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DETERMINATION OF SIGMA IN THE MODEL FFAG MARK Ib ACCELERATOR

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I Introduction

The methods for determining σ in the Mark Ib Model, which are to be described in this report, have grown out of a number of discussions by the members of the MURA Summer Study group. A proposed source of nearly mono-energetic electrons, eminating very nearly from one point, is accomplished by means of cold field point emission. This source is placed at a point of symmetry of the machine, and the subsequent location of the beam is accomplished by use of a flourescent screen.

An equilibrium orbit is located approximately by varying either the radial location of the source or the accelerating voltage until the beam is located at the same radial distance at corresponding points in two sectors. The equilibrium orbit need not be located precisely, for within the validity of the linear approximation for oscillations about the equilibrium orbit the difference between two oscillations obeys the same equation as an oscillation about the equilibrium orbit. Thus if we use two electron beams near an equilibrium orbit and measure the difference in velocities and positions, we may apply the theory

developed here for oscillations about a known equilibrium orbit.

II First Method

In this method the source is fitted with a screen which has a small hole so that the source emits electrons primarily in one direction. It is placed on an equilibrium orbit at the center of a positive (negative) sector and set so that it emits along the orbit, as shown in Fig. I,

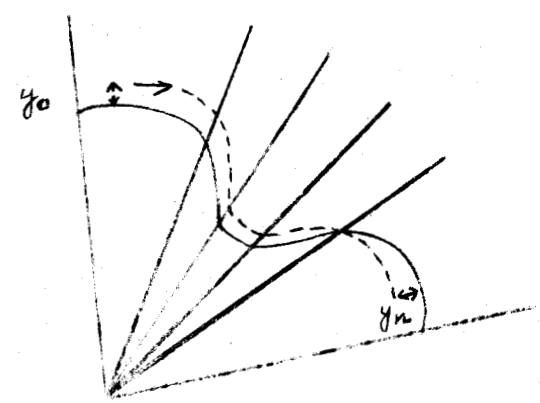

Fig. I

The source is displaced radially by an amount y_o. The displacement y_n, at the center of the nth positive sector after the source, is measured, clearly:

$$y_n = y_o \cos n\sigma$$

Because of the linearity theorem, y need not be measured exactly from an equilibrium orbit, and if the source does not

emit in a direction exactly tangent to an equilibrium orbit there is no error, for as long as $\dot{y}_o$ is the same in both positions (as it is, in practice) the terms in $\dot{y}_o \beta \sin\sigma S$ cancel.

$$\text{Let } R_n = \frac{y_n}{y_o} = \cos n\sigma$$

Then the tolerable error in R_n is given by:

$$\Delta R_n = - n \sin n\sigma (\Delta\sigma)$$

Using the approximate values $\nu_x = 2.75$, $\nu_z = 1.75$, and setting $\Delta\sigma = \pm .1$, which should be sufficient to ensure not being on a resonance, the values R_n and ΔR_n given in Table I were evaluated.

n	$(R_n)_x$	$(R_n)_y$	$(\Delta R_n)_x$	$(\Delta R_n)_y$
1	-.81	+.19	-.08	-.10
3	+.98	-.56	-.06	+.25
5	-.17	+.83	+.49	-.28
7	-.97	-.98	-.28	-.15
8	+.04	+.04	+.80	+.80

Table I

Error in R arises from error in y_n and y_o. The error in y_o is probably very small since one can locate the source quite accurately. Error in y_n, leads in an error in R, given by $\Delta R = \frac{\Delta y_n}{y_o}$. Corrections due to orbit distortion in the source are shown to be negligible in Appendix I.

This method may be used when only 1 1/2 sectors are available.

It is subject to difficulty in that it is hard to make a source which will define a beam sufficiently accurately.

III Second Method

For this, and the third method, the source consists simply of the accelerating mechanism with a defining aperture which is rather wide. This is placed so that the emitting point, A, is at the center of a positive (negative) sector. At the center of the next positive sector a grid with two small holes (of known separation) is located. The distance between beams is measured at the center of the next positive sector.

