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RADIAL AND PHASE MOTION IN A SYNCHROTRON

Norman C. Francis*

Department of Physics, Indiana University
Bloomington, Indiana

and
Member of MURA Working Group

ABSTRACT

In Part I the adiabatic solution to the coupled linear phase and radial equations of motion is obtained. The solution presented is a good approximation when the momentum gain per turn is small compared with the momentum of the particle.

In Part II the effect of an accelerating electric field which depends on the radial position is discussed.

PART I

In the median plane the coupled equations of radial and phase motion can be derived from the following Hamiltonian:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{k_1 x_1^2}{2} + \frac{k_2 x_2^2}{2} - c x_1 p_2 \cdots \quad (1)$$

where $m_1 = \gamma$, $m_2 = \gamma^3$, $k_1 = (1 - n)\gamma w_s^2$, $k_2 = \frac{eV}{2\pi} \frac{\cos \phi_s}{R_o^2}$ and $c = w_s$. γ is the total energy of the particle, w_s is the frequency of the R.F. accelerating field, e is the charge of the electron, V is the maximum voltage across the R.F. gap, R_o is the radius of the equilibrium orbit and ϕ_s is the relative phase of the particle with respect to the R.F. when the particle is synchronous. The field index n is defined in the usual way.

$$n = -\left(\frac{r}{H}\frac{\partial H}{\partial r}\right)_{r = R_o}$$

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*Now at the Knolls Atomic Power Laboratory, Schenectady, New York.

With the exception of k_2, all of the coefficients appearing in Eq. (1) may be considered to be slowly varying functions of the time

$$x_1 = r - R_o$$

where r is the radial position of the particle and

$$x_2 = R_o \chi$$

where χ is defined to be the difference between the relative phase and the synchronous phase, ϕ_s.

The equations of motion generated by Eq. (1) are linear. Consequently the solution obtained is accurate when

$$/x_i/ < R_o \qquad i = 1,2.$$

Betatron oscillations introduced when the particle crosses the R.F. gap have been neglected. This error is small when the energy gain per time is small.

Although the variables in Eq. (1) may be separated with one complicated time dependent canonical transformation, it is instructive to exhibit the separation with the product of 3 canonical transformations. The first of these is time independent while the last two transformations are time dependent. In the third transformation the adiabatic assumption is used explicitly.

Transformation 1

Let S be the generating function for the transformation. Let

$$S_1 = X_2 x_2 - P_1 x_1.$$

The new coordinates, X_i, and new momenta P_i are related to the old ones as follows:

$$p_1 = P_1, \; X_2 = p_2, \; P_2 = - x_2, \; x_1 = X_1$$

The hamiltonian then becomes

$$H_1 = \frac{P_1^2}{2m_1} + \frac{k_1}{2} X_1^2 - \frac{k_2 P_2^2}{2} + \frac{X_2^2}{2m_2} - cX_1X_2.$$

Transformation 2

Let $S_2 = (u_1 + a_2u_2)P_1 + (a_1u_1 + u_2)P_2$ where $a_2 = \frac{1}{\gamma w_s(1 - n)}$ and $a_1 = \frac{eV \cos \phi_s}{2\pi R_o^2 w_s(1 - n)}$. u_1 and u_2 are the coordinates conjugate to the momenta π_1 and π_2 respectively.

$$X_1 = u_1 + a_2u_2$$

$$X_2 = a_1u_1 + u_2$$

$$\pi_1 = P_1 + a_1P_2$$

$$\pi_2 = P_1a_2 + P_2.$$

The hamiltonian then becomes

$$H_2 = \frac{\pi_1^2}{2M_1} + \frac{K_1u_1^2}{2} - \frac{\pi_2^2}{2M_2} - \frac{K_2u_2^2}{2} - \dot{\gamma}\left\{\pi_1u_2F_2 + \pi_2u_1F_1\right\}$$

where $M_1 = \gamma$, $M_2 = \frac{2\pi R_o^2}{eV \cos \phi_s}$, $K_1 = \gamma(1 - n)w_s^2$, $K_2 = \frac{\Lambda}{\gamma^3(1 - n)}$,

$F_2 = \frac{da_2}{d\gamma}$, $F_1 = \frac{da_1}{d\gamma}$, and $\Lambda = \left[\gamma^2 - (1 - n)\right]$.

Transformation 3

In the above two transformations it was assumed that the V is small. We shall now make the further assumption that $(\dot{\gamma})^2$ and $\ddot{\gamma}$ may be neglected. The generating function that diagonalizes the hamiltonian is

$$S = u_1\phi_1 + u_2\phi_2 + \dot{\gamma}(au_1u_2 + b\phi_1\phi_2)$$

where $a = \dfrac{K_2F_1 - K_1F_2}{\dfrac{K_1}{M_1} - \dfrac{K_2}{M_2}}$ and $b = \dfrac{\dfrac{F_2}{M_2} - \dfrac{F_1}{M_2}}{\dfrac{K_1}{M_1} - \dfrac{K_2}{M_2}}$

Then

$$\pi_1 = \phi_1 + \dot{\gamma} a u_2$$

$$\pi_2 = \phi_2 + \dot{\gamma} a u_1$$

$$\psi_1 = u_1 + \dot{\gamma} b \phi_2$$

$$\psi_2 = u_2 + \dot{\gamma} b \phi_1$$

and

$$H_3 = \frac{\phi_1^2}{2M_1} + \frac{K_1}{2} \psi_1^2 - \frac{\phi_2^2}{2M_2} - \frac{K_2}{2} \psi_2 .$$

The hamiltonian is now separated and therefore the adiabatic theorem for one variable may be employed. This procedure is valid when the change in the frequency of the oscillation in one period is small compared with the frequency. This criterion clearly fails when the frequency vanishes.

