison with known results in one case, we take the predicted amplitude (about the fixed point) for the forced oscillation as

$$\mp \frac{f}{N^2 - (k+1)}$$

The phase shift, σ_u, experienced by the small-amplitude radial betatron oscillations in traversing one period is given by the smooth approximation as

$$\sigma_u = \frac{2\pi\sqrt{k+1}}{N}.$$

For a more reliable estimate, we determine the coefficients A, B, and C in the standard Hill equation

$$\frac{d^2u}{d\tau^2} + (A + B\cos 2\tau + C\cos 4\tau)u = 0,$$

using the relations

$$A = \left(\frac{2}{N}\right)^2\left[k+1 - \frac{1}{2}\frac{(f/w)^2}{N^2-(k+1)}\right]$$

$$B = \left(\frac{2}{N}\right)^2 \frac{f}{w}$$

$$C = \left(\frac{2}{N}\right)^2 \frac{1}{2}\ \frac{(f/w)^2}{N^2-(k+1)},$$

and then interpolate σ_u from the graphs mentioned in Appendix III.

As one measure of the extent to which one might expect in advance accurate results from the theory, we list the quantity $\frac{f/w}{N^2-(k+1)}$, which should be small in comparison to unity.

It should also be mentioned that the examples given do not necessarily represent practicable combinations of machine parameters, the first example being in fact axially unstable and others having possibly undesirably large values for σ_u.

APPENDIX V

DIAGRAM OF STABILITY REGION

The first stability region has been plotted (Fig. 2) as a function of machine parameters on the basis of the theory presented in this report and assisted by the graphs[16] describing the character of solutions to the Hill equation. The basic variables are k/N^2 and $f/(wN^2)$, for $k \gg 1$, and the computed results are expected to apply for small-amplitude betatron oscillations most accurately when the ordinates are small in comparison to unity (say $\frac{f}{wN^2} < \frac{1}{3}$). A more accurate plot of this character could be prepared, if required, by use of ILLIAC solutions of more accurate equations of motion.

COMPARISON WITH ILLIAC RESULTS

Machine Parameters						Fixed Points			Forced Amplitude			σ_u	
k	k' = k+1	f	1/w	N	$\frac{f/w}{N^2-(k+1)}$	Theor.	x x'	Observed	Theoretical	Observed	A B C	Smooth Approx. Hill Eqn. Graphs	Observed
75	76	0.25	167	27	0.064	-0.000043 -0.01034		-0.00004 -0.0104			0.4097 0.2291 0.0073	0.65π 0.65π	0.65π
75	76	0.25	1047	27	0.401	-0.000043 -0.01034		-0.00004 -0.0108 Approx.			0.1291 1.4372 0.2879	0.65π 0.82π	0.79π
299	300	0.25	4000	52	0.416	-0.0000116 -0.00541		-0.0000144 -0.00564	∓0.000104	∓0.0000987	0.137 1.48 0.307		
150	151	0.25	2094	37	0.430	-0.000023 -0.0076		-0.00004 -0.0079 Approx.			0.1125 1.53 0.3287	0.66π 0.91π	0.86π

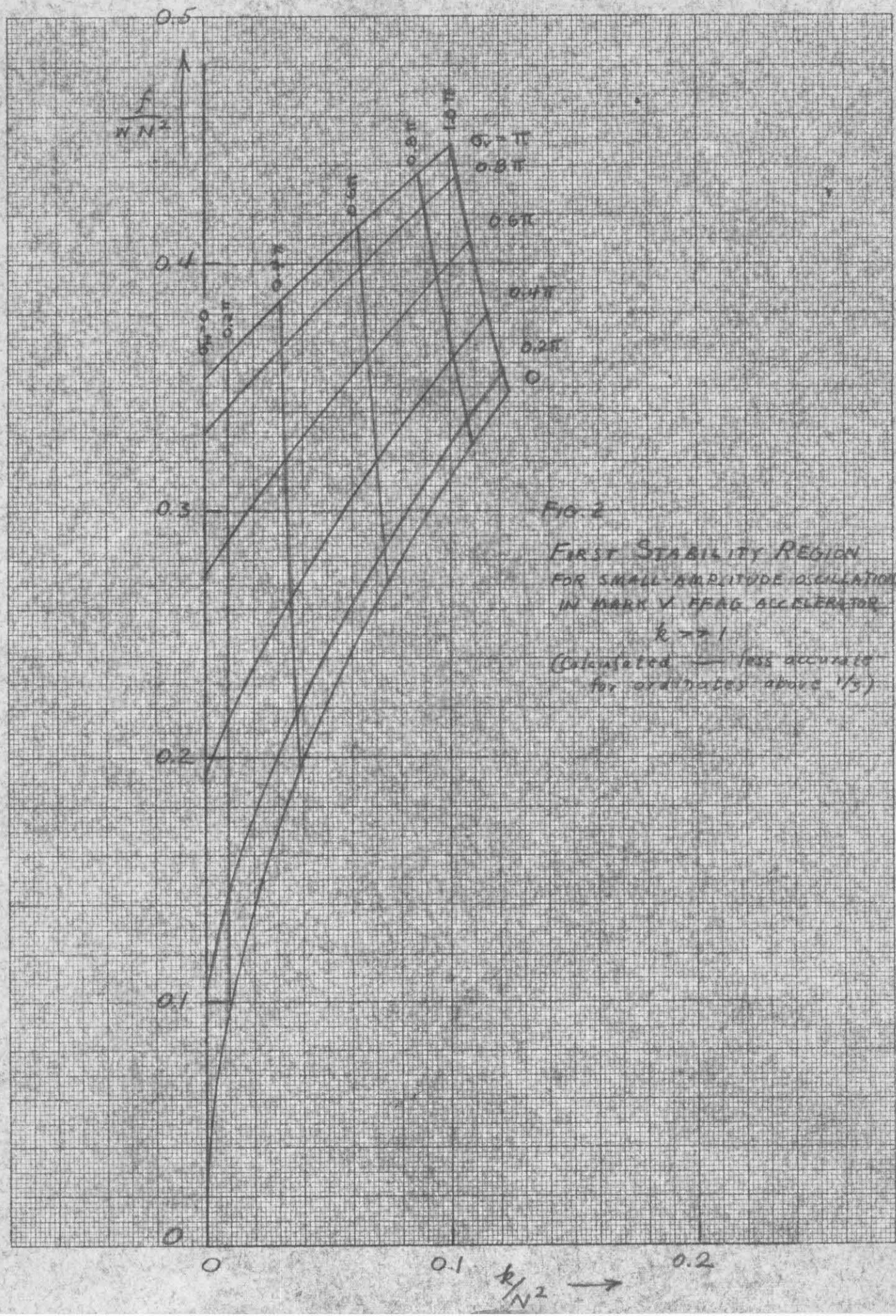

FIG. 2

FIRST STABILITY REGION
FOR SMALL-AMPLITUDE OSCILLATIONS
IN MARK V FFAG ACCELERATOR
$k \gg 1$
(Calculated — less accurate for ordinates above 1/3)

K&E 10 X 10 TO THE CM. 359-14 KEUFFEL & ESSER CO. MADE IN U.S.A.