The displacement of a beam from the equilibrium orbit at the center of the nth positive sector after the source is:

$$y_n = (\cos n\sigma) y_0 + \beta (\sin n\sigma) \dot{y}_0$$

Consider the rays which leave point A, $y_0 = 0$, $\dot{y}_0 \neq 0$:

$$y_1 = \beta \sin \sigma \; \dot{y}_0 \qquad \bar{y}_1 = \beta \sin \sigma \; \bar{\dot{y}}_0$$

$$y_2 = \beta \sin 2\sigma \; \dot{y}_0 \qquad \bar{y}_2 = \beta \sin 2\sigma \; \bar{\dot{y}}_0$$

Hence:

$$\frac{y_2 - \bar{y}_2}{y_1 - \bar{y}_1} = 2 \cos \sigma$$

or

$$\cos \sigma = \tfrac{1}{2} \; \frac{y_2 - \bar{y}_2}{y_1 - \bar{y}_1} \equiv \tfrac{1}{2} R$$

Thus a measurement of $y_2 - \bar{y}_2$, the distance between beams in

the second sector from the source, yields σ directly. Once again the linearity argument ensures that the above is valid even if the source is not exactly on an equilibrium orbit, since the terms in $\cos\sigma\ y_0$ will cancel as y_0 is the same for both orbits.

The limit of error tolerable in R is given by:

$$\Delta R = -2 \sin \sigma \ (\Delta \sigma)$$

For $\Delta\sigma = \pm .1$ this yields $(\Delta R_x) = .17$, $(\Delta R_y) = .20$. This can be improved by measuring at sectors which are further apart.

This analysis assumes the electrons have constant speed throughout their motion. This is not correct, and thus we should consider trajectories corresponding to constant velocity which deviate from the true trajectories within the accelerating mechanism. Fig. II indicates the true trajectories and projected curves of constant energy. These intersect at the point α,

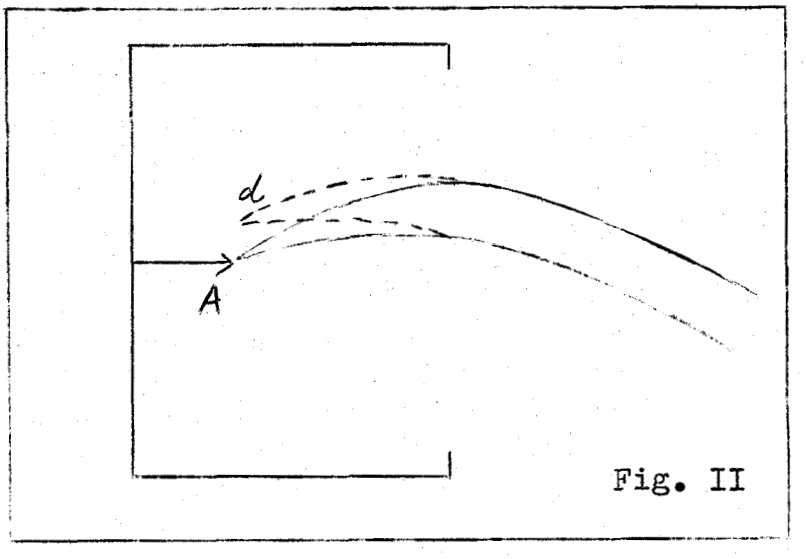

Fig. II

which should be at a point of symmetry of the machine. We shall show in Appendix II that α is the same for all orbits leaving within a small solid angle, and that α is displaced from A by a negligible amount.

This method needs at least 2 1/2 sectors, but yields the most accurate values of σ .Note that one only has to measure the distance between two simultaneously viewed beams.

IV Third Method

In this method the source is located at the center of a positive (negative) sector, and the screen is located in the center of the air gap between this sector and the next. The observing screen is placed in the next air gap.