After using the invariance of the action integral to determine the amplitude of motion and transforming to the original coordinates, we obtain

$$x_1 = \frac{A_1 \sin(w_1 t + \delta_1)}{(\gamma w_s)^{1/2}} \quad \frac{A_2 \sin(w_2 t + \delta_2)}{\Gamma^{1/2} \gamma^{1/2} w_s}$$

$$x_2 = \frac{A_1 \cos(w_1 t + \delta_1)}{(1 - n)^{1/2} (\gamma w_s)^{1/2}} \quad \frac{A_2 \Gamma^{1/4} (2\pi R_o^2)^{1/2} (1 - n)^{1/2} \cos(w_2 t + \delta_2)}{\gamma^{3/4} (eV \cos \phi_s)^{1/2}}$$

where

$$w_1^2 = (1 - n)w_s^2$$

$$w_2^2 = \frac{eV \cos \phi_s}{2\pi R_o^2 \gamma^3 (1 - n)}$$

and A_i and δ_i are constants. w_1 and w_2 are the familiar radial and phase oscillation frequences.

PART II

The effect of a radial variation of the voltage gain per turn will now be considered. For simplicity we shall assume that the electric field in the accelerating gap is localized in a region from $\theta_o - \frac{\varepsilon}{2}$ to $\theta_o + \frac{\varepsilon}{2}$ where ε is small. Then the electric field in the azimuthal direction is

$$E_\theta = f(\theta)E_o(1 + \frac{bx}{R_o}) \frac{R_o}{r} \sin w_s t$$

where

$$f(\theta) = f(\theta + 2\pi i) = 1 \text{ when } \theta_o + \frac{\varepsilon}{2} > \theta > \theta_o - \frac{\varepsilon}{2}$$

$$= 0 \text{ otherwise.}$$

E_o and b are constants. The torque is then

$$L = eV_o \sum \delta(\theta - 2\pi i)(1 + \frac{bx}{R_o})\sin w_s t.$$

The δ function is obtained when we take the limit in which $t \to 0$.

$$\sum \delta(\theta - 2\pi i) = \lim_{e \to 0} \left(\frac{f(\theta)}{t}\right)$$

and

$$V_o = R_o \varepsilon E_o.$$

Maxwell's equations require that a magnetic field in the z direction accompany the electric field, E_θ.

$$H_z = \frac{1}{r} \frac{b}{R_o} V_o \Sigma \delta(\theta - 2\pi i) \cos w_s t.$$

The resulting radial force F_r is

$$F_r = eV_o\dot{\theta} \frac{b}{R_o} \Sigma \delta(\theta - 2\pi i) \cos w_s t$$

In the linear approximation

$$F_r = eV_o \frac{b}{R_o} \left\{ \dot{\chi} \cos \phi_s - w_s \chi \sin \phi_s \right\} \Sigma \delta(2\pi i - (w_s t + \phi_s))$$

A term proportional to the derivative of the δ function has been neglected. This term is expected to be small, but it should be considered in detail for any particular gap field configuration. The $\dot{\chi}$ term is small compared with a similar term already present in the radial equation of motion. In the smooth approximation the δ function is replaced by $\frac{1}{2\pi}$. This procedure, of course, neglects effects which depend on the abrupt change of energy at the gap. The error however is small when V_o is small. Similarly, the torque, L, is

$$L = \frac{eV_o}{2\pi} \left\{ \chi \cos \phi_s - \frac{bx}{R_o} \sin \phi_s \right\} .$$

The coupled equations of motion may now be written as follows:

$$\ddot{x}_1 - a_1 x_1 - \gamma^2 w_s \dot{x}_2 + \gamma^2 c x_2 = 0 \qquad (2)$$

$$\ddot{x}_2 - a_2 x_2 + w_s \dot{x}_1 + c x_1 = 0$$

where

$$a_1 = w_s^2 \gamma, \quad c = \frac{eV_o}{2\pi} \frac{b}{R_o^2} \frac{\sin \phi_s}{\gamma^3} ,$$

$$a_2 = \frac{eV_o}{2\pi R_o^2} \frac{\cos \phi_s}{\gamma^3}, \quad b_2 = \gamma^2 w_s .$$

The time dependence of the slowly varying coefficients has been neglected in Eq. (2). To lowest order in V_o, the solution to Eq. (2) is independent of c. In fact the solution to Eq. (2) and to the equations of motion derived from the hamiltonian, Eq. (1), are the same. Consequently, the introduction of a torque which depends linearly on x_1 has no effect on the motion of a particle when V_o is small.