The general expression for an oscillation is:

$$y = \phi_0 (s) \cos \sigma s + \phi_1 (s) \sin \sigma s$$

where s is the path length measured in units of the path length through one sector and ϕ_0 and ϕ_1 are periodic functions.

Consider a ray which has y = o at s = o (the source) and then goes through a hole in the screen in the middle of the first straight section, and is located in the middle of the second straight section.

$$y_0 = \phi_0 (o) = o$$

$$y_1 = \phi_0 (s_1) \cos \sigma s_1 + \phi_1 (s_1) \sin \sigma s_1$$

$$y_2 = \phi_0 (s_2) \cos \sigma s_2 + \phi_1 (s_2) \sin \sigma s_2$$

Thus

$$y_1 = \phi_1 (s_1) \sin \sigma s_1$$

$$y_2 = \phi_1 (s_2) \sin \sigma s_2$$

By symmetry $\phi_0 (s_2) = \phi_1 (s_1)$ $\phi_1 (s_2) = -\phi (s_1)$ and thus:

$$R_{2,1} = \frac{-y_2}{y_1} = \frac{\sin \sigma s_1}{\sin \sigma s_2}$$

Again the linearity theorem holds. One obtains:

$$\Delta\sigma = \frac{\sin^2 \sigma S_2 \qquad (\Delta R_{2,1})}{\left[S_1 \sin \sigma S_2 \cos \sigma S_1 - S_2 \sin \sigma S_1 \cos \sigma S_2 \right]}$$

From Cole's work one can obtain very accurate values of s_1 and s_2.

$$s_1 = .342$$
$$s_2 = .658$$

Thus approximately (using $\nu_x = 2.75$, $\nu_z = 1.75$,) we obtain

$$(R_{2,1})_x = .682 \qquad\qquad (R_{2,1})_z = .577$$

$$(\Delta\sigma)_x = 5.34\ (\Delta R_{2,1})_x \qquad\qquad (\Delta\sigma)_z = 11.24\ (\Delta R_{2,1})_z$$

These formulas indicate that this method will not be good to obtain accurate values of σ. Because it is tedious to obtain σ from $R_{2,1}$, the last expression may be used to calculate σ if the measured values of $R_{2,1}$ are close to the estimated values.

Appendix I

In Method I a correction must be made for the change in y_o due to distortion of orbits within the source.

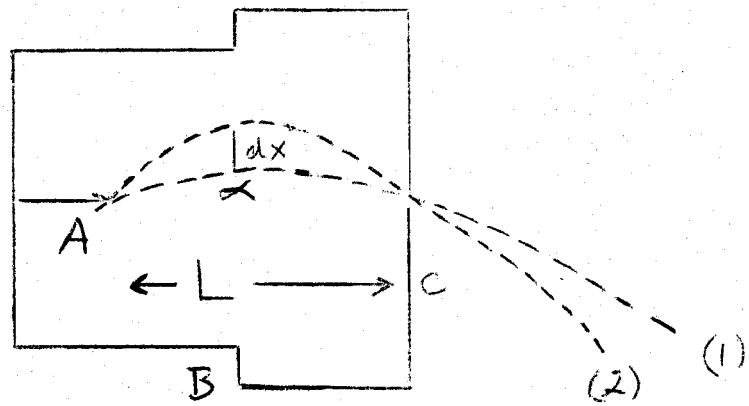

Fig. III Potential applied between A & B

The source is located so that x is at the center of a positive sector. The curvature in orbits is due to the magnetic field H, and we shall calculate assuming the velocity is constant over the distance from A to C. When the source is displaced a distance y_o, the average magnetic field is increased by amount dH, distorting orbit (1) into orbit (2).

$$dH = k\,H\left(\frac{y_o}{r}\right)$$

The radius of curvature ρ is changed by

$$d\rho = -\rho\,\frac{dH}{H} = -k\rho\left(\frac{y_o}{r}\right)$$

By simple geometry

$$dx = \left[1 - \frac{\rho}{\sqrt{\rho^2 - \frac{L^2}{4}}}\right] d\rho$$

which must be added to the displacement y_o of the source in order to obtain the true y_o. Employing the fact that $L << \rho$

$$dx = \frac{1}{8} \frac{kL}{\rho r} y_o$$

Typically r = 50 cm, k = 3.5, ρ = 20 cm, L = 1 cm, which yields

$$dx = 4.4 \times 10^{-4} y_o$$

Thus dx is, in practice, a negligible correction.

Appendix II

In Methods II and III it is essential that the electron trajectories intersect at a point of symmetry of the machine. Since the electrons start from a point, but undergo a change of velocity in the electric field of the source, orbits corresponding to constant speed electrons do not intersect at the point source. We shall show that this shift in intersection point is negligible.

An electron in the presence of a uniform magnetic field H, and a radial electric field (potential $V(\rho)$,) is described by the Lagrangian L:

$$L = \tfrac{1}{2} m \left[\dot{\rho}^2 + \rho^2 \dot{\theta}^2 \right] - V(\rho) + \frac{e H_o}{2} \rho^2 \dot{\theta}$$

where (ρ, θ, z) are cylindrical coordinates, and we are confining our attention to the z = o plane. Since θ is a cyclic variable, $\frac{\partial L}{\partial \theta}$ = constant. Conservation of energy supplies another constant of the motion. Employing the boundary conditions at t = o, ρ = a, $\dot{\theta}$ = o, $\dot{\rho}$ = 0, θ = 0, we obtain:

$$\dot{\theta} = \frac{\omega}{m} \left[\frac{a^2}{\rho^2} - 1 \right]$$

$$\dot{\rho} = \sqrt{\frac{2}{m}\left[V(a) - V(\rho)\right] - \frac{\omega^2}{m^2}\left[\frac{a^2-\rho^2}{\rho}\right]^2}$$

where: $\omega = \frac{eH_0}{2}$

Thus the trajectory is defined by

$$\theta(\rho) = \frac{\omega}{m}\int_a^{\rho} \frac{(a^2-\rho^2)\,d\rho}{\rho^2\sqrt{\frac{2}{m}\left[V(a)-V(\rho)\right] - \frac{\omega^2}{m^2}\left[\frac{a^2-\rho^2}{\rho}\right]^2}}$$

We are interested in $V(\rho) = \frac{V_0 a}{\rho}$ for $\rho > a$, but this proves difficult to evaluate. We shall choose instead $V(\rho) = V_0\,(1 - \rho/r_0)$, $a = 0$.

Evaluation of the integral yields:

$$\rho = \frac{m V_0}{\omega^2 r_0}\left[1 - \cos\theta\right] \qquad \text{for} \quad \rho \leq r_0$$

Let $\rho_0 = \sqrt{\frac{m V_0}{2\omega^2}}$ = radius of curvature of an electron with a velocity equal to that obtained from falling through the potential V_0.

Then $$\rho = \frac{2\rho_0^2}{r_0}\left[1 - \cos\theta\right] \qquad \text{for} \quad \rho \leq r_0$$

For our injector $r_0 \approx 1$ cm., $\rho_0 = 20$ cm. and thus $\theta \approx 0$. We now wish to match this orbit at $\rho = r_0$ to a circle of radius ρ_0. This yields a circle, whose equation is

$$y = \frac{(x + .5)^2 - 2.25}{20} \qquad \text{for } y \approx 0.$$

Now consider another trajectory which differs from the above in that it starts from the same point, but in a different direction, specified by θ. For our machine $\theta \approx 1^\circ$. The equation of the

circle corresponding to the projected orbit is given by replacing x and y by $x + y\theta$ and $y - x\theta$ in the above.

These curves intersect at a point which is independent of θ and given by $x = .005$ cm., $y = -.9$ cm. This change in x is negligible. It is easy to show that the use of $V(\rho) = \frac{V_0 a}{\rho}$ will decrease